Proc. R. Soc. Lond. A **410**, 19–34 (1987)
Printed in Great Britain

A stochastic model of sea-surface roughness
I. Wave crests

BY M. S. LONGUET-HIGGINS, F.R.S.

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, U.K., and Institute of Oceanographic Sciences, Wormley, Surrey GU8 5UB, U.K.

(*Received* 26 *June* 1986)

In this paper we develop a two-scale model of sea-surface roughness, in which for the first time the randomness of both long and short waves is taken fully into account. The model includes long-wave–short-wave interactions, dissipation of the short-wave energy by breaking, and regeneration by the wind. This leads to an integral equation for the short-wave steepness, which is solved by iteration.

The effects of wind speed and of long-wave steepness upon the distribution of roughness at the long-wave crests are calculated and discussed. Also the effect of a band-width parameter for the long-wave spectrum. A random noise source can be included.

1. INTRODUCTION

In almost all sea states under the action of wind there is a rather broad spectrum of wavelengths, ranging from long gravity waves of period several seconds down to short gravity waves and capillary waves, which are highly responsive to the local wind speed. The latter are known to be modulated by the longer gravity waves on which they ride (Evans & Shemdin 1980) and it has been shown by numerous laboratory and field studies (Keller & Wright 1975; Wright *et al.* 1980; Plant & Keller 1983; Hoogeboom 1985) that these modulations are responsible for backscattering of centimetric radar waves from the ocean surface, and hence are the principal factor in imaging the surface by X-band and L-band radars. Nevertheless, the problem of how to account for the observed distribution of short-wave energy with regard to the phase of the longer waves has remained unsolved.

It has long been known theoretically that straining of the short waves by the orbital motion of the long waves, together with work done against the short-wave radiation stresses, tends to produce shortening and steepening of the shorter waves near the long-wave crests (Longuet-Higgins & Stewart 1960). Recently, some much more accurate calculations of this effect according to the principle of action conservation (Longuet-Higgins 1987) have shown the importance of including full nonlinearity of the longer waves, in some circumstances. A linear model for short-surface waves, which includes both growth and dissipation, has been proposed by Smith (1986). Nevertheless, all models so far proposed are unrealistic, in that it has been assumed, first, that the short-wave steepness is somehow rigidly determined, rather than having a distributed probability, and second, that the height of the longer waves is uniform and given. The purpose of the present paper is to replace these by more realistic assumptions, and indeed to show that the randomness of the longer waves plays an essential role in distributing the short wave steepness.

The basic idea, introduced in §3, is to suppose that the short-wave steepness s at the crest of a long wave of amplitude A has some probability density depending on A, and then to relate the density on one wave crest to the density on the wave crest immediately before. This procedure is made possible by the fact that we already know, at least approximately, the joint probability density of the amplitudes A_1 and A_2 of two successive waves. (The approximate joint distribution, sometimes called the bivariate Rayleigh distribution, has been used with success in predicting the properties of wave groups.) With suitable assumptions regarding the history of the short waves over the intervening time, including growth by wind action and possible dissipation by breaking, we are able to show that the density $p(s|A)$ of the short-wave slopes on a long wave of given amplitude A satisfies a certain integral equation, which can be solved in a straightforward manner by iteration (§4). In fact, the presence of both growth and dissipation are essential to the convergence of the solution.

In the subsequent sections, we present some results for typical values of the parameters. These show how the solutions depend upon the RMS steepness of the longer waves, and on the wind-induced growth rate. It is notable that the short-wave density $p(s|A)$ can be biomodal, with a peak both at the limiting steepness s_0 and at some lower value of s. In §6 we investigate the effect of group length of the longer waves. Somewhat different distributions are obtained for ocean swell (long-wave groups) and wind waves (short groups), respectively. In §7, we investigate the result of adding a certain amount of noise to the process of wave growth.

Sections 8 and 9 contain a physical discussion and a statement of the conclusions.

2. Description of the model

Suppose that the longer waves consist of a fairly narrow-band, gaussian disturbance, in which the time-interval τ between successive crests is nearly constant, as shown in figure 1. The amplitudes of two typical successive waves are denoted by A_1 and A_2. Superposed on the longer waves is a group of shorter waves whose RMS surface slopes, at the crests of longer waves, are denoted by s_1 and s_2 respectively. It is convenient at first to consider the time t in figure 1 as the time measured at the short-wave group. Thus, the elapsed time between the appearance of the short-wave group at crests 1 and 2 is given by

$$\tau = t_2 - t_1 = L/(C - c_g), \tag{2.1}$$

where L and C are the wavelength and phase speed of the long waves, and c_g is the group velocity of the short waves. Assuming $c_g \ll C$ we have

$$\tau \approx L/C = T, \tag{2.2}$$

approximately.

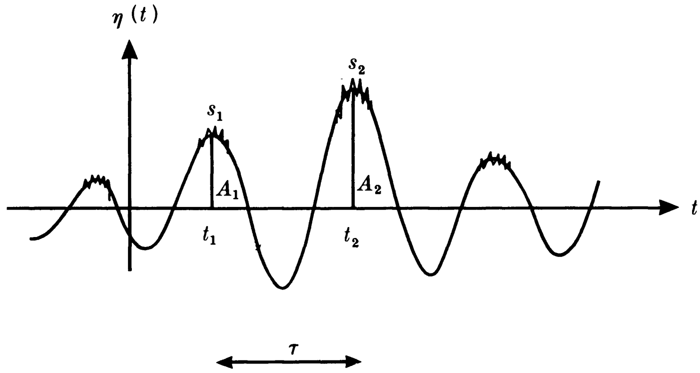

FIGURE 1. Time history of a short-wave group, with steepness s_1, s_2 at times $t = t_1$, t_2.

During this time the wind will generally act on the short waves so as to increase their slope, and if there were no dissipation by wave breaking the short waves would grow by a factor $e^{\beta\tau}$, where β is a time rate of growth, related to the wind speed (see, for example, Plant 1982). That is to say

$$s_2/s_1 = e^B, \quad B = \beta\tau. \tag{2.3}$$

On the other hand, if there were no generation or dissipation of short-wave action, the straining of the long waves would induce a change in the short-wave steepness depending only on the vertical elevation η of the waves, for a given long wavelength (Longuet-Higgins 1987, figure 9). Thus we should have

$$s_2/s_1 = f(KA_2)/f(KA_1), \tag{2.4}$$

where $K = 2\pi/L$ is the wavenumber of the long waves. In fact, it is a good approximation to take

$$f(K\eta) = e^{2.08(K\eta)+2.94(K\eta)^2}. \tag{2.5}$$

It will be sufficient initially to retain only the first term in this expression. Then from (2.4) and (2.5) we have

$$s_2/s_1 = e^{2.08K(A_2-A_1)}. \tag{2.6}$$

Supposing now that in the absence of wave breaking the rate of growth due to the wind is independent of the long waves, we shall have altogether

$$s_2 = s_1\,e^{B+\gamma(A_2-A_1)}, \quad \gamma = 2.08K, \tag{2.7}$$

provided that neither s_1 or s_2 is limited by breaking. In such a case it is convenient to write (2.7) reciprocally in the form

$$s_1 = s_2\,F(A_1, A_2), \tag{2.8}$$

where
$$F(A_1, A_2) = e^{-B+\gamma(A_1-A_2)}. \tag{2.9}$$

However, the RMS steepness s of the short waves is necessarily limited by breaking. Here we shall assume a sharp limit

$$s_1 \leqslant s_0, \quad s_2 \leqslant s_0, \tag{2.10}$$

an appropriate value of s_0 being given by (5.3). Hence s_1 is given by the smaller of (2.8) and (2.10), and s_2 by the smaller of (2.7) and (2.10). The situation is shown schematically in figure 2.

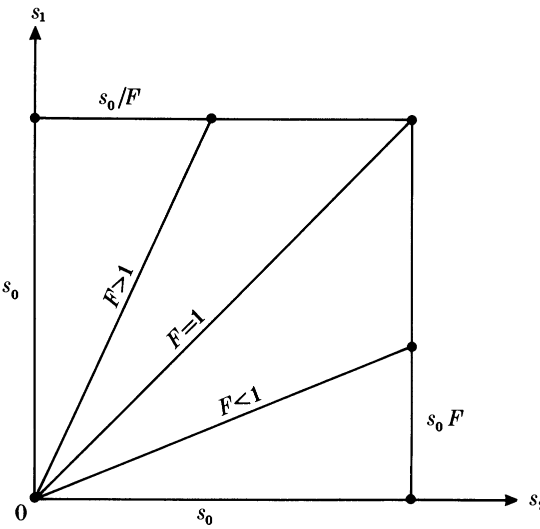

FIGURE 2. Schematic diagram of the relation between s_1 and s_2.

Physically, the introduction of both wave generation by wind and energy dissipation by breaking provides the means for attaining a statistically steady state.

3. PROBABILITY DENSITIES

We propose now to treat both the slope s and the amplitude A as random variables. The distribution of the long-wave heights will be regarded as given, and independent of the short waves over a sufficiently long period of time. The problem then is to determine the probability density of s at given wave amplitude A.

Now the density $p(A_1, A_2)$ of successive wave amplitudes A_1 and A_2 for narrow spectra has been derived and used in connection with the theory of group lengths (Rice 1944–1945, 1958; Kimura 1980; Longuet-Higgins 1984). In fact it is the 'two-dimensional Rayleigh' density

$$p(A_1, A_2) = \frac{A_1 A_2}{(1-\kappa^2)\bar{A}^4} \mathrm{e}^{-(A_1^2+A_2^2)/2(1-\kappa^2)\bar{A}^2} \times I_0\left(\frac{\kappa A_1 A_2}{(1-\kappa^2)\bar{A}^2}\right) \tag{3.1}$$

in which $\bar{A}$ and κ are constants and $I_0(z)$ is the modified Bessel function of order zero. $\bar{A}$ is equal to the RMS value of the surface elevation η and κ is a 'groupiness parameter' explicitly related to the spectral density of η. It can be shown that for narrow spectra $\kappa^2 \approx 1-4\pi^2\nu^2$, where ν is a dimensionless bandwidth (Longuet-Higgins 1984). The density of A_1 or A_2 alone is the Rayleigh density

$$p(A) = (A/\bar{A}^2)\,\mathrm{e}^{-A^2/2\bar{A}^2}. \tag{3.2}$$

Hence the probability density of A_1 *given* A_2, which is obtained by dividing $p(A_1, A_2)$ by $p(A_2)$, is

$$\left.\begin{aligned} p(A_1|A_2) &= \frac{1}{(1-\kappa^2)^{\frac{1}{2}}\bar{A}}\xi_1\,\mathrm{e}^{-\frac{1}{2}\xi_1^2}\left[\mathrm{e}^{-\frac{1}{2}\kappa^2\xi_2^2}I_0(\kappa\xi_1\xi_2)\right] \\ \text{where}\quad \xi_1 &= A_1/(1-\kappa^2)^{\frac{1}{2}}\bar{A}, \quad \xi_2 = A_2/(1-\kappa^2)^{\frac{1}{2}}\bar{A}. \end{aligned}\right\} \tag{3.3}$$

By using the model of §2 we shall now derive an equation for the probability density $p(s|A)$ of the short-wave slope on long waves of a given amplitude A.

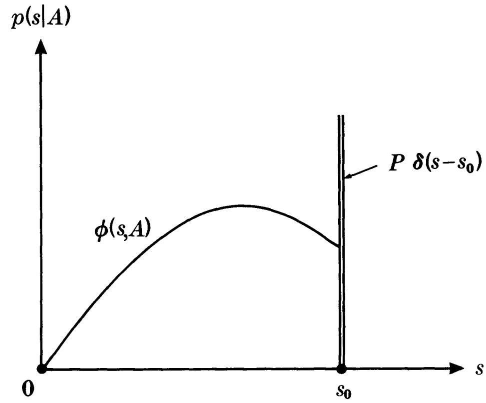

FIGURE 3. Sketch of the density $p(s|A)$, showing the continuous part $\phi(s, A)$ and the singular part $P(A)\,\delta(s-s_0)$.

First, the fact that the slope s is limited by breaking means that, for a typical value of A, there must be an exceptionally large probability density in the neighbourhood of $s = s_0$, as shown in figure 3. In fact, the density $p(s|A)$ has two components, a continuous component, which we denote by $\phi(s, A)$ in $0 < s < s_0$, and a singular component $P(A)\,\delta(s-s_0)$ in the neighbourhood of $s = s_0$, $\delta(x)$ being the Dirac delta function. Normalization of $p(s|A)$ clearly requires that

$$\int_0^{s_0} \phi(s, A)\,\mathrm{d}s + P(A) = 1. \tag{3.4}$$

We now need to relate $p(s_2|A_2)$ to $p(s_1|A_1)$. We shall regard $p(s_2|A_2)$ as being determined by the values of $p(s_1|A_1)$ on the previous wave, over the whole possible range of A_1. Thus,

$$p(s_2|A_2)\,\Delta s_2 = \int p(s_1|A_1)\,\Delta s_1\, p(A_1|A_2)\,\mathrm{d}A_1, \tag{3.5}$$

but this takes somewhat different forms according as the continuous or the singular parts of $p(s_1|A_1)$ or $p(s_2|A_2)$ are considered. First, the contribution to $\phi(s_2, A_2)$ from $\phi(s_1, A_1)$ presents no difficulties. s_1 is related to s_2 by (2.8), so that we have

$$\Delta s_1/\Delta s_2 \approx \mathrm{d}s_1/\mathrm{d}s_2 = F(A_1, A_2). \tag{3.6}$$

Moreover, the integral in (3.4) must be taken over the range of A_1 for which $s_1 < s_0$, hence

$$0 < A_1 < A_2 + \gamma^{-1}[B + \ln(s_0/s_2)] \equiv H(s_2, A_2), \tag{3.7}$$

say. The function H is shown schematically in figure 4.

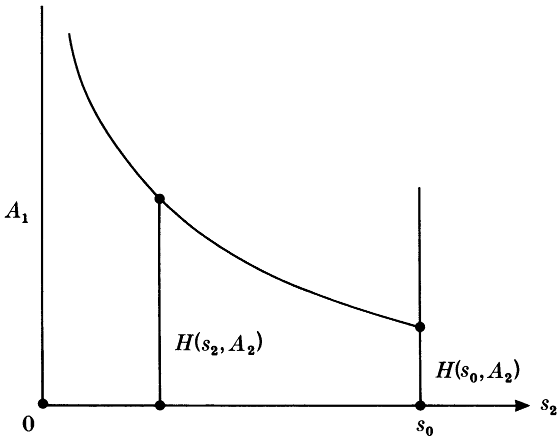

FIGURE 4. Sketch of the function $H(s_2, A_2)$ of equation (3.7).

The contribution to $\phi(s_2, A_2)$ from $P(A_1)$ can be expressed as a double integral

$$\iint [P(A_1)\,\delta(s_1 - s_0)\,\delta(s_2 F - s_1)\,\mathrm{d}s_1\, F(A_1, A_2)\, p(A_1|A_2)\,\mathrm{d}A_1 \tag{3.8}$$

and from the formula

$$\int \delta[y(x)]\,\mathrm{d}x = \frac{1}{(\mathrm{d}y/\mathrm{d}x)_{y=0}} \tag{3.9}$$

we have

$$\int \delta(s_2 F - s_1)\,\mathrm{d}A_1 = \frac{1}{[\mathrm{d}(s_2 F)/\mathrm{d}A_1]_{s_1 = s_2 F}} = \frac{1}{\gamma s_2 F} \tag{3.10}$$

from (2.10). Therefore, on reversing the order of integration in (3.8) the double integral becomes

$$(1/\gamma s_2)\, P(A_1)\, p(A_1|A_2), \quad A_1 = H(s_2, A_2). \tag{3.11}$$

Altogether, then, when $0 < s_2 < s_0$ we have

$$\left.\begin{aligned} \phi(s_2, A_2) &= \int_0^{H(s_0, A_2)} \phi(s_1, A_1)\, F(A_1, A_2)\, p(A_1|A_2)\,\mathrm{d}A_1 + \frac{1}{\gamma s_2} P(A_1)\, p(A_1|A_2), \\ A_1 &= H(s_2, A_2). \end{aligned}\right\} \tag{3.12}$$

On the other hand, when $s_2 = s_0$ we may integrate each side of (3.5) to obtain

$$P(A_2) = \int_0^{H(s_0, A_2)} \left[\int_{s_0 F}^{s_0} \phi(s_1, A_1)\,\mathrm{d}s_1 + P(A_1)\right] p(A_1|A_2)\,\mathrm{d}A_1. \tag{3.13}$$

Physically, this expresses the probability $P(A_2)$ as the sum of two parts. The first comes from the lower long waves, as a result of steepening by straining and by the effect of the wind; the second part comes from the slightly higher long waves, as a result of regeneration by the wind only, and in spite of the straining.

Equations (3.12) and (3.13) are clearly a coupled pair of integral equations to determine $\phi(s, A)$ and $P(A)$ simultaneously.

4. Solution of equations (3.12) and (3.13)

To find $\phi(s, A)$ and $P(A)$ we may proceed by successive approximation. Arbitrary starting values $\phi^{(1)}(s, A)$ and $P^{(1)}(A)$ satisfying the condition (3.4) may be substituted on the right of (3.12) and (3.13) and used to calculate $\phi^{(2)}(s, A)$ and $P^{(2)}(A)$. Substituting these again on the right-hand sides of the equations, we may calculate $\phi^{(3)}(s, A)$ and $P^{(3)}(A)$, and so on, the process imitating the development of the short wave probabilities from one wave to the next. If the functions $\phi^{(n)}(s, A)$ and $P^{(n)}(A)$ converge, we have a solution.

This was tried with two different sets of initial conditions, namely

$$\phi^{(1)}(s, A) = s_0^{-1}, \quad P^{(1)}(A) = 0 \tag{4.1}$$

and

$$\phi^{(1)}(s, A) = 0, \quad P^{(1)}(A) = 1. \tag{4.2}$$

With either set, the procedure was found to converge to the same solution, although with (4.2) the convergence was somewhat faster; typically four decimal places were obtained after only seven or eight iterations.

At each step, the accuracy of the integrations was checked by evaluating the left-hand side of (3.4) for $\phi = \phi^{(n+1)}$ and $P = P^{(n+1)}$; for it may be shown that if any pair $\phi^{(n)}$, $P^{(n)}$ satisfy (3.4) then $\phi^{(n+1)}$, $P^{(n+1)}$, if calculated by the iteration procedure, should also satisfy (3.4). For graphical accuracy (0.1 %) it was found sufficient to take 101 integration points in the range $0 \leqslant s/s_0 \leqslant 1$ and 51 integration points in the range $0 \leqslant A/\bar{A} \leqslant 5$.

5. The ranges of $\bar{A}K$, s and B

The steepness AK of the longer waves will generally be limited by the fact that the 'significant waves', i.e. those with amplitude $2\bar{A}$, cannot have a steepness much exceeding the limiting steepness $AK = 0.443$ for steady progressive waves. Thus we would expect

$$2\bar{A}K \leqslant 0.44, \quad \bar{A}K \leqslant 0.22 \tag{5.1}$$

approximately. The above estimate is consistent with the laboratory wind-wave data of Lake & Yuen (1978), who found a mean value of AK for the dominant waves not exceeding 0.28. For a Rayleigh distribution this would correspond to $(\frac{1}{2}\pi)^{\frac{1}{2}}\bar{A}K$; hence $\bar{A}K \leqslant 0.224$.

We note that in experiments with unsteady, plunger-generated waves, Ochi & Tsai (1983) found signs of breaking in individual waves for which $AK = 0.35$, less than the limiting value 0.443, but this is not inconsistent with (5.1).

In a similar way, if we consider the short waves as a narrow-band process in which the local wave steepness $\alpha = ak$ has a Rayleigh distribution,

$$p(\alpha) = (\alpha/s^2)\,\mathrm{e}^{-\alpha^2/2s^2} \tag{5.2}$$

it is reasonable to suppose that local breaking of the short waves imposes a limit

$$s \leqslant s_0 = 0.22. \tag{5.3}$$

We note that according to (5.2) and (5.3) the local mean-square slope surface slope s^2 satisfies

$$s^2 \leqslant 0.05, \tag{5.4}$$

which is not far from the limiting value 0.04 ± 0.02 found by Plant (1982).

For the rate of growth of surface slopes under the action of the wind, Plant (1982) has shown that the formula

$$\beta = 0.02(u_*/c)^2 \sigma \cos\theta \tag{5.5}$$

fits many different observations. Here u_* denotes the friction velocity, taken to equal 0.04 times the wind speed U at a standard elevation. c and σ are the phase speed and the radian frequency of the short waves, and θ is the angle between the directions of the short waves and the wind. Because our analysis refers to the short-wave slopes rather than to their spectral density, we have here divided the right-hand side of Plant's equation (1) by a factor 2. In the special case $\theta = 0$, when wind and waves are in the same direction, (5.5) reduces to

$$\beta = 3.2 \times 10^{-5}(U/c)^2 \sigma. \tag{5.6}$$

For example, if we consider short-gravity waves of length $l = 20$ cm, we have $\sigma = (2\pi g/l)^{\frac{1}{2}} = 17.6$ rad s^{-1} and $c = g/\sigma = 0.56$ m s^{-1}. Hence with $U = 6.0$ m s^{-1} we find $\beta = 0.065$ s^{-1}. With the period of the long waves about 7.5 s (Evans & Shemdin 1980, table 2), we have $B = 0.49$ and $e^B = 1.63$. With the long waves having period 2 s, we have $B = 0.129$ and $e^B = 1.13$ only. Thus swell and wind-waves may correspond to rather different values of the wind amplification, even for the same wind-speed.

With shorter (X-band) short waves, B is generally much greater, but the amplification of the short waves is then more limited by breaking.

6. Results

To explore the behaviour of the solutions, calculations were at first carried out with representative values of $\bar{A}K$ and B, and with $\kappa = 0$. The latter condition implies zero correlation between successive waves, so that $p(A_1|A_2)$ is independent of A_2 and is, in fact, given by the Rayleigh density $p(A)$ of equation (3.2), with $A = A_1$.

Figure 5 shows the solution when $\bar{A}K = 0.1$ and $B = 0.1$. It will be seen that as the height A of the waves increases, so the curves of $\phi(s, A)$ tend to move to the right, that is to say the short waves on the whole become steeper. The probability $P(A)$ of the short waves attaining their limiting steepness also increases monotonically with A. For the lowest values of $A/\bar{A}$ the probability of finding breaking waves at the crests of the longer waves is quite low; when $A/\bar{A} = 0.5$ the probability is less than 20%. When $A/\bar{A} = 2$, $P(A)$ is more than 90%.

The probability density $\phi(s, A)$ does not always increase monotonically with s. In fact, when $A/\bar{A} = 0$ and 0.5, the density has maxima near $s/s_0 = 0.83$ and 0.92 respectively, as well as the delta-function peak at $s/s_0 = 1$. The distribution of

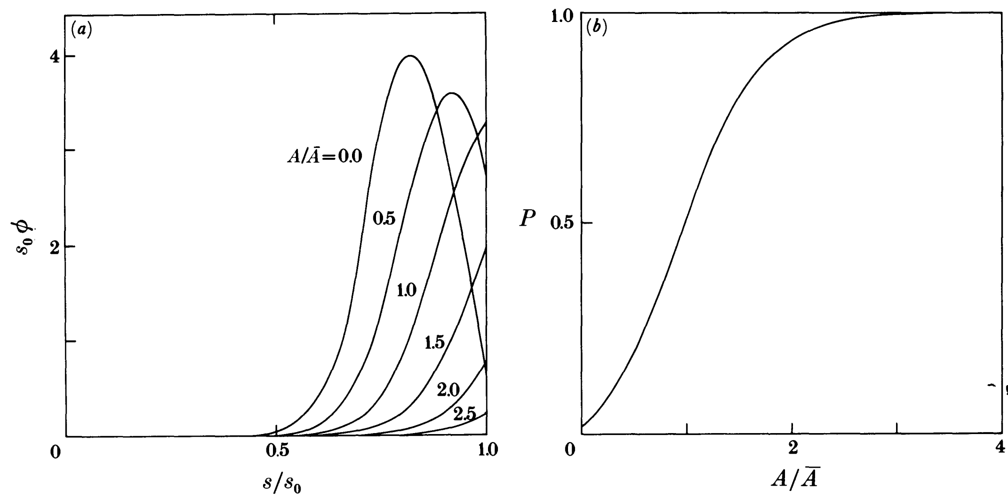

FIGURE 5. (*a*) The calculated probability density $\phi(s, A)$. (*b*) The probability of breaking $P(A)$ when $\bar{A}K = 0.1$, $B = 0.1$ and $\kappa = 0$.

slopes is thus bimodal in these cases. When $s/s_0 < 0.5$ we see that $\phi(s, A)$ is generally quite negligible.

In figure 6 we see the result of increasing the wave-generation parameter B to 0.2, while keeping $\bar{A}K = 0.1$. This clearly shifts the curves $\phi(s, A)$ to the right, increasing the short-wave steepnesses in general. The probability P of breaking is now never less than 0.3.

In figures 7 and 8 we see similar results, but for steeper long waves: $\bar{A}K = 0.2$. When $B = 0.1$ most of the densities $p(s|A)$ are now bimodal. Physically, this is because the distributions of short-wave steepness on the crests of the lower long

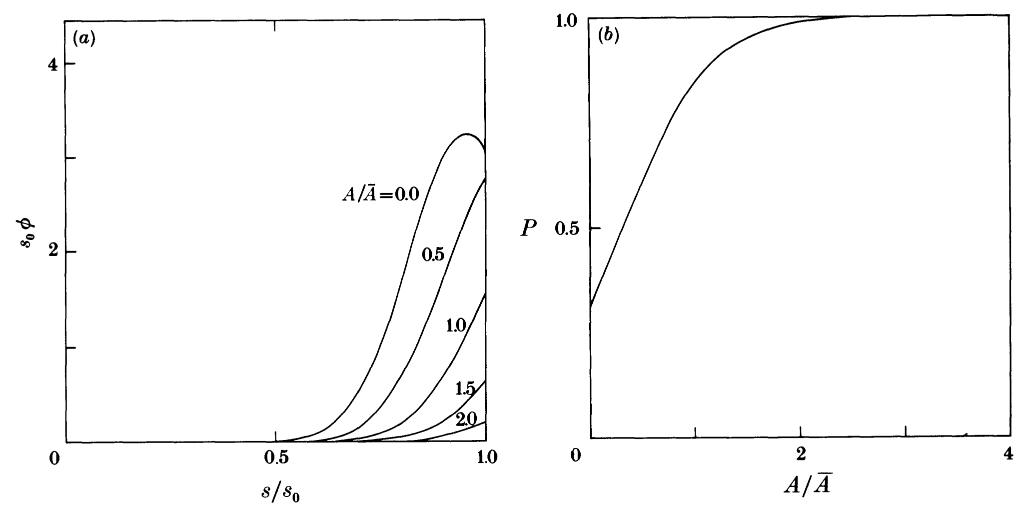

FIGURE 6. As figure 5, when $\bar{A}K = 0.1$, $B = 0.2$, $\kappa = 0$.

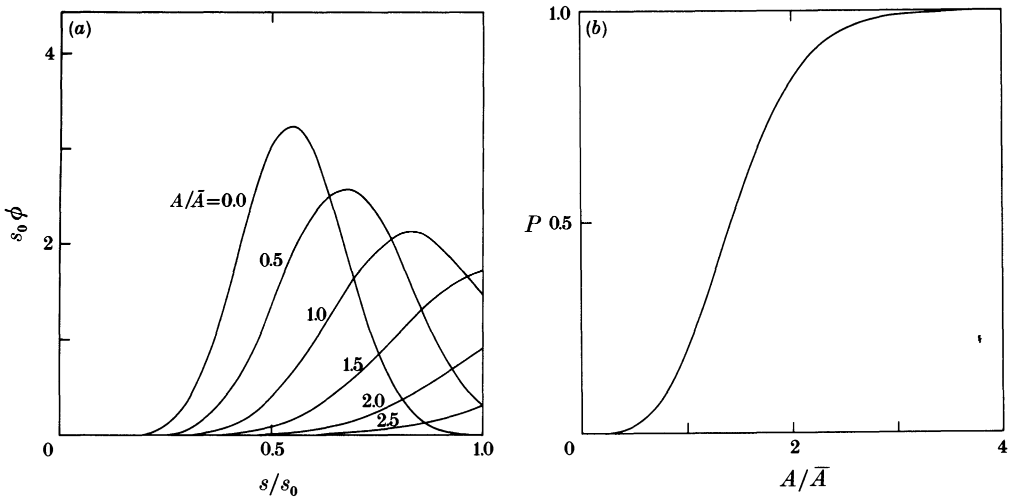

FIGURE 7. As figure 5, when $\bar{A}K = 0.2$, $B = 0.1$, $\kappa = 0$.

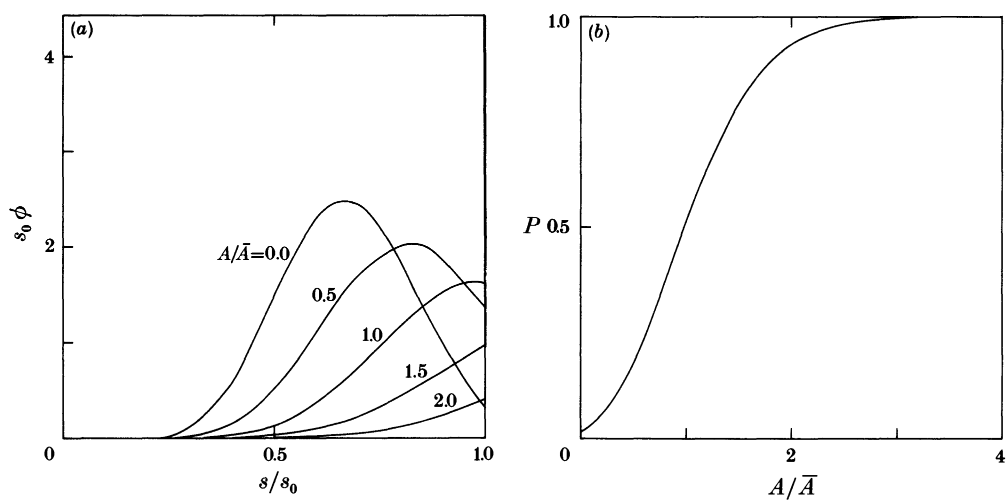

FIGURE 8. As figure 5, when $\bar{A}K = 0.2$, $B = 0.2$, $\kappa = 0$.

waves, say $A/\bar{A} \leqslant 1$, are determined by the fact that the short-wave steepness is limiting on the higher long waves and is then reduced by the subsequent fall in long-wave height. The regeneration of short-wave steepness by the wind is insufficient to overcome this effect.

In figure 8, however, when B is increased to 0.2, the curves for $\phi(s, A)$ are shifted back again toward the right.

Variation of κ

In the results so far, the correlation parameter κ has been set equal to zero. Figures 9, 10 and 11 show the effect of increasing κ by stages, while keeping $\bar{A}K$ and B constant at 0.2 as in figure 8.

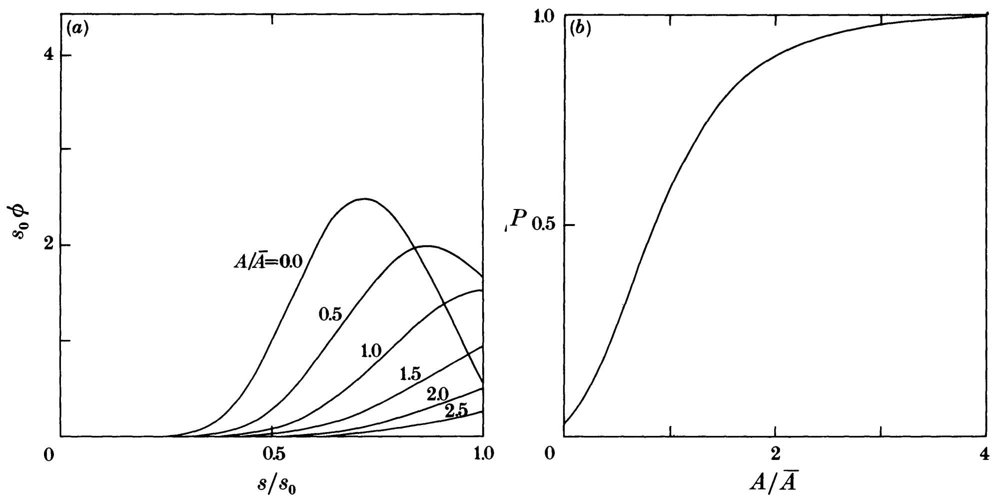

FIGURE 9. As figure 8, when $\kappa = 0.5$.

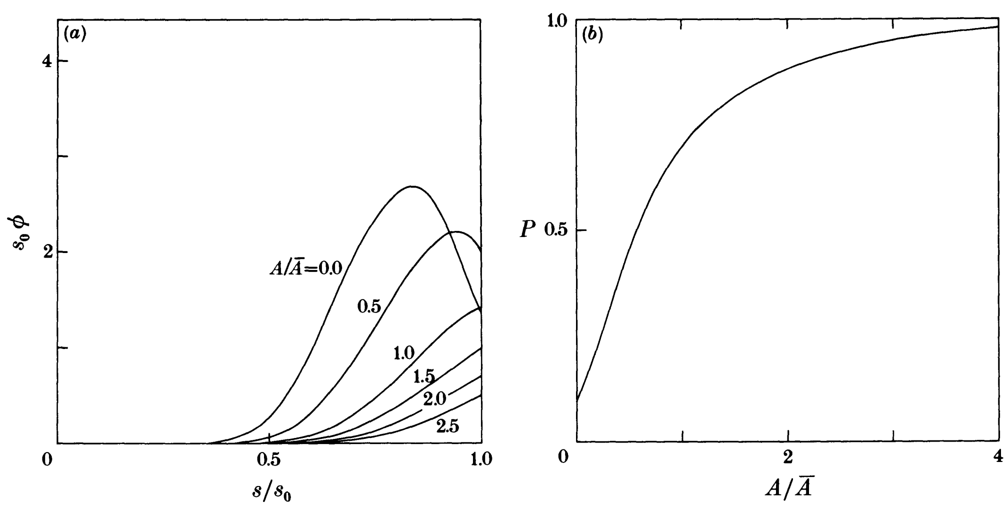

FIGURE 10. As figure 8, when $\kappa = 0.75$.

When $\kappa = 0.5$ (figure 9) there is little effect on either $\phi(s, A)$ or $P(A)$, though the curves for $\phi(s, A)$ will be seen to have shifted slightly to the right. This value of κ, corresponding to $\kappa^2 = 0.25$ or $\nu \approx 0.2$ is typical of local wind-waves (Longuet-Higgins 1984, figure 14).

However, when $\kappa = 0.75$ (figure 10) the shift to the right is very distinct, and even more so when $\kappa = 0.85$ (figure 11). The corresponding values of κ^2 and ν are typical of ocean swell (Longuet-Higgins 1984, figure 13).

Physically, the effect of increasing κ toward unity is to increase indefinitely the mean length of the wave groups, and so to make the long waves appear locally of uniform height. In the limit $\kappa \rightarrow 1$ it is clear that the regeneration of short-wave steepness by the wind will prevail over the redistribution of short waves by

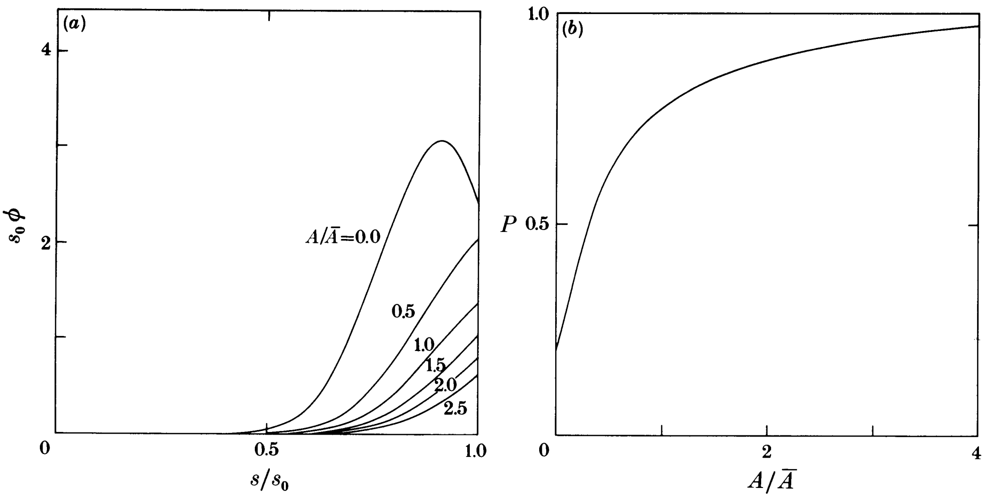

FIGURE 11. As figure 8, when $\kappa = 0.85$.

variability of the long waves, and so will force the short waves up to the breaking steepness at every long-wave crest. Thus, we shall have

$$\phi(s, A) \to 0, \quad P(A) \to 1. \tag{6.1}$$

This is the situation that has been assumed, in effect, by previous authors (Phillips & Banner 1974).

Generally, however, we see that this condition is by no means attained, and that the variation of long-wave heights is an important factor in reducing the probable short-wave steepness, even at the long-wave crests.

7. Effect of background 'noise'

So far we have assumed that the short waves are present initially and, in spite of some dissipation through breaking, are always regenerated by the wind. In other words, we have a 'boot-strap' situation, the regeneration of the waves depending on their initial presence. However, apart from the exponential growth rate of the waves, as expressed through the factor in (2.8), there may in fact be other sources of short-wave energy, arising from splashing, nonlinear wave interactions, etc. Without specifying the exact mechanism, we may represent such a noisy source by a constant input of probability density to the small slopes, a sort of 'probability rain'. For such an input, $\Delta\phi$, a plausible form is

$$\Delta\phi = N(s/\bar{s}^2)\,\mathrm{e}^{-s^2/2\bar{s}^2}, \tag{7.1}$$

in other words a Rayleigh distribution of the RMS slope s and total amplitude N per wave period T.

Accordingly, we tried the effect of adding on a small extra probability, $\Delta\phi$, having the above form, at each iteration and at the same time normalizing by dividing the probabilities by $(1 + \int \Delta\phi\,\mathrm{d}s)$. For numerical values we chose $\bar{s} = 0.1s_0$ (so that $\int \Delta\phi\,\mathrm{d}s \approx N$), and $N = 0.05$ or 0.1. Also $\bar{A}K = 0.2 = B$, as in figure 8.

The iteration was found to converge just as before (although at a slower rate, comparable to the initial conditions (4.1)). The resulting steady solution in the case $N = 0.05$ is shown in figure 12. This may be compared with figure 8. Instead of the almost vanishing values of ϕ found previously for low slopes ($s/s_0 < 0.25$, say) we now see a substantial 'background probability' extending over the whole range of low slopes, well beyond the original RMS value $\bar{s}$ of the noise input (7.1). This, of course, is caused by amplification of the noise by the wind stress.

Figure 13 shows the result when N is increased from 0.05 to 0.1. Now the two maxima in ϕ are of comparable magnitude, when $A/\bar{A} \leqslant 2$. At the larger values of $A/\bar{A}$ the short waves are almost uniformly distributed over $0 < s < s_0$. The probability P of breaking is correspondingly reduced.

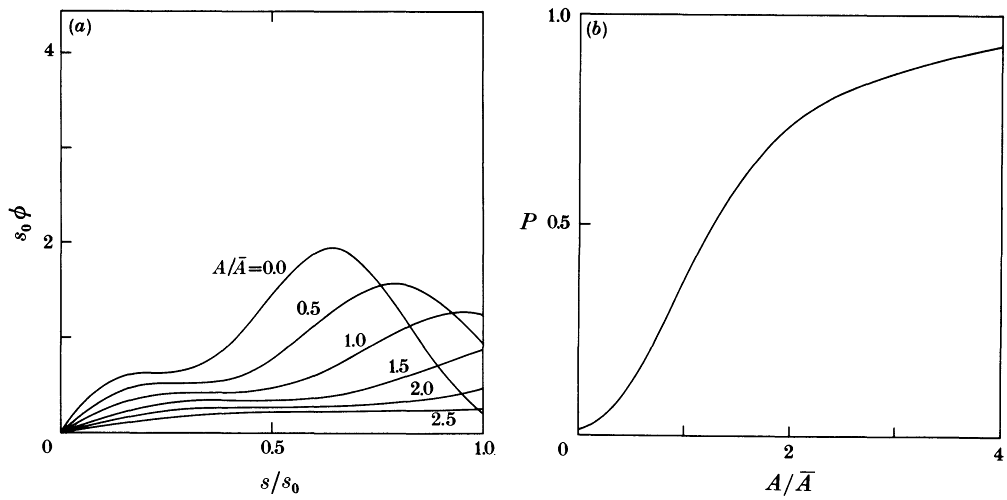

FIGURE 12. As figure 8, but showing the effect of added 'noise'; $N = 0.05$.

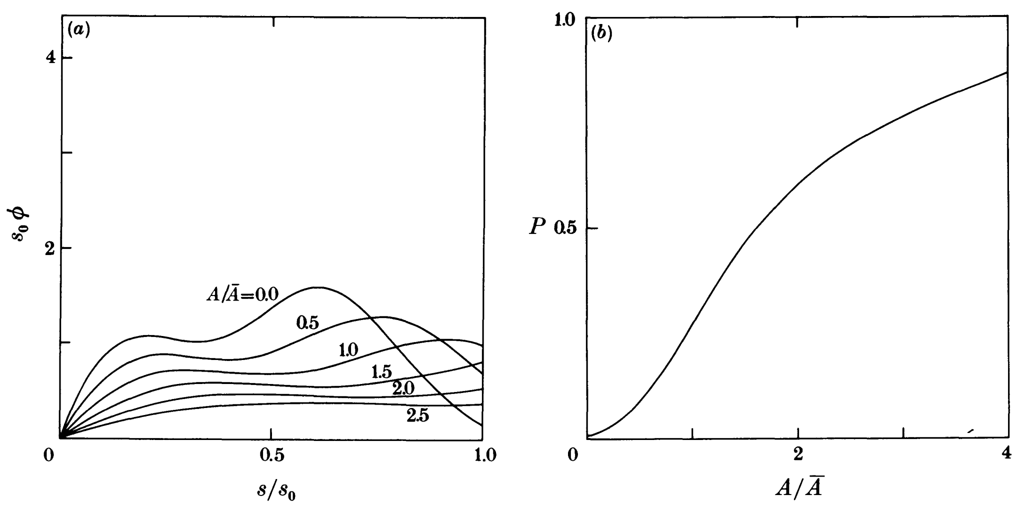

FIGURE 13. As figure 12; $N = 0.10$.

8. Discussion

One of the simplifying assumptions of the model is the existence of a sharp critical steepness for wave breaking. In reality, there may be a range of possible short-wave steepnesses, rather than a single value, leading to a smoother distribution for $p(s, A)$ at high values of s. A modified assumption might be made that at high steepnesses a high degree of damping occurs, to represent the wave breaking. In that case, a similar type of integral equation for $p(s, A)$ could still be formulated.

A second limitation of the model is the simplified relation (2.7) between s_1 and s_2. This includes only the linear part of the expression (2.5), and the nonlinear term becomes significant when AK exceeds 2, say. Inclusion of this term leads to a modification of (3.6), but no essential difficulty.

A more subtle deficiency of (2.7) is that it implies a possibly steady rate of growth due to the wind, even though the slope s_1 may already be limiting. Thus (2.7) strictly would imply that near the long-wave crest, where the rate of straining is small, s would have to grow beyond s_0, at least initially. This difficulty may be overcome by determining s through a step-by-step integration throughout the interval $t_1 \leqslant t \leqslant t_2$. If this is done, then (2.7) is replaced by a more consistent, but at the same time less simple, relation.

A fourth limitation of our model is that we have used the joint density $p(A_1, A_2)$ for consecutive long waves as given by (3.1). Strictly, this is justified only for narrow wave spectra, associated with fairly long groups of (long) waves, but, like the Rayleigh distribution for single waves, it may well have an unexpectedly wide range of validity. Again, it might theoretically be replaced by a more general expression, but only with a consequent loss of simplicity in the calculations.

9. Conclusions

We have developed a simplified model of sea-surface roughness that takes into account the randomness of both the longer waves and of the short-scale waves superposed on them. Essentially the model depends upon three parameters: (1) the RMS steepness $\bar{A}K$ of the longer waves; (2) the natural rate of growth B due to the wind; and (3) a correlation parameter κ for the long wave amplitudes. The model demonstrates that:

(1) the density $p(s|A)$ of the short-wave slopes s is not necessarily unimodal, especially at low wave amplitudes A;

(2) as $A/\bar{A}$ increases, so the density $p(s|A)$ tends to become monotonic, and the probability $P(A)$ of breaking at the crest increases;

(3) the effect of increasing B is to move the curves for $\phi(s, A)$ (the continuous part of $p(s|A)$) to the right and to increase the probability of breaking at the crest;

(4) the effect of decreasing the long-wave band width ν, or letting κ approach closer to 1, is again to increase the probability of breaking at the crests of both higher and lower long waves;

(5) the effect of adding a small amount of noise to the short waves is to increase

the probabilities of low short-wave slopes. Paradoxically, the probability $P(A)$ of breaking is reduced.

In a further paper (in preparation) we shall show how the analysis can be extended so as to calculate the probable distribution of surface slope, s, over the complete profile of the long waves. This leads to expressions for the mean-square short-wave slope, the average phase lag between the surface slope and the long waves, and other observed quantities.

The calculations presented in this paper were begun in Cambridge, England during November 1985, and were continued during a visit by the author to the University of Florida, Gainesville, in February and March 1986. To Dr K. Millsaps, Head of the Department of Aerospace Sciences at the University of Florida, and to members of his staff, the author expresses his appreciation of their hospitality and assistance.

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