Proc. R. Soc. Lond. A **396**, 269–280 (1984)
Printed in Great Britain

On the stability of steep gravity waves

BY M. S. LONGUET-HIGGINS, F.R.S.
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, U.K.
and Institute of Oceanographic Sciences, Wormley, Godalming, Surrey GU8 5UB, U.K.

(*Received* 9 *July* 1984 – *Revised* 12 *November* 1984)

Previous calculations of the normal mode perturbations of steep gravity waves have suggested that the lowest superharmonic mode $n = 2$ becomes unstable at around $ak = 0.436$, where $2a$ is the crest-to-trough height of the unperturbed wave and k is the wavenumber. This would correspond to the wave steepness at which the phase speed c is a maximum (considered as a function of ak). However, numerical calculations at such high wave steepnesses can become inaccurate. The present paper studies analytically the conditions for the existence of a normal mode at zero limiting frequency. It is proved that for superharmonic perturbations such conditions will occur only for a pure phase-shift (corresponding to $n = 1$) or when the speed c is stationary with respect to the wave steepness, that is when $\mathrm{d}c = 0$. Hence the limiting form of the instability found by Tanaka (*J. phys. Soc. Japan* **52**, 3047–3055 (1983)) near the value $ak = 0.429$ must be a pure phase-shift.

1. INTRODUCTION

Since the pioneering work of Whitham (1967) and Benjamin & Feir (1967) it has been known that gravity waves on water of infinite depth are unstable, at least to certain subharmonic perturbations. Precise calculations of the normal modes by the present author (1978*a*) and by McLean (1982) have shown also the existence of some unstable perturbations with length scales shorter than the basic wavelength.

While many of the instabilities are three-dimensional, some important and interesting examples occur even in two dimensions (one horizontal and one vertical). Thus it was shown in Longuet-Higgins (1978*a*; hereinafter referred to as I) that some *superharmonic* disturbances, with a dominant wavelength one half that of the original wave (i.e. having wavenumber $n = 2$) tend to become unstable when the steepness ak of the unperturbed wave exceeds about 0.43. Although the calculations could not be carried accurately beyond about $ak = 0.42$, nevertheless they were consistent with the suggestion (made on physical grounds) that a transition to instability occurs at the wave steepness ak for which the phase speed c is a maximum. From independent calculations (Longuet-Higgins & Fox 1978) this was known to be at about $ak = 0.436$ (less than the maximum steepness $ak = 0.4434$).

Recently, however, Tanaka (1983) has proposed a different method of calculation, claiming to have overcome the difficulties inherent in using Fourier series expansions at high values of ak. His numerical results agreed closely with those of I up to $ak = 0.41$, but then tended to diverge, so that instability of the mode $n = 2$ appeared to occur at around $ak = 0.429$. This corresponds closely to the steepness for which the total *energy density* E is a maximum. However, no physical reason for this coincidence could be found.

Since numerical calculations, however accurate, often lack the certainty of mathematical analysis, the present author sought an analytical approach to the problem. An opportunity came with the discovery that by making use of certain identities between the coefficients a_n in the Stokes series for deep-water waves it was possible to simplify considerably both the expression of some integral quantities (Longuet-Higgins 1984*a*) and the numerical calculations of finite-amplitude gravity waves and their points of bifurcation (Longuet-Higgins 1984*b*). In this paper we show that the analysis of the normal-mode perturbations can be simplified also.

In brief, we shall show that if there exists a normal instability with radian frequency σ tending to zero at a certain wave steepness ak, then at the critical point either (i) the phase speed is stationary: $\mathrm{d}c = 0$, or (ii) the limiting form of the perturbation is a pure phase-shift. The points at which $\mathrm{d}c = 0$ and $\mathrm{d}E = 0$ are quite distinct (see Longuet-Higgins 1984*b*). Therefore if a zero-frequency normal mode occurs at $\mathrm{d}E = 0$, its limiting form must be a pure phase-shift.

The present paper further illustrates the utility of a matrix analysis for gravity waves on deep water, which in turn is based on the quadratic relations (3.12) for the Fourier coefficients a_n. Although the analysis applies strictly only to waves in two dimensions, it would seem worthwhile to investigate the possibility of a three-dimensional analogue.

2. Equations for normal modes

In any two-dimensional, irrotational flow of an inviscid, incompressible fluid, the rectangular coordinates (x, y) may be expressed as functions of the velocity potential ϕ, the streamfunction ψ and the time t. In the interior of the fluid $(x+\mathrm{i}y)$ is everywhere an analytic function of $(\phi+\mathrm{i}\psi)$, and at a free surface, where

$$\psi = F(\phi, t), \tag{2.1}$$

the two boundary conditions expressing that the pressure p is constant and that a particle at the surface remains at the surface become respectively

$$-(y_\phi y_t + y_\psi x_t) + gy(y_\phi^2 + y_\psi^2) + \tfrac{1}{2} = 0 \tag{2.2}$$

and

$$(y_\psi y_t - y_\phi x_t) + [1 - (y_\phi y_t + y_\psi x_t)]\, F_\phi + (y_\phi^2 + y_\psi^2)\, F_t = 0 \tag{2.3}$$

(see I, §2). Here suffixes are used to denote partial derivation. In (2.2) g denotes gravity, the y-axis being taken vertically upwards. We shall choose units so that $g = 1$.

We seek solutions to these equations in the form

$$\left.\begin{aligned} x &= X(\phi,\psi)+\epsilon\xi(\phi,\psi)\,\mathrm{e}^{-\mathrm{i}\sigma t},\\ y &= Y(\phi,\psi)+\epsilon\eta(\phi,\psi)\,\mathrm{e}^{-\mathrm{i}\sigma t},\\ F &= \epsilon f(\phi)\,\mathrm{e}^{-\mathrm{i}\sigma t}, \end{aligned}\right\} \tag{2.4}$$

in which (X, Y) represents a steady, progressive wave travelling with speed c in the positive x-direction, as seen in a frame of reference travelling with the waves. The flow then appears steady. The terms $\epsilon(\xi,\eta,f)$ represent a small perturbation of the steady wave, varying harmonically with the time t.

We now substitute (2.4) into (2.2) and (2.3) and expand each of the left-hand sides in a Taylor series in ψ about $\psi = 0$ (the unperturbed surface). From the terms independent of ϵ we obtain

$$2Y(Y_\phi^2+Y_\psi^2)+1 = 0 \tag{2.5}$$

and

$$F_\phi = 0 \tag{2.6}$$

to be satisfied when $\psi = 0$. For waves in deep water we must also have

$$Y \sim -c\psi \tag{2.7}$$

as $\psi \to \infty$.

Assuming these satisfied, the terms in ϵ then give us the following equations for the perturbations:

$$-\mathrm{i}\sigma(Y_\psi\,\xi+Y_\phi\,\eta) = P\eta+Q\eta_\psi+R\eta_\phi+Sf, \tag{2.8}$$

$$-\mathrm{i}\sigma(Y_\phi\,\xi-Y_\psi\,\eta) = f_\phi, \tag{2.9}$$

where we have written

$$\left.\begin{aligned} P &= Y_\phi^2+Y_\psi^2,\\ Q &= (Y^2)_\psi = 2YY_\psi,\\ R &= (Y^2)_\phi = 2YY_\phi,\\ S &= [Y(Y_\phi^2+Y_\psi^2)]_\psi = Y_\psi P+Y(Y_\phi^2+Y_\psi^2)_\psi, \end{aligned}\right\} \tag{2.10}$$

all to be evaluated on $\psi = 0$. In addition

$$\xi+\mathrm{i}\eta \to 0 \quad \text{as} \quad \psi\to\infty \tag{2.11}$$

(see I, §3).

3. The basic wave

In the unperturbed flow, $F = 0$ by (2.4) so that (2.6) is already satisfied. To satisfy (2.5) and (2.7) we assume a steady, symmetric wave given by the Stokes expansion

$$(Y-\mathrm{i}X)+(\psi-\mathrm{i}\phi)/c = \tfrac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\,\mathrm{e}^{\mathrm{i}n(\phi+\mathrm{i}\psi)/c} \tag{3.1}$$

in which the coefficients a_n are all real, and units have been chosen so that the wavelength λ equals 2π. As $\psi \to \infty$ so the right side of (3.1) tends to a constant $\frac{1}{2}a_0$. On the unperturbed surface ($\psi = 0$) we have, from the real part of (3.1),

$$Y = \sum_{m=0}^{\infty}{}^{*}\, a_m \cos m\theta \tag{3.2}$$

where

$$\theta = \phi/c \tag{3.3}$$

and an asterisk means that whenever a_0 occurs in the sum it is to be replaced by $\frac{1}{2}a_0$. Further

$$\left.\begin{aligned} -cY_\psi &= \sum_{n=0}^{\infty} b_n \cos n\theta, \\ -cY_\phi &= \sum_{n=0}^{\infty} b_n \sin n\theta, \end{aligned}\right\} \tag{3.4}$$

where

$$b_0 = 1, \quad b_n = na_n, \quad n = 1, 2, 3, \ldots. \tag{3.5}$$

Hence, when $\psi = 0$,

$$c^2(Y_\phi^2 + Y_\psi^2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_n b_m \cos(m-n)\theta, \tag{3.6}$$

that is

$$c^2 P = \sum_{l=0}^{\infty}{}^{*}\, 2P_l \cos l\theta, \tag{3.7}$$

where

$$P_l = \sum_{m=0}^{\infty} b_{m+l} b_m, \quad l = 0, 1, 2, \ldots. \tag{3.8}$$

The boundary condition (2.5) can be written

$$2c^2 YP = -c^2. \tag{3.9}$$

On substituting for Y and c^2P from (3.2) and (3.7) we have

$$2 \sum_{m=0}^{\infty}{}^{*} \sum_{l=0}^{\infty}{}^{*}\, a_m P_l[\cos(m+l)\theta + \cos(m-l)\theta] = -c^2 \tag{3.10}$$

and on equating the coefficients of $\cos n\theta$, $n = 0, 1, 2, \ldots$, on each side we obtain the infinite set of equations

$$\left.\begin{aligned} a_0 P_0 + (a_1+a_1) P_1 + (a_2+a_2) P_2 + (a_3+a_3) P_3 + \ldots &= -c^2, \\ a_1 P_0 + (a_0+a_2) P_1 + (a_1+a_3) P_2 + (a_2+a_4) P_3 + \ldots &= 0, \\ a_2 P_0 + (a_1+a_3) P_1 + (a_0+a_4) P_2 + (a_1+a_5) P_3 + \ldots &= 0, \\ \ldots, \end{aligned}\right\} \tag{3.11}$$

between c, and the original coefficients a_n.

Formally, equations (3.11) are cubic in the a_n. It is a remarkable fact, first noted explicitly in Longuet-Higgins (1978*b*), that these equations are equivalent to a simpler, quadratic set of equations, namely

$$\left.\begin{aligned} a_0 b_0 + a_1 b_1 + a_2 b_3 + a_3 b_3 + \ldots &= -c^2, \\ a_1 b_0 + a_0 b_1 + a_1 b_2 + a_2 b_3 + \ldots &= 0, \\ a_2 b_0 + a_1 b_1 + a_0 b_2 + a_1 b_3 + \ldots &= 0, \\ \ldots. \end{aligned}\right\} \tag{3.12}$$

For, if we denote the left-hand sides of equations (3.12) by F_n, $n = 0, 1, 2, \ldots$, and the corresponding expressions in (3.11) by G_n then we have identically

$$\left.\begin{aligned} b_0 F_0 + b_1 F_1 + b_2 F_2 + b_3 F_3 + \ldots &= G_0, \\ b_0 F_1 + b_1 F_2 + b_2 F_3 + \ldots &= G_1, \\ b_0 F_2 + b_1 F_3 + \ldots &= G_2, \\ \ldots, \end{aligned}\right\} \tag{3.13}$$

that is to say

$$\boldsymbol{B} \times \boldsymbol{F} = \boldsymbol{G}, \tag{3.14}$$

where

$$\boldsymbol{B} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \ldots \\ 0 & b_0 & b_1 & b_2 \ldots \\ 0 & 0 & b_0 & b_1 \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{3.15}$$

and

$$\left.\begin{aligned} \boldsymbol{F} &= (F_0, F_1, F_2, \ldots)^{\mathrm{T}}, \\ \boldsymbol{G} &= (G_0, G_1, G_2, \ldots)^{\mathrm{T}}. \end{aligned}\right\} \tag{3.16}$$

Since $b_0 = 1$ we have

$$|\boldsymbol{B}| = 1 \tag{3.17}$$

and the matrix $\boldsymbol{B}$ is non-singular. Hence any non-vanishing solution to (3.11) is equivalent to a non-vanishing solution to (3.12) and vice versa.

The above proof, which was given in Longuet-Higgins (1978*b*), relies on the convergence of all the series involved, which is, however, assured at all interior points of the fluid domain. An alternative proof due essentially to J. G. B. Byatt-Smith, is given in the same paper (see also Longuet-Higgins 1984*a*, *b*).

4. The normal modes; limit as $\sigma \to 0$

To solve the perturbation equations (2.8) and (2.9) it was noted in I that (ξ, η) being conjugate functions, tending to 0 as $\psi/c \to \infty$, may be expanded in the form

$$(\eta - \mathrm{i}\xi) = \sum_{n=0}^{\infty} (\alpha_n + \mathrm{i}\beta_n)\, \mathrm{e}^{\mathrm{i}n(\phi + \mathrm{i}\psi)/c} \tag{4.1}$$

and that in general

$$f = \sum_{n=0}^{\infty} (\gamma_n + \mathrm{i}\delta_n)\, \mathrm{e}^{\mathrm{i}n\phi/c}, \tag{4.2}$$

where α_n, β_n, γ_n and δ_n are real constants. Substitution of these expressions into (2.8) and (2.9) and equating the coefficients of $\cos n\theta$ $(n = 0, 1, 2, \ldots)$ and $\sin n\theta$ $(n = 1, 2, 3, \ldots)$ yields a system of equations for the constants $\alpha_n, \beta_n, \gamma_n, \delta_n$ and the radian frequency σ, which can be solved numerically by successive truncations.

However, we are here interested chiefly in the limiting case when $\sigma \to 0$. From (2.9) we see immediately that in the limit

$$f_\phi = 0, \quad f = \gamma_0. \tag{4.3}$$

Equation (2.8) then reduces to

$$P\eta + Q\eta_\psi + R\eta_\phi + S\gamma_0 = 0, \tag{4.4}$$

to be satisfied when $\psi = 0$.

We remark that this equation can be derived directly from the steady boundary condition

$$2y(y_\phi^2+y_\psi^2) = -1 \quad \text{on} \quad \psi = F \tag{4.5}$$

(compare (2.5)) by writing

$$y = Y+\epsilon\eta, \quad F = \epsilon\gamma_0 \tag{4.6}$$

and then considering the coefficient of ϵ.

5. Normal modes (continued)

To proceed further with (2.8) we must evaluate Q, R and S. From (2.10) and (3.4) we have

$$-cQ = \sum_{n=0}^{\infty}{}^{*} \sum_{m=0}^{\infty} a_n b_n[\cos(m+n)\theta+\cos(m-n)\theta], \tag{5.1}$$

$$-cR = \sum_{n=0}^{\infty}{}^{*} \sum_{m=0}^{\infty} a_n b_m[\sin(m+n)\theta+\sin(m-n)\theta], \tag{5.2}$$

that is

$$-cQ = \sum_0^\infty Q_l \cos l\theta, \tag{5.3}$$

$$-cR = \sum_0^\infty R_l \sin l\theta, \tag{5.4}$$

where

$$Q_0 = \sum_{m=0}^{\infty} a_m b_m \tag{5.5}$$

and when $l > 0$,

$$\left.\begin{aligned} Q_l &= \sum_{m=0}^{l-1} a_{l-m} b_m + \sum_{m=0}^{\infty} (a_m b_{l+m} + a_{l+m} b_m), \\ R_l &= \sum_{m=0}^{l-1} a_{l-m} b_m + \sum_{m=0}^{\infty} (a_m b_{l-m} - a_{l+m} b_m). \end{aligned}\right\} \tag{5.6}$$

Therefore when $l > 0$,

$$Q_l+R_l = 2\sum_{m=0}^{l-1} a_{l-m} b_m + 2\sum_{m=0}^{\infty} a_m b_{l+m} = 0 \tag{5.7}$$

by (3.12). Hence for $l = 0, 1, 2, 3, \ldots$ we have

$$Q_l = \sum_{m=0}^{\infty} a_{l+m} b_m \tag{5.8}$$

and in particular

$$Q_0 = \sum_{m=0}^{\infty} a_m b_m = -c^2. \tag{5.9}$$

Thirdly we have from (2.10)

$$S = Y_\psi P + Y_{\psi\psi} Q + Y_{\phi\psi} R. \tag{5.10}$$

Hence

$$\begin{aligned} -c^3 S = &\sum_{m=0}^{\infty} b_m \cos m\theta \sum_{n=0}^{\infty}{}^{*}\, 2P_n \cos n\theta \\ &+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m b_m (Q_n \cos m\theta \cos n\theta + R_n \sin m\theta \sin n\theta). \end{aligned} \tag{5.11}$$

On making use of the relations (5.7) we obtain

$$-c^3 S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}{}^{*}\, b_m P_n[\cos(m+n)\,\theta + \cos(m-n)\,\theta]$$

$$+\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m b_m Q_n \cos(m+n)\,\theta \tag{5.12}$$

$$= \sum_{l=0}^{\infty}{}^{*}\, 2S_l \cos l\theta, \tag{5.13}$$

say.

Now adopting the expansions (4.1) and (4.2) we have when $\psi = 0$

$$\left.\begin{aligned} \eta &= \sum_0^\infty (\alpha_n \cos n\theta - \beta_n \sin n\theta), \\ -c\eta_\psi &= \sum_0^\infty (n\alpha_n \cos n\theta - n\beta_n \sin n\theta) \\ -c\eta_\phi &= \sum_0^\infty (n\alpha_n \sin n\theta + n\beta_n \cos n\theta) \end{aligned}\right\} \tag{5.14}$$

On substituting these expressions into the boundary condition (4.4) and equating to zero the coefficients of $\frac{1}{2}$, $\cos\theta$, $\cos 2\theta$, ... and $\sin\theta$, $\sin 2\theta$, ... we obtain a linear system of equations for

$$\left.\begin{aligned} \boldsymbol{\alpha} &= (\alpha_0, \alpha_1, \alpha_2, \ldots), \\ \boldsymbol{\beta} &= (\beta_1, \beta_2, \beta_3, \ldots) \end{aligned}\right\} \tag{5.15}$$

and γ_0, which may be written in the form

$$\left(\begin{array}{c|c|c} \mathbf{M} & \mathbf{O} & \mathbf{S} \\ \hline \mathbf{O} & \mathbf{N} & \mathbf{0} \end{array}\right) \times (\boldsymbol{\alpha} \;;\; \boldsymbol{\beta} \;;\; \gamma_0)^{\mathrm{T}} = \left(\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \end{array}\right), \tag{5.16}$$

where

$$\boldsymbol{M} = \begin{bmatrix} (P_0+P_0) & (P_1+P_1) & (P_2+P_2) & (P_3+P_3) & \cdots \\ (P_1+P_1) & (P_0+P_2+Q_0) & (P_1+P_3) & (P_2+P_4) & \cdots \\ (P_2+P_2) & (P_1+P_3+Q_1) & (P_0+P_4+2Q_0) & (P_1+P_5) & \cdots \\ (P_3+P_3) & (P_2+P_4+Q_2) & (P_1+P_5+2Q_1) & (P_0+P_6+3Q_0) & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix} \tag{5.17}$$

and

$$\boldsymbol{N} = \begin{bmatrix} (P_0-P_2+Q_0) & (P_1-P_3) & (P_2-P_4) & \cdots \\ (P_1-P_3+Q_1) & (P_0-P_4+2Q_0) & (P_1-P_5) & \cdots \\ (P_2-P_4+Q_2) & (P_1-P_5+2Q_1) & (P_0-P_6+3Q_0) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{5.18}$$

and $\mathbf{O}$ denotes the zero square matrix, $\mathbf{0}$ a zero column vector. Finally

$$\boldsymbol{S} = (S_0, S_1, S_2, \ldots)^{\mathrm{T}} \tag{5.19}$$

where S_n is given by (5.13).

6. Conditions when $\sigma = 0$

We shall make use of the following lemmas:

$$\boldsymbol{M} = \boldsymbol{B} \times \boldsymbol{C}, \tag{6.1}$$

$$\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{D}, \tag{6.2}$$

in which $\boldsymbol{B}$ denotes the matrix (3.15) and $\boldsymbol{C}, \boldsymbol{D}$ are the symmetric matrices:

$$\boldsymbol{C} = \begin{bmatrix} 2 & (a_1+a_1) & (2a_2+2a_2) & (3a_3+3a_3) & \cdots \\ (a_1+a_1) & (1+a_0+2a_2) & (2a_1+3a_3) & (3a_2+4a_4) & \cdots \\ (2a_2+2a_2) & (2a_1+3a_3) & (1+2a_0+4a_4) & (3a_1+5a_5) & \cdots \\ (3a_3+3a_3) & (3a_2+4a_4) & (3a_1+5a_5) & (1+3a_0+6a_6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6.3}$$

and

$$\boldsymbol{D} = \begin{bmatrix} (1+a_0-2a_2) & (2a_1-3a_3) & (3a_2-4a_4) & \cdots \\ (2a_1-3a_3) & (1+2a_0-4a_4) & (3a_1-5a_5) & \cdots \\ (3a_2-4a_4) & (3a_2-5a_5) & (1+3a_0-6a_6) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}. \tag{6.4}$$

Moreover,

$$\boldsymbol{M} \times \boldsymbol{b}^{\mathrm{T}} = \boldsymbol{S}, \tag{6.5}$$

$$\boldsymbol{N} \times \boldsymbol{b}'^{\mathrm{T}} = \boldsymbol{0}, \tag{6.6}$$

where $\boldsymbol{S}$ is given by (5.19) and

$$\boldsymbol{b} = (b_0, b_1, b_2, \ldots), \tag{6.7}$$

$$\boldsymbol{b}' = (b_1, b_2, b_3, \ldots). \tag{6.8}$$

These results are all proved in the Appendix.

Consider the consequences. Equation (6.5) implies that the last column of the matrix in (5.16) depends linearly on the columns of $\boldsymbol{M}$, and hence is redundant. The value of γ_0 is thus arbitrary. This is not obvious *a priori*, since although a non-zero value of γ_0 implies the addition of a mere constant to the streamfunction ψ, it also changes the position of the free surface, to order ϵ, hence the values of the coefficients a_n in the Fourier expansion (3.1).

On omitting the last column of the matrix, (5.16) reduces to the 'square' form

$$\left(\begin{array}{c|c} \mathbf{M} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{N} \end{array}\right) \times \left(\begin{array}{c} \boldsymbol{\alpha}^{\mathrm{T}} \\ \hline \boldsymbol{\beta}^{\mathrm{T}} \end{array}\right) = \left(\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \end{array}\right). \tag{6.9}$$

This again reduces to the two independent systems

$$\boldsymbol{M} \times \boldsymbol{\alpha}^{\mathrm{T}} = \boldsymbol{0} \tag{6.10}$$

and

$$\boldsymbol{N} \times \boldsymbol{\beta}^{\mathrm{T}} = \boldsymbol{0} \tag{6.11}$$

for the in-phase and quadrature components of the perturbation (4.1).

Consider first the symmetric perturbations. The system (6.10) can have a non-vanishing solution only if $|M| = 0$. But by (6.1)

$$|\boldsymbol{M}| = |\boldsymbol{B}| \times |\boldsymbol{C}| = |\boldsymbol{C}| \tag{6.12}$$

since $|\boldsymbol{B}| = 1$ by (3.15). Therefore for a non-vanishing solution we require

$$|\boldsymbol{C}| = 0. \tag{6.13}$$

Now it was shown in Longuet-Higgins (1984*b*) and can indeed be seen from equations (3.12), that the vanishing of $|\boldsymbol{C}|$ is the necessary condition that the phase speed c be stationary with respect to changes in the wave amplitude, that is

$$\mathrm{d}c = 0. \tag{6.14}$$

This then shows that there can exist symmetric normal perturbations at zero frequency only if $\mathrm{d}c = 0$.

Consider on the other hand the asymmetric perturbations. From (6.6) it follows that there always exists an antisymmetric perturbation, given by

$$\boldsymbol{\beta} = \boldsymbol{b}'. \tag{6.15}$$

This represents a simple phase-shift of the original wave through the horizontal distance ϵc.

Now (6.11) implies also that the determinant of $\boldsymbol{N}$ vanishes. Hence the first row of $\boldsymbol{N}$ is linearly dependent on the others. The corresponding equation is thus redundant and may be replaced by a condition on the phase, for example that

$$\beta_1 = 0. \tag{6.16}$$

The modified system then becomes

$$\boldsymbol{N}' \times \boldsymbol{\beta}'^{\mathrm{T}} = \boldsymbol{0}, \tag{6.17}$$

where $\boldsymbol{\beta}' = (\beta_2, \beta_3, \ldots)$ and $\boldsymbol{N}'$ is the matrix derived from $\boldsymbol{N}$ by omitting the first row and column, that is

$$\boldsymbol{N}' = \begin{bmatrix} (P_0 - P_4 + 2Q_0) & (P_1 - P_3) & \cdots \\ (P_1 - P_5 + 2Q_1) & (P_0 - P_6 + 3Q_0) & \cdots \\ \vdots & \vdots & \end{bmatrix}. \tag{6.18}$$

Thus an antisymmetric perturbation exists only if

$$|\boldsymbol{N}'| = 0. \tag{6.19}$$

Now, as in the proof of (6.1) given in the Appendix, it may be shown that

$$\boldsymbol{N}' = \boldsymbol{B} \times \boldsymbol{D}', \tag{6.20}$$

where $\boldsymbol{D}'$ is the matrix derived from $\boldsymbol{D}$ by omitting the first row and column, that is

$$\boldsymbol{D}' = \begin{bmatrix} (1 + 2a_0 - 4a_4) & (3a_1 - 5a_5) & \cdots \\ (3a_1 - 5a_5) & (1 + 3a_0 - 6a_6) & \cdots \\ \vdots & \vdots & \end{bmatrix}. \tag{6.21}$$

But it has already been shown numerically that $|\boldsymbol{D}'|$ does not vanish anywhere in the range of uniform waves (see Longuet-Higgins 1984*b*), and particularly not when $E = E_{\max}$. Hence $|\boldsymbol{N}'|$ does not vanish anywhere in the range, and there are therefore no antisymmetric normal modes at zero frequency apart from a pure phase-shift.

Since any perturbation to the original wave form may be expressed as the sum of a symmetric part and antisymmetric part, this shows that there are no other antisymmetric normal modes of *any* form at zero frequency, except at points where the phase speed has a stationary value, i.e. when $\mathrm{d}c = 0$.

The form of the normal mode, at points close to a point of stationary phase speed, is presumably given by

$$\boldsymbol{a} = [a_0, a_1, a_2, \ldots] = \boldsymbol{a}^{(c)} + \epsilon(\lambda\boldsymbol{\alpha} + \mu\boldsymbol{\beta}) + O(\epsilon^2), \tag{6.22}$$

where $\boldsymbol{a}^{(c)}$ denotes the values of the Fourier coefficients at the critical wave amplitude, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ correspond to solutions of (6.10) and (6.11), and λ, μ are constants. To determine the ratio $\lambda:\mu$ and the corresponding (non-zero) value of the radian frequency σ the full equations (2.8) and (2.9) may be carried to higher order in ϵ.

7. Discussion

We have shown, essentially, that for irrotational waves on deep water the conditions for a normal-mode perturbation at zero limiting frequency are the same as those for a bifurcation of the steady motion. But it was already shown previously (Longuet-Higgins 1984*b*) that the only possible $+1$ bifurcation consists of a pure phase-shift, except possibly at a stationary value of the phase speed, that is when $\mathrm{d}c = 0$. It follows that if there exists a superharmonic normal mode with zero limiting frequency at any value of ak other than $\mathrm{d}c = 0$, then its limiting form must be a pure phase-shift.

This conclusion applies in particular to the numerical result of Tanaka (1983), who found a zero-frequency mode at around $ak = 0.429$.

In a recent correspondence Dr Tanaka has kindly informed the author that the limiting form of the instability found by him is indeed a pure phase-shift. However, it is still not clear why this situation should occur so close to the point at which $\mathrm{d}E = 0$. A further investigation of the question is in progress.

Appendix A. Proof of equations (6.1), (6.2), (6.5) and (6.6)

To prove (6.1) let the matrix $\boldsymbol{C}$, defined by (6.3), be written in the apparently asymmetric form

$$\boldsymbol{C} = \begin{bmatrix} (b_0+b_0) & (0+b_1+a_1) & (0+b_2+2a_2) & (0+b_3+3a_3) & \ldots \\ (b_1+b_1) & (b_0+b_2+a_0) & (0+b_3+2a_1) & (0+b_4+3a_2) & \ldots \\ (b_2+b_2) & (b_1+b_3+a_1) & (b_0+b_4+2a_0) & (0+b_5+3a_1) & \ldots \\ (b_3+b_3) & (b_2+b_4+a_2) & (b_1+b_5+2a_1) & (b_0+b_6+3a_0) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{A 1}$$

On multiplying the ith row of $\boldsymbol{B}$ (see equation (3.15)) by the jth column of (A 1) and using the definition of P_l in (3.8), we see that the terms in b_n alone yield the inner product $(P_{|i-j|}+P_{i+j})$. On the other hand, from equations (3.12) and (5.3) the terms in b_i and a_j yield 0 if $i<j$, and $(j-1)\,Q_{i-j}$ if $i \geqslant j$. This proves (6.1).

To prove (6.2) let us define

$$\boldsymbol{D}_1 = \begin{bmatrix} (b_0-b_0) & (0-b_1+a_1) & (0-b_2+2a_2) & (0-b_3+3a_3) & \cdots \\ (b_1-b_1) & (b_0-b_2+a_0) & (0-b_3+2a_1) & (0-b_4+3a_2) & \cdots \\ (b_2-b_2) & (b_1-b_3+a_1) & (b_0-b_4+2a_0) & (0-b_5+3a_1) & \cdots \\ (b_3-b_3) & (b_2-b_4+a_2) & (b_1-b_5+2a_1) & (b_0-b_6+3a_0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{A 2}$$

Then it is clear that

$$\boldsymbol{D}_1' = \boldsymbol{D} \tag{A 3}$$

(where a prime denotes the matrix derived from a given matrix by omitting the first row and first column). As before, we now have

$$\boldsymbol{B}\times\boldsymbol{D}_1 = \boldsymbol{N}_1, \tag{A 4}$$

where

$$\boldsymbol{N}_1 = \begin{bmatrix} (P_0-P_0) & (P_1-P_1+0) & (P_2-P_2+0) & (P_3-P_3+0) & \cdots \\ (P_1-P_1) & (P_0-P_2+Q_0) & (P_1-P_3+0) & (P_2-P_4+0) & \cdots \\ (P_2-P_2) & (P_1-P_3+Q_1) & (P_0-P_4+2Q_0) & (P_1-P_5+0) & \cdots \\ (P_3-P_3) & (P_2-P_4+Q_2) & (P_1-P_5+2Q_1) & (P_0-P_6+3Q_0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{A 5}$$

But in $\boldsymbol{D}_1$ the elements of the first row all vanish. So from (A 4) we have

$$\boldsymbol{B}'\times\boldsymbol{D}_1' = \boldsymbol{N}_1' \tag{A 6}$$

and since $\boldsymbol{B}'=\boldsymbol{B}$ (see equation (3.15)) this is (6.2).

To prove (6.6) we note that since the elements in the first row of $\boldsymbol{N}_1$ all vanish, it follows from (A 4) that

$$\boldsymbol{b}'\times\boldsymbol{D} = \boldsymbol{0}^{\mathrm{T}} \tag{A 7}$$

and on taking the transpose matrix of each side of this equation we obtain

$$\boldsymbol{D}\times\boldsymbol{b}'^{\mathrm{T}} = \boldsymbol{0}, \tag{A 8}$$

$\boldsymbol{D}$ being symmetric. We now multiply on the left by $\boldsymbol{B}$, using the commutative law for matrices, to obtain

$$(\boldsymbol{B}\times\boldsymbol{D})\times\boldsymbol{b}'^{\mathrm{T}} = \boldsymbol{0} \tag{A 9}$$

and (6.6) follows, by equation (6.2).

Lastly, to prove (6.5) note first that from the definition of S_l in (5.13) we have

$$\left.\begin{aligned} S_0 &= (P_0+P_0)\,b_0+(P_1+P_1+0)\,b_1+(P_2+P_2+0)\,b_2+\ldots, \\ S_1 &= (P_1+P_1)\,b_0+(P_0+P_2+Q_0)\,b_1+(P_1+P_3+0)\,b_2+\ldots, \\ S_3 &= (P_2+P_2)\,b_0+(P_1+P_3+Q_1)\,b_1+(P_0+P_4+2Q_0)\,b_2+\ldots, \\ &\ldots. \end{aligned}\right\} \tag{A 10}$$

From (5.17) and (6.6) it is clear that S_l is the inner product of $\boldsymbol{b}$ with the $(l-1)$th row of $\boldsymbol{M}$. This proves the lemma.

This paper was begun while the author was visiting Cal. Tech. Jet Propulsion Laboratory, Pasadena. These and other results were presented at the Workshop on Surface Gravity Waves at U.C. Santa Barbara on 31 May–1 June 1984; also at a meeting at the University of Bristol on 3 July 1984. To Dr M. Chahine, Head of the Earth Sciences Division of J.P.L., and to Professor M. Tulin, U.C.S.B., the author expresses thanks for their hospitality and assistance.

I am indebted to Dr P. A. Saffman and Dr M. Tanaka for interesting correspondence and discussion.

References

Benjamin, T. B. & Feir 1967 The disintegration of wave trains on deep water. *J. Fluid Mech.* **27**, 417–430.

Longuet-Higgins, M. S. 1975 Integral properties of periodic gravity waves of finite amplitude. *Proc. R. Soc. Lond.* A **342**, 157–174.

Longuet-Higgins, M. S. 1978*a* The instabilities of gravity waves of finite amplitude in deep water. I. Superharmonics. *Proc. R. Soc. Lond.* A **360**, 471–488.

Longuet-Higgins, M. S. 1978*b* Some new relations between Stokes's coefficients in the theory of gravity waves. *J. Inst. Maths. Applics.* **22**, 261–273.

Longuet-Higgins, M. S. 1984*a* New integral relations for gravity waves of finite amplitude. *J. Fluid Mech.* (In the press.)

Longuet-Higgins, M. S. 1984*b* Bifurcation in gravity waves. *J. Fluid Mech.* (In the press.)

Longuet-Higgins, M. S. & Fox, M. J. H. 1978 Theory of the almost-highest wave. Part 2. Matching and analytic extension. *J. Fluid Mech.* **85**, 769–786.

McLean, J. W. 1982 Instabilities of finite-amplitude water waves. *J. Fluid Mech.* **114**, 315–330.

Tanaka, M. 1983 The stability of steep gravity waves. *J. phys. Soc. Japan* **52**, 3047–3055.

Whitham, G. B. 1967 Variational methods and applications to water waves. *Proc. R. Soc. Lond.* A **299**, 6–25.