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On the joint distribution of wave periods and amplitudes in a random wave field

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A theoretical probability density is derived for the joint distribution of wave periods and amplitudes which has the following properties: (1) the distribution is asymmetric, in accordance with observation; (2) it depends only on three lowest moments m_0, m_1, m_2 of the spectral density function. It is therefore independent of the fourth moment m_4, which previously was used to define the spectral width (Cavanié *et al.* 1976). In the present model the width is defined by the lower-order parameter

$$\nu = (m_0 m_2/m_1^2 - 1)^{\frac{1}{2}}.$$

The distribution agrees quite well with wave data taken in the North Atlantic (Chakrabarti & Cooley 1977) and with other data from the Sea of Japan (Goda 1978). Among the features predicted is that the total distribution of wave heights is slightly non-Rayleigh, and that the interquartile range of the conditional wave period distribution tends to zero as the wave amplitude diminishes.

The analytic expressions are simpler than those derived previously, and may be useful in handling real statistical data.

1. INTRODUCTION

In a previous contribution (Longuet-Higgins 1975; to be referred to as I) the author proposed a theoretical expression for the joint distribution of the periods and amplitudes of sea waves, which was based on a narrow-band approximation applied to the well known linear theory of gaussian noise. While giving a fairly good fit to wave data with a narrow spectrum such as those of Bretschneider (1959), the model did not account for the asymmetry in the distribution of wave period τ which is commonly observed in wave spectra with a broader bandwidth (see, for example, Goda 1978).

At about the same time, Cavanié *et al.* (1976) proposed a theoretical distribution, also based on a narrow-band, gaussian model, which accounted very successfully for the asymmetry in the distribution of τ. However it involved the use of the well known spectral width parameter ϵ where $\epsilon^2 = 1 - m_2^2/m_0 m_4$, and m_n denotes the nth moment of the spectral density. For some practical purposes this parameter is inconvenient, since the fourth moment, m_4, may depend rather critically on the behaviour of the spectrum at high frequencies.

Some lengthy and perhaps accurate approximations to the distribution of wavelength and amplitude in gaussian noise have been given by Lindgren (1972) and by Lindgren & Rychlik (1982), but for their evaluation these require a great deal of computation. Moreover, these expressions also involve high moments of the spectrum.

The purpose of this note is to present an alternative theoretical distribution, also based on narrow-band theory, which has the same merit as the Cavanié distribution in being asymmetric in τ, but which depends only on the lower-order moments m_0, m_1, m_2, as in paper I. A measure of the spectral width is provided by the parameter ν, where $\nu^2 = m_0 m_2/m_1^2 - 1$. As we shall see, this also accounts well for the observations, and in addition the theoretical expressions are somewhat simpler to handle than those in Cananié's distribution.

2. Theory

As in paper I we begin with the representation of the sea surface elevation ζ in the form

$$\zeta = \operatorname{Re} A\, e^{i\bar{\sigma}t}, \tag{2.1}$$

where $A(t)$ is a complex-valued envelope function:

$$A = \rho\, e^{i\phi}, \tag{2.2}$$

with amplitude ρ and phase ϕ both real but slowly varying functions of the time t. It is convenient to choose the carrier frequency $\bar{\sigma}$ so that

$$\bar{\sigma} = m_1/m_0, \tag{2.3}$$

where m_n denotes the nth moment of the spectral density $E(\sigma)$:

$$m_n = \int_0^\infty \sigma^n E(\sigma)\, d\sigma. \tag{2.4}$$

A spectral width parameter ν can then be defined in terms of the variance of $E(\sigma)$ about the mean:

$$\nu^2 = \mu_2/\bar{\sigma}^2 m_0, \tag{2.5}$$

where

$$\mu_n = \int_0^\infty (\sigma - \bar{\sigma})^n E(\sigma)\, d\sigma. \tag{2.6}$$

Clearly,

$$\mu_0 = m_0, \qquad \mu_1 = 0, \qquad \mu_2 = m_2 - m_1^2/m_0, \tag{2.7}$$

and so

$$\nu^2 = m_0 m_2/m_1^2 - 1. \tag{2.8}$$

We shall adopt the narrow-band hypothesis, namely,

$$\nu^2 \ll 1. \tag{2.9}$$

In practice we assume that $\nu^2 \leqslant 0.36$. This ensures (as we shall see) that the envelope function varies slowly compared with the carrier wave $\exp(\mathrm{i}\,\overline{\sigma}\,t)$ so that the wave crests lie almost on the envelope $\zeta = \rho$. Also, the rate of change of the total phase, $\chi = \phi + \overline{\sigma}t$, that is,

$$\dot{\chi} = \dot{\phi} + \overline{\sigma} \tag{2.10}$$

(where a dot denotes differentiation with respect to t) is almost equal to $\overline{\sigma}$. In other words $\dot{\phi} \ll \overline{\sigma}$, in general. We shall assume further that $\ddot{\phi} \ll \overline{\sigma}\dot{\phi}$, so that $\dot{\phi}$ varies little over a wave period (an assumption discussed below). Then the local wave period τ can be approximated by

$$\tau = 2\pi/\dot{\chi} = 2\pi/(\overline{\sigma} + \dot{\phi}). \tag{2.11}$$

The wave amplitude ρ and the wave period τ may be normalized by writing

$$R = \rho/(2m_0)^{\frac{1}{2}}, \quad T = \tau/\overline{\tau}, \tag{2.12}$$

where we define

$$\overline{\tau} = 2\pi/\overline{\sigma} = 2\pi m_0/m_1. \tag{2.13}$$

Now it can be shown rigorously (see paper I, and earlier papers referred to therein) that the joint probability density of ρ and $\dot{\phi}$ is given by

$$p(\rho, \dot{\phi}) = \{\rho^2/(2\pi\mu_0^2\mu_2)^{\frac{1}{2}}\}\,\mathrm{e}^{-\frac{1}{2}\rho^2(1/\mu_0 + \dot{\phi}^2/\mu_2)}. \tag{2.14}$$

We can now find the joint density of R and T from

$$p(R, T) = p(\rho, \dot{\phi})\,|\partial(\rho, \dot{\phi})/\partial(R, T)|. \tag{2.15}$$

Applying the above formulae we obtain immediately

$$p(R, T) = (2/\pi^{\frac{1}{2}}\nu)\,(R^2/T^2)\,\mathrm{e}^{-R^2[1+(1-1/T)^2/\nu^2]}\,L(\nu), \tag{2.16}$$

where $L(\nu)$ is a normalization factor introduced to take account of the fact that we consider only positive values of T:

$$\frac{1}{L} = \frac{2}{\pi^{\frac{1}{2}}}\int_0^\infty\int_0^\infty \frac{R^2}{T^2}\mathrm{e}^{-R^2[1+(1-1/T)^2/\nu^2]}\,\mathrm{d}R\,\mathrm{d}T. \tag{2.17}$$

On evaluating the integral (see the Appendix) we find,

$$1/L = \tfrac{1}{2}[1 + (1+\nu^2)^{-\frac{1}{2}}]. \tag{2.18}$$

For small values of ν this is close to unity:

$$L \approx 1 + \tfrac{1}{4}\nu^2. \tag{2.19}$$

Some values of L are listed in table 1.

TABLE 1. PARAMETERS OF $p(R, T)$

		mode		
ν	L	R	T	$p_{\max}$
0.1	1.0025	0.955	0.990	4.203
0.2	1.0098	.981	.962	2.180
0.3	1.0215	.958	.917	1.541
0.4	1.0371	.928	.862	1.248
0.5	1.0557	.894	.800	1.096
0.6	1.0767	.857	.735	1.013

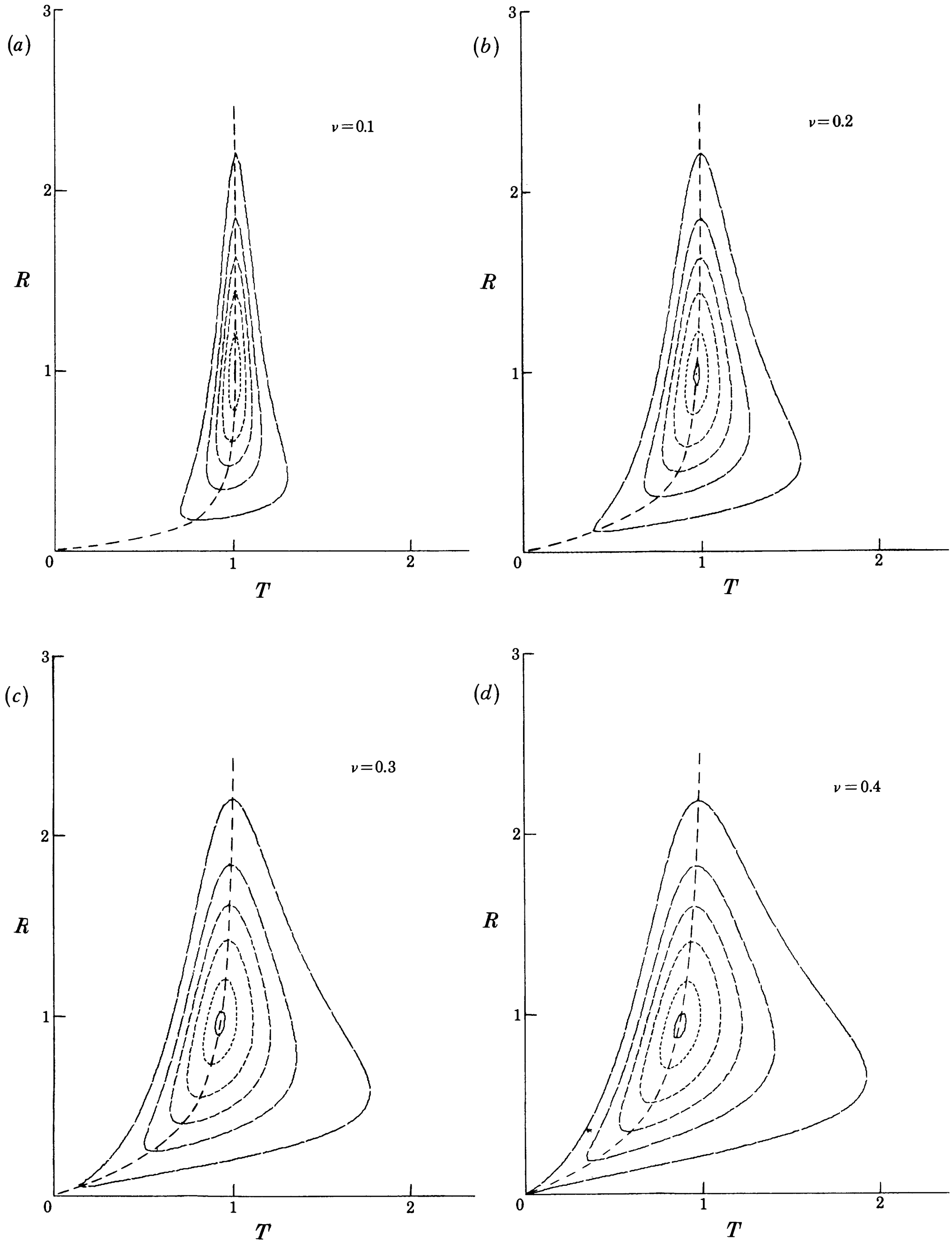


FIGURE 1*a–f*. Contours of $p(R, T)/p_{max}$, where $p(R, T)$ is the joint density of the normalized wave amplitude R and normalized period T and p_{max} is the density at the mode, see (4.2); p/p_{max} takes the values 0.99, 0.90, 0.70, 0.50, 0.30 and 0.10 respectively from the centre contour outwards.

3. Discussion

Strictly speaking, (2.16) gives the probability density of R and T (the dimensionless wave amplitude and period) at points uniformly distributed with regard to t. To find the density of R and T at particular points, say, the maxima of ζ, we would have to consider the joint density of ζ, $\dot{\zeta}$ and $\ddot{\zeta}$ at least, as is done by Arhan *et al.* (1976), or equivalently the joint distribution of ρ, $\dot{\rho}$, $\ddot{\rho}$ and $\dot{\phi}$, $\ddot{\phi}$. But the variance of $\dot{\rho}$

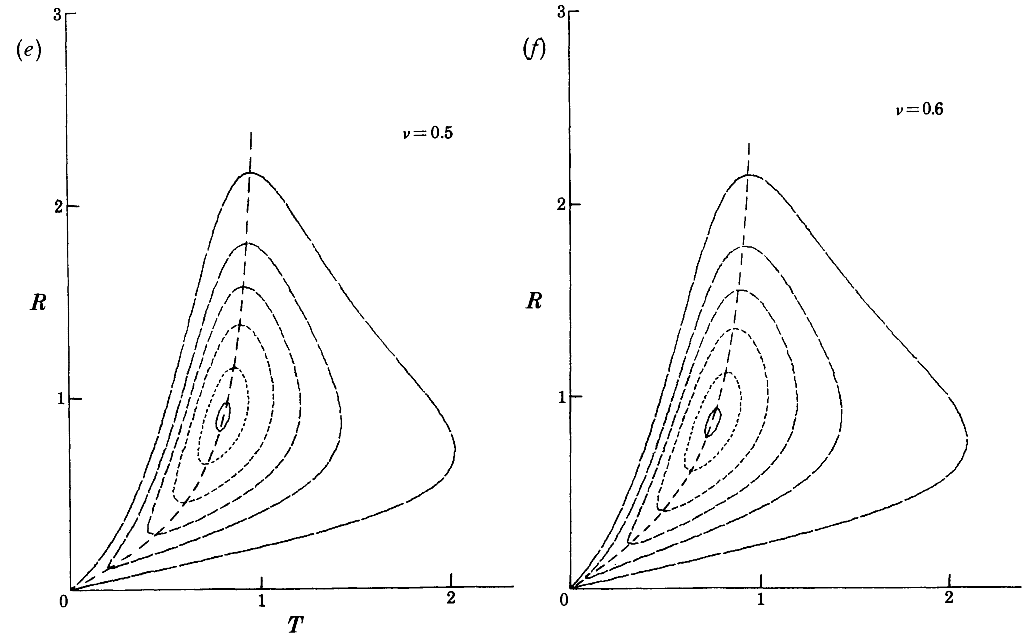


FIGURE 1 *e–f*. For legend see opposite.

is proportional to ν^2 or μ_2 (see Longuet-Higgins 1957) so that $\dot{\rho}$ is negligible, by our assumption (2.9). This implies that ρ varies slowly compared with $\overline{\sigma}$. Similarly, we have assumed $\ddot{\phi} \ll \overline{\sigma}\dot{\phi}$, that is, $\dot{\phi}$ is also slowly varying. Hence the crests will occur at almost regularly spaced intervals in time, and it matters not, in this approximation, whether the density is for values at the crests of the waves, or values uniformly distributed with regard to t as in (2.16).

In figure 1 we show contours of $p(R, T)$ for a sequence of values of the parameter ν. The density clearly shows some asymmetry with regard to T, in general. However, in the limit as $\nu \to 0$ if we write, as in paper I,

$$\xi = \rho/m_0^{\frac{1}{2}} = 2^{\frac{1}{2}}R, \qquad \eta = (T-1)/\nu, \tag{3.1}$$

and assume $|T-1|$ is of order ν, then (2.16) reduces to

$$p(\xi, \eta) = (2\pi)^{-\frac{1}{2}}\, \xi^2\, \mathrm{e}^{-\frac{1}{2}\xi^2(1+\eta^2)}, \tag{3.2}$$

as in I, equation (5). In other words, in the neighbourhood of $T = 1$ the distribution becomes symmetric about the mean wave period, independently of R.

It is pertinent to enquire whether the general expression (2.16) is any more accurate, theoretically, than the expression (3.2) which is restricted to small values of ν and of $|T-1|$. One reason why this may be so is that in the derivation of (3.2) in paper I, the 'period', τ, as defined by (2.11) of the present paper, was approximated by $\bar{\tau}(1-\dot{\phi}/\bar{\sigma})$ (see paper I, equation (A 20)). In the present paper this approximation, though formally legitimate, has not been made.

Another way of stating the situation is that though the distribution of the time derivative $\dot{\chi}$ is exactly symmetric about its mean value $\bar{\sigma}$, the distribution of the reciprocal $2\pi/\dot{\chi}$ is asymmetric. In other words, the distribution of apparent wave 'frequencies' is symmetric, but the distribution of wave periods is not.

In making the approximation (2.11) we assumed implicitly that $(\bar{\sigma}+\dot{\phi})$ was positive, for it is difficult to attach any meaning to a negative period. We therefore agree to ignore the part of the density (2.16) for which $\dot{\phi} < -\sigma$, or $T < 0$.

4. Properties of $p(R, T)$

The position of the mode, or maximum value of $p(R,T)$ is found from the condition that $\partial p/\partial R$ and $\partial p/\partial T$ both vanish. Hence we find

$$R = 1/(1+\nu^2)^{\frac{1}{2}}, \qquad T = 1/(1+\nu^2). \tag{4.1}$$

The value of $p(R,T)$ at this point is therefore

$$p_{\max} = (2L/\pi^{\frac{1}{2}}\mathrm{e})\,(1+\nu^2)/\nu = 0.415(\nu+\nu^{-1})\,L(\nu). \tag{4.2}$$

The effect of broadening the spectrum is therefore to reduce the 'most probable' joint values of the wave period and amplitude, and also to reduce their probability density (when $\nu < 1$).

Consider now the behaviour of $p(R,T)$ near the origin. When R and T are both small, we have

$$p(R,T) \approx (2L/\pi^{\frac{1}{2}}\nu)\,(R^2/T^2)\,\mathrm{e}^{-R^2/\nu^2T^2}, \tag{4.3}$$

that is,

$$p(R,T) \approx (2\nu L/\pi^{\frac{1}{2}})\,\lambda^2\,\mathrm{e}^{-\lambda^2}, \tag{4.4}$$

where $\lambda = R/\nu T$. Hence the contours of p become tangent to the radii $R/T = \lambda\nu$, constant. The axes $R=0$ and $T=0$ both correspond to $p=0$. The direction from 0 in which p is greatest is given by the maximum of (4.4), which corresponds to $\lambda^2 = 1$, hence

$$R/T = \nu, \qquad p = p_0 = (2/\pi^{\frac{1}{2}}\mathrm{e})\,\nu L, \tag{4.5}$$

(compare (4.2)). The contour $p = p_0$ actually has a cusp at the origin. When $p < p_0$, the contours $p = $ constant all pass through the origin. On the other hand when $p_0 < p < p_{\max}$, the contours enclose the mode once, but do not pass through the origin.

From (4.2) and (4.5) it follows that

$$p_0/p_{\max} = \nu^2/(1+\nu^2), \tag{4.6}$$

so that in figures 1 (*a–c*) no contours pass through the origin, and in figures 1 (*d–f*) only the lowest contour: $p = 0.1$.

5. The density of R

The density of the wave amplitude R by itself is found on integrating $p(R, T)$ with respect to T over $0 < T < \infty$, that is,

$$p(R) = \frac{2L(\nu)}{\pi^{\frac{1}{2}}\nu} R^2 \mathrm{e}^{-R^2} \int_0^\infty \frac{1}{T^2} \mathrm{e}^{-R^2(1-1/T)^2/\nu^2} \,\mathrm{d}T. \tag{5.1}$$

Setting

$$R(1-1/T) = \nu\beta, \tag{5.2}$$

we have

$$p(R) = \frac{2L}{\pi^{\frac{1}{2}}} R\,\mathrm{e}^{-R^2} \int_{-\infty}^{R/\nu} \mathrm{e}^{-\beta^2}\,\mathrm{d}\beta \tag{5.3}$$

$$= 2R\,\mathrm{e}^{-R^2} L(\nu)\,F(R/\nu), \tag{5.4}$$

where

$$F(R/\nu) = \frac{1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{R/\nu} \mathrm{e}^{-\beta^2}\,\mathrm{d}\beta, \tag{5.5}$$

a well known error function. Equation (5.4) states that the density of R is almost Rayleigh, but must be corrected by the factor $LF(R/\nu)$. For values of R that are of order 1 or larger, the correction will be exponentially small. However when R is of order ν, that is, close to the origin, the correction becomes significant. Figure 2

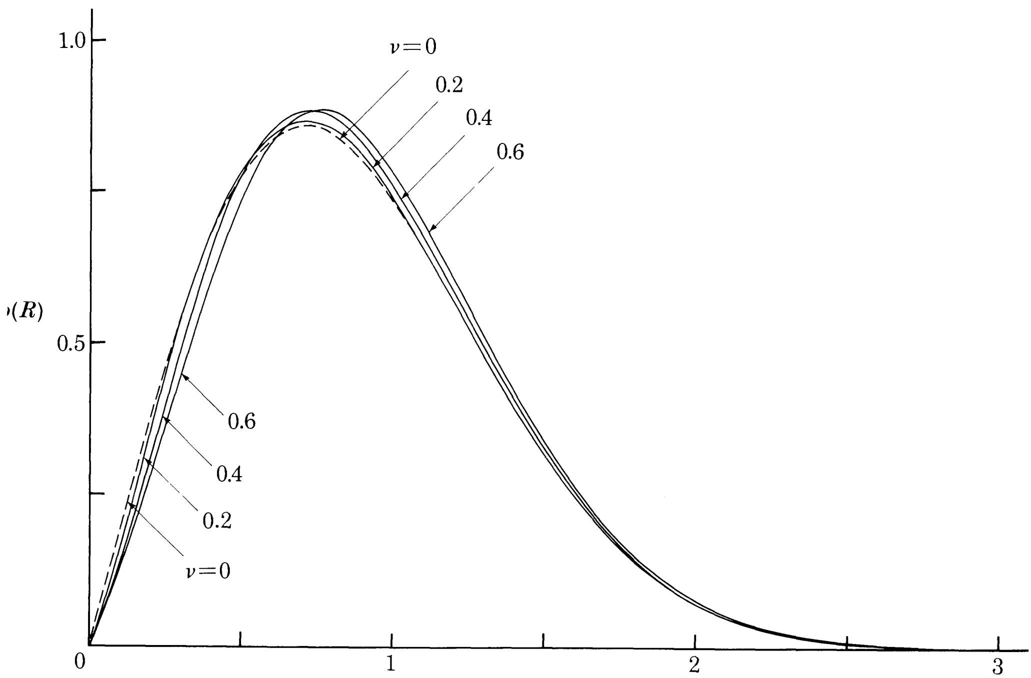


Figure 2. The density of R (see (5.4)) when ν = 0.2, 0.4 and 0.6 (full curves) compared with the Rayleigh distribution (broken curve).

shows some examples. The effect of the correction factor is to reduce the number of very low waves, and to shift the mode of the distribution, which otherwise is at $R = 2^{-\frac{1}{2}}$, somewhat to the right of this point.

TABLE 2. PARAMETERS OF $p(R)$

ν	R_{av}	R^2_{rms}	$(R_{av} - \frac{1}{2}\pi^{\frac{1}{2}})$	$(R_{rms} - 1)$
0.1	0.8832	1.0025	0.0020	0.0012
0.2	.8935	1.0095	.0072	.0047
0.3	.9006	1.0202	.0144	.0100
0.4	.9087	1.0332	.0224	.0165
0.5	.9167	1.0472	.0304	.0233
0.6	.9241	1.0611	.0378	.0301

The lower order moments of $p(R)$, found by numerical integration, are shown in table 2. From this it will be seen that the r.m.s. value of R differs only slightly from unity. When $\nu = 0.3$, for example, the difference is only 1 %.

6. THE CONDITIONAL DISTRIBUTION OF WAVE PERIODS $p(T/R)$

The distribution of T at fixed values of the wave amplitude R is found on dividing $p(R, T)$ by $p(R)$; hence

$$p(T|R) = (\pi^{\frac{1}{2}}\,\nu\, F(R/\nu))^{-1}\,(R/T^2)\,e^{-R^2(1-1/T)^2/\nu^2}. \tag{6.1}$$

To find the mode, or peak, of this function we set $\partial p/\partial T = 0$ to obtain

$$(1/T)(1/T-1) = \nu^2/R^2, \tag{6.2}$$

and so

$$T = 2/[1+(1+4\nu^2/R^2)^{\frac{1}{2}}]. \tag{6.3}$$

This curve is shown by the dashed lines in figure 1. It must clearly pass through the mode (4.1) and where it intersects any countour p = constant, the tangent to that contour is parallel to the axis of T. For small R we have $T \approx R/\nu$, so that the curve touches the contour (4.5). On the other hand, for large R the curve is asymptotic to the vertical line $T = 1$. In general, the curve expresses very well the asymmetry in the distribution of T.

Now the quartiles of $p(T/R)$ are given by

$$\int_0^{Q_n} p(T|R)\,dT = \tfrac{1}{4}n, \qquad n = 1, 2, 3, \tag{6.4}$$

that is

$$\frac{1}{\pi^{\frac{1}{2}}}\int_\beta^\infty e^{-\beta^2}\,d\beta = \tfrac{1}{4}n\,F(R/\nu), \tag{6.5}$$

where

$$\beta = R(1-1/T)/\nu. \tag{6.6}$$

So we have to solve numerically

$$F(\beta) = \tfrac{1}{4}n\,F(R/\nu) \tag{6.7}$$

for β, and then

$$Q_n = 1/(1-\beta\nu R). \tag{6.8}$$

These curves are illustrated in figure 3, in the case $\nu = 0.3$. Clearly all the quartiles

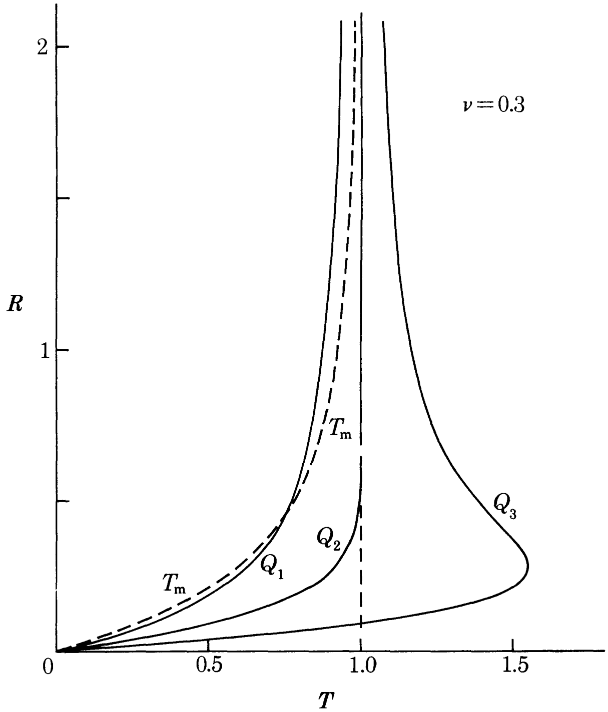


FIGURE 3. Curves showing the mode and quartiles of the conditional density of wave periods when $\nu = 0.3$.

are now asymmetric, and pass through the origin. Moreover, the interquartile range $(Q_3 - Q_1)$, instead of being proportional to $1/R$ for all values of R, as in paper I, has a maximum at around $R = 0.22\,(\xi = 0.31)$ and tends to 0 both as $R \to \infty$ and as $R \to 0$.

7. The total density $p(T)$

The density of T regardless of R is found by integrating $p(R, T)$ with respect to R over $0 < R < \infty$, to give

$$p(T) = (L/2\nu T^2)\,[1 + (1 - 1/T)^2/\nu^2]^{-\frac{3}{2}}. \tag{7.1}$$

This is shown in figure 4 for some representative values of ν. The median and quartiles are found by the substitution

$$\alpha = (1 - 1/T)/\nu, \tag{7.2}$$

leading to

$$\tfrac{1}{2}L\int_{-\infty}^{\alpha} \frac{\mathrm{d}\alpha}{(1+\alpha^2)^{\frac{3}{2}}} = \tfrac{1}{4}, \tfrac{1}{2} \text{ or } \tfrac{3}{4}, \tag{7.3}$$

hence

$$\alpha/(1+\alpha^2)^{\frac{1}{2}} = n/2L - 1, \quad n = 1, 2, 3. \tag{7.4}$$

Solving for α we find,

$$\alpha = (n/2L - 1)/[1 - (n/2L - 1)^2]^{\frac{1}{2}}, \tag{7.5}$$

and then

$$Q_n = T_{\cdot} = 1/(1 - \nu\alpha), \qquad n = 1, 2, 3. \tag{7.6}$$

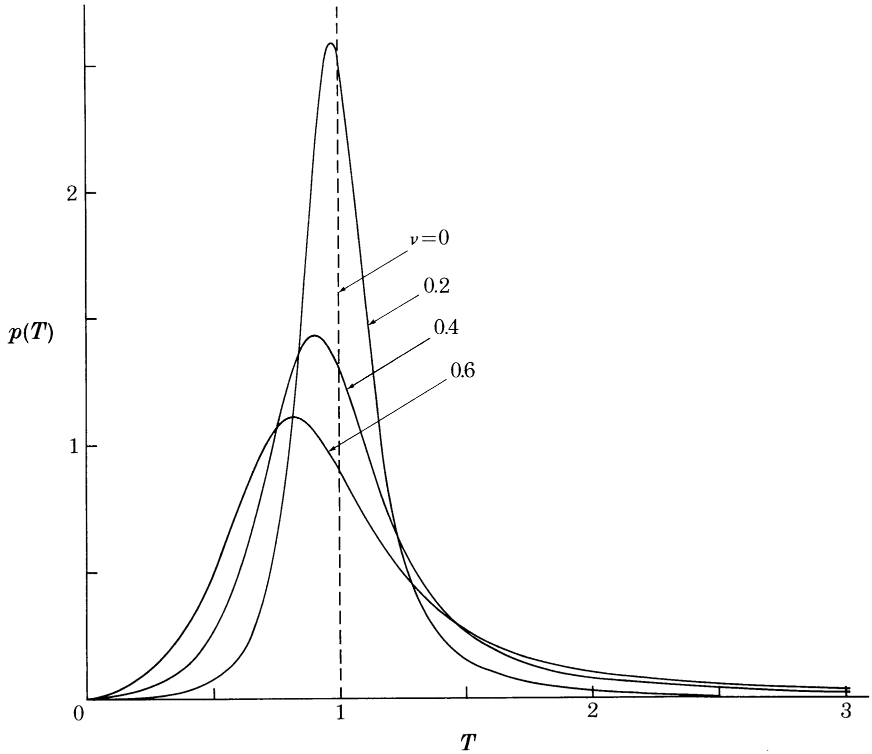


FIGURE 4. The density of the wave period T (see (6.9)) when ν = 0.2, 0.4 and 0.6.

Some representative values of Q_n are given in table 3. Also shown is the interquartile range

$$\mathrm{IQR} = Q_3 - Q_1. \tag{7.7}$$

The mode of the distribution is also easily found and is given by

$$T_{\mathrm{m}} = 2/[(9+8\nu^2)^{\frac{1}{2}} - 1]. \tag{7.8}$$

Note that the mean of the distribution is theoretically infinite, since for large values of T the density $p(T)$ behaves like T^{-2}. This implies only that as $T \to \infty$ the

TABLE 3. PARAMETERS OF THE DISTRIBUTION OF PERIODS, $p(T)$

ν	T_{m}	Q_1	Q_2	Q_3	IQR	T_{av}
0.1	0.9934	0.9452	0.9998	1.0606	0.1154	0.9950
0.2	.9742	.8953	.9981	1.1249	.2296	.9806
0.3	.9444	.8488	.9937	1.1891	.3403	.9578
0.4	.9065	.8050	.9859	1.2492	.4442	.9285
0.5	.8633	.7636	.9743	1.3020	.5384	.8944
0.6	.8174	.7243	.9589	1.3450	.6207	.8574

integrated error becomes infinite. An alternative estimate of the mean does, however, exist. For we know the exact result that the average frequency of up-crossings of the mean level is

$$N = (2\pi)^{-1}(m_2/m_0)^{\frac{1}{2}}. \tag{7.9}$$

From the relations (2.3), (2.5) and (2.7), this can be written

$$N = (1+\nu^2)^{\frac{1}{2}}\bar{\sigma}/2\pi. \tag{7.10}$$

Hence

$$\tau_{av} = N^{-1} = \bar{\tau}/(1+\nu^2)^{\frac{1}{2}}, \tag{7.11}$$

and so

$$T_{av} = \tau_{av}/\bar{\tau} = (1+\nu^2)^{-\frac{1}{2}}. \tag{7.12}$$

These parameters are all shown in table 3.

8. Comparison with observation

The measurement of the local wave height and period from a wave record is liable to some ambiguities. For example, in figure 5, should the 'period' be taken as the crest-to-crest interval τ_1 or the up-crossing interval τ_2? Not all authors specify their choice precisely. It may be that for the Cavanié distribution, depending on the higher moment m_4, the choice of τ_1 is more appropriate, whereas for the present distribution, depending only on m_2, it is more appropriate to choose τ_2.

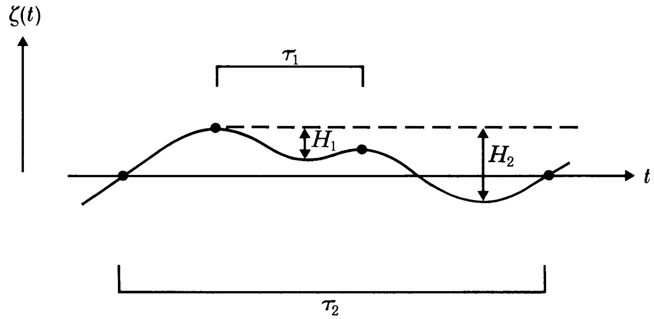


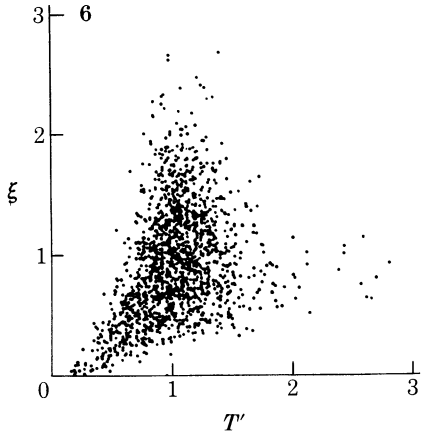
Figure 5. Alternative measures of the local wave height and period.

Without knowing precisely the authors' procedure we shall nevertheless compare the theoretical model described in §§ 2–6 with some previous observations.

Chakrabarti & Cooley (1977) measured 1624 waves from a North Atlantic storm, over an interval of 3.5 days. The scatter diagram of their observations is shown in figure 6, where the vertical scale is the wave height normalized by the 'r.m.s. wave height' H_{rms}; the horizontal scale is the wave period normalized by the 'mean period' T_{av}. To judge by the spread of wave periods (see figure 7 of their paper) an appropriate value of ν for these data was 0.30.

To make a comparison with $p(R, T)$, we replot the contours to a new vertical scale $R' = R/R_{rms}$, and a new horizontal scale $T' = T/T_{av}$ where $T_{av} = (1+\nu^2)^{-\frac{1}{2}}$. There will be a new value of p_{max}, namely $p'_{max} = R_{rms}\,T_{av}\,p_{max}$, but the relative values of p, namely $p'(R', T')/p'_{max}$, will be unchanged.

This is done in figure 7, and it will be seen that the resemblance between figures 6 and 7 is close, in particular as regards the shape of the distributions and the tendency for plotted points to be drawn down towards the origin. However, the absolute densities are not easy to determine from the scatter diagram.



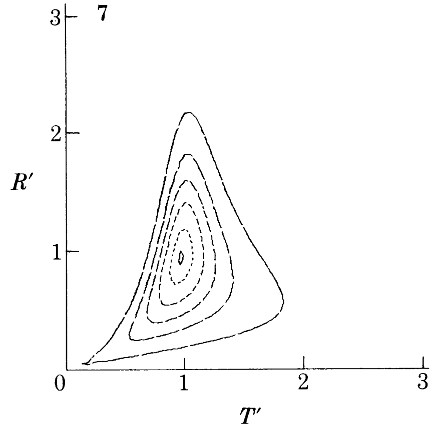


FIGURE 6. (From Chakrabarti & Cooley 1977.) A scatter diagram of normalized heights against normalized wave period, for a storm in the North Atlantic.

FIGURE 7. Contours of $p'(\xi, T')/p_{\max}$ when $\nu = 0.3$, for comparison with the data of figure 6. The contour values are as in figure 1.

Figure 8 shows a histogram of the wave heights measured by Chakrabarti & Cooley (1977). The horizontal scale has been normalized by the r.m.s. value of the observations. In the same diagram, the full curve indicates the theoretical density

$$p(R/R_{\mathrm{rms}}) = R_{\mathrm{rms}}\, p(R), \tag{8.1}$$

where $p(R)$ is given by (5.4) and $\nu = 0.30$. The broken curve shows the Rayleigh distribution corresponding to $\nu = 0$. It will be seen that the curve for $\nu = 0.30$ is a slightly better fit to the observations when R is small, and near the peak of the distribution.

In figure 9 we show a comparison of the interquartile range $(Q_3 - Q_1)$, corresponding to the theory of figure 3, and the data plotted by Chakrabarti & Cooley (1977). At large values of ξ, the theoretical curve is asymptotic to the hyperbola given by the narrow-band theory: $Q_3 - Q_1 = 1.35\, \nu/\xi$, but at lower values of ξ the curve reaches a maximum and then returns to the origin. The plotted observations follow the theory down to about $\xi = 1$, and then lie inside the curve. The discrepancy between theory and observation is less than previously, but is still appreciable.

Goda (1978) has presented diagrams of the relative wave height H/H_{av} against the relative wave period T/T_{av}, as in figure 10, the data being classified according to the value of a certain 'skewness parameter' r. Goda has found a fairly good correlation

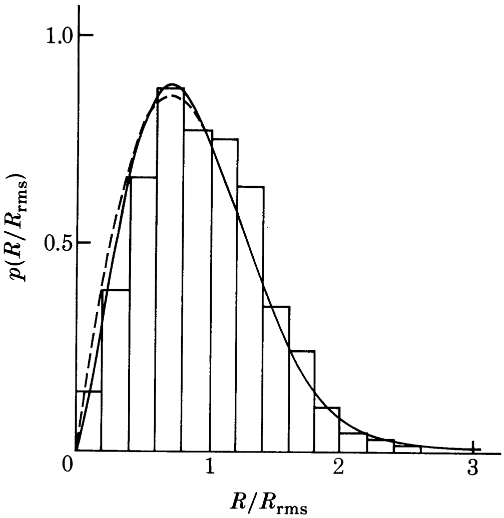


FIGURE 8. Histogram of wave heights from Chakrabarti & Cooley (1977) normalized by the r.m.s. value. The full curve represents $p = R_{rms}\,p(R)$ (equation (5.4)) when $\nu = 0.3$. The broken curve is the Rayleigh distribution: $p = 2R\,e^{-R^2}$.

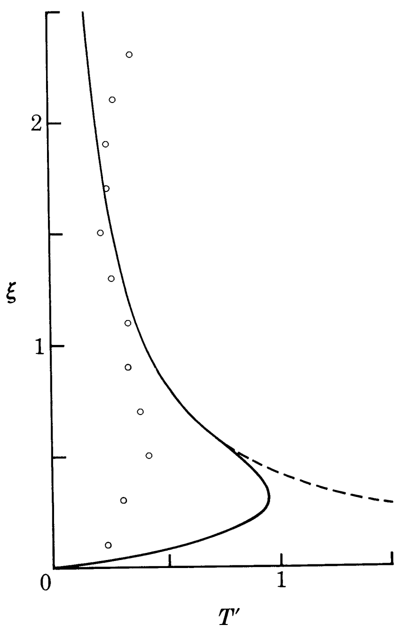


FIGURE 9. The interquartile range of the wave periods, as a function of the normalized wave height ξ. Data are from Chakrabarti & Cooley (1977). The full curve represents the difference $(Q_3 - Q_1)$ in figure 3. The dashed curve is the narrow-band asymptote.

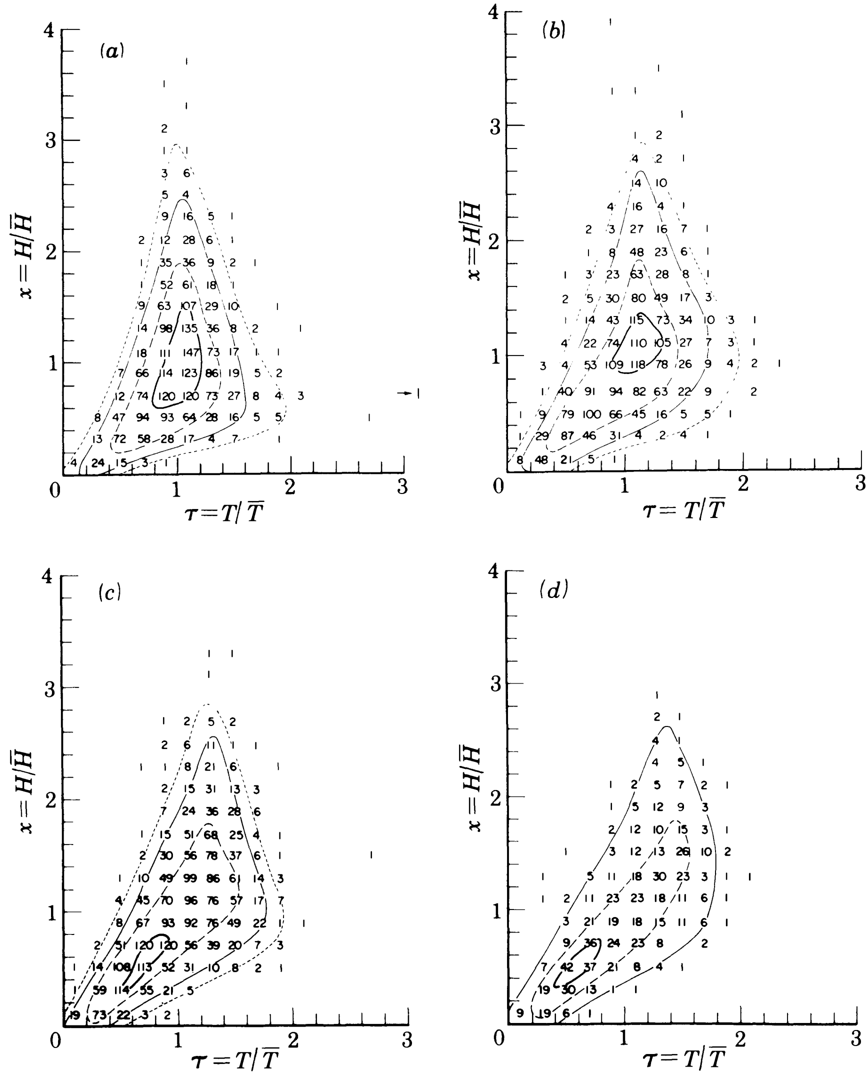


FIGURE 10. (From Goda 1978.) Scatter diagrams of H/H_{av} against T/T_{av} for different ranges of r. The contours of $p(x, t)$ take the values 1.0, 0.5, 0.1, 0.03 respectively from the centre curve outwards. The parameter $r(H, T)$ lies in the range (*a*) 0.20–0.39; (*b*) 0.40–0.59; (*c*) 0.60–0.69; (*d*) 0.70–0.79.

between r and the parameter ν_T (derived from the distribution of wave periods) which corresponds roughly to ν. In table 4 we indicate the average values of ν chosen (from Goda's figure 10) to correspond to the stated ranges of r.

We note also that since Goda plotted H/H_{av} rather than ξ, the vertical scale of his plots is different from that of Chakrabarti & Cooley (1977). Accordingly the scale must be modified by the factor $1/R_{av}$; see table 4. The horizontal scale has also to be modified by the factor $1/T_{av}$, which we assume is given by the formula $T_{av} = (1+\nu^2)^{-\frac{1}{2}}$ derived from zero-crossings (see § 7). This factor also is shown in table 4. Finally the theoretical value

$$p''_{max} = R_{av} T_{av} p_{max}, \tag{8.2}$$

TABLE 4. PARAMETERS FOR THE DATA OF GODA (1978)

	r	ν	R_{av}	T_{av}	$p''_{\max}$ theor.	$p''_{\max}$ obs.
(*a*)	0.20–0.39	0.29	0.8999	0.9604	1.34	1.37
(*b*)	0.40–0.59	0.38	.9070	.9348	1.06	1.14
(*c*)	0.60–0.69	0.50	.9167	.8944	0.85	1.05
(*d*)	0.70–0.79	0.58	.9226	.8650	0.76	0.96

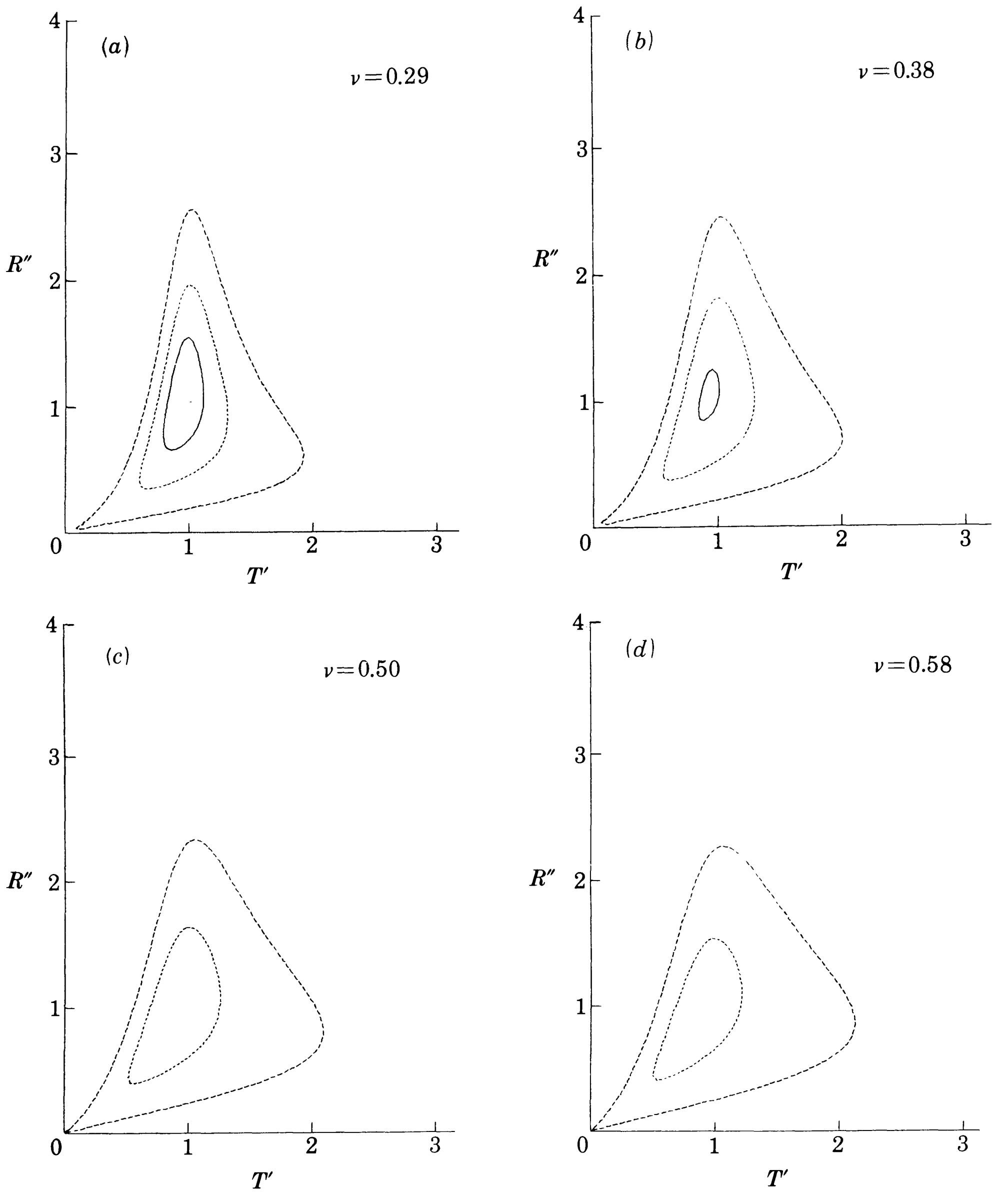


FIGURE 11. Contours of $R_{\mathrm{av}} T_{\mathrm{av}}\, p(R, T)$ for values of ν corresponding to figure 7. Contours take values 1.0, 0.5, 0.1 in (*a*) and (*b*) and 0.5, 0.1 in (*c*) and (*d*) from inner to outer.

where p_{max} is given by (4.2) is shown in the next-to-last column of table 4, compared with the maximum observed value, from Goda's figure 10. The agreement is reasonable, except that the theory is consistently lower. Most of the discrepancy seems due to our choice of T_{av}, and it is possible that a value nearer to unity would be more appropriate.

In figure 11 we show contours of $R_{av} T_{av} p(R, T)$, which may be compared with the corresponding contours in figure 10. We have not plotted any contours corresponding to $p'' = 0.03$, since Goda's data appear insufficient for him to trace the corresponding curves with any accuracy.

The agreement between figures 10 and 11 seems reasonable. In particular the position of the modes agrees fairly well, though in figure 10 there is an indication that for the larger values of ν the mode of the distribution splits into two, one further from and one nearer to the origin.

Other authors, for example, Cavanié *et al.* (1976), have combined data for many different spectra, which precludes any precise comparison with theory. However Cavanié's data, containing 28 240 waves with a mean value $\epsilon = 0.865$ do appear to resemble in a general way the contours of figures 1 (*d–f*).

9. Conclusion

We have derived an approximation to the joint distribution of wave periods and amplitudes that gives a reasonably good fit to some typical data, and that depends only on the low-order parameter ν. Technically the approximation is correct only to order ν, but by not making certain approximations, legitimate to this order, which were made in a previous paper I, the distribution is given an asymmetry with respect to T in agreement with the observations. Undoubtedly there are further corrections of order ν^2 to be made if the distribution is to be entirely correct to this order, but the observational evidence suggests that such corrections are small and not very significant for practical applications.

In comparison with the analysis of Cavanié *et al.* (1976) the present model has the advantage of comparative simplicity, and in depending only on ν rather than on the higher-order parameter ϵ. This seems desirable, since for many spectra that behave like σ^{-4} or σ^{-5} at infinity (such as the Pierson–Moskowitz spectrum) m_4 becomes infinite, making $\epsilon = 1$. Thus ϵ becomes insensitive to other parts of the spectrum. On the other hand, ν, which depends only on m_2, is less subject to this difficulty. Rye & Svee (1976) have suggested that even ν is unduly influenced by the high-frequency cut-off but the examples given are for rather broad spectra. The most satisfactory procedure may be to estimate ν not from the spectrum $E(\sigma)$ but from the measured distribution of wave periods, as is done by Longuet-Higgins (1975) and Goda (1978). Indeed Goda finds that ν_T is highly correlated with certain other parameters which shows that it is reasonably stable. Apparently the reason for the success of this method of estimating ν is that it reflects whatever subjective choices are made by the observer when measuring τ from the wave record, choices which may amount to applying a subjective low-pass filter to the record.

Finally, the parameter ν has one clear advantage in being related theoretically to other statistical properties of the record and in particular to the lengths of the wave groups, (see, for example, Longuet-Higgins 1957, 1983).

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Appendix. Evaluation of $L(\nu)$

To carry out the integration in (2.17) set

$$(1-1/T)/\nu = \alpha, \tag{A 1}$$

so that

$$\frac{1}{L} = \frac{2}{\pi^{\frac{1}{2}}}\int_0^\infty \int_{-\infty}^{1/\nu} R^2\,\mathrm{e}^{-R^2(1+\alpha^2)}\,\mathrm{d}R\,\mathrm{d}\alpha. \tag{A 2}$$

Then

$$\begin{aligned}\frac{1}{L} &= \frac{1}{2}\int_{-\infty}^{1/\nu} \frac{\mathrm{d}\alpha}{(1+\alpha^2)^{\frac{3}{2}}}\\ &= \frac{1}{2}\left[\frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\right]_{-\infty}^{1/\nu}\\ &= \tfrac{1}{2}[1+(1+\nu^2)^{-\frac{1}{2}}].\end{aligned} \tag{A 3}$$

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