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A technique for time-dependent free-surface flows

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In this paper we present a reformulation of the problem of time-dependent irrotational free-surface flows under gravity. Compact expressions are derived for the pressure p and its rate of change $\mathrm{D}p/\mathrm{D}t$ following a particle. Some exact but special solutions are discussed, and a method of approach for the general case is proposed.

1. INTRODUCTION

Over 130 years after the formulation by Stokes (1845) of the general equations for inviscid irrotational flow, it is remarkable that few if any exact solutions to the problem of time-dependent free-surface flows under gravity are known. Most known solutions refer to flows that are steady, or, like progressive waves, are easily reducible to steady flows by an appropriate choice of the frame of reference. Among the time-dependent gravitational flows that have been considered are those for shallow-water waves and standing waves, but these all appear to involve expansions in powers of some small parameter, such as surface slope, and hence must be considered as approximations at best.

Other time-dependent solutions, such as the Dirichlet parabola (John 1953), ellipse (Taylor 1960) and hyperbola (Longuet-Higgins 1972) are exact but do not contain gravity in an essential way.

An interesting method for obtaining solutions to time-dependent free-surface flows was indeed suggested by John (1953), who in this way derived some special flows (see also Longuet-Higgins 1976). However, the method cannot be general, for it assumes a Lagrangian coordinate ω which is an analytic function of the velocity potential. This would not include, for instance, a progressive irrotational wave with plane bottom, in which the mass-transport velocity induces a strong vertical gradient of the mean displacement.

The present situation is highlighted by the absence of any satisfactory analytic solution to the problem of an overturning wave, although the possibility of accurate numerical calculation has been demonstrated by Longuet-Higgins & Cokelet (1976, 1978).

In our quest for a general method, we may be guided by the following considerations. For free-surface flows there are two boundary conditions. One of these, the kinematic condition, states that a particle in the free surface must move with the surface. This is most easily expressed in terms of Lagrangian coordinates. However, in the Lagrangian analysis the equations of continuity and of irrotationality for the *interior* of the fluid are both highly nonlinear and difficult to handle, whereas in the Eulerian description both conditions are automatically ensured by the assumption of a velocity potential ϕ satisfying $\nabla^2\phi = 0$, or equivalently the assumption that

$$\chi \equiv \phi + \mathrm{i}\psi \tag{1.1}$$

is an analytic function of the space coordinate z. This overwhelming advantage of the Eulerian description suggests strongly that it be incorporated in our attack on the problem.

The second essential boundary condition for free-surface flows is the condition that the *pressure* at the free surface be constant (say zero). Now the pressure, for a time-dependent irrotational motion, is expressed quite simply in terms of the complex potential $\chi(z,t)$ by the Bernoulli equation, so that the equation of the free surface can be expressed very simply in Eulerian coordinates as $p = 0$.

Returning to the first condition, it is important to realize that the free surface, in unsteady flow, is not a streamline. (Indeed the trajectories of particles throughout the fluid are not streamlines, but only tangent to the streamlines instantaneously.) We may however express the *kinematic* boundary condition by saying that for a particle in the free surface $\mathrm{D}p/\mathrm{D}t = 0$, that is to say the rate of change of the pressure *following a particle* must vanish. This also can be expressed in Eulerian coordinates. However until the present investigation (described in § 2) it was possibly not realized how compact this boundary condition becomes when expressed in terms of the complex coordinate z.

Thus the problem, as formulated in the present paper, reduces to ensuring that the two surfaces $p = 0$ and $\mathrm{D}p/\mathrm{D}t = 0$ are coincident.

As we shall see later, this formulation, together with the fact that calculations in complex coordinates are very easily performed on a modern computer, places in our hands a powerful tool for investigating time-dependent free-surface flows.

2. z AND t AS INDEPENDENT VARIABLES

Throughout this paper we assume the flow to be two-dimensional and work in terms of the complex variables

$$z = x + \mathrm{i}y, \quad z^* = x - \mathrm{i}y, \tag{2.1}$$

where x and y are rectangular coordinates, x being directed vertically downwards. The flow being irrotational and incompressible there exists a complex velocity potential

$$\chi = \phi + \mathrm{i}\psi \tag{2.2}$$

which is an analytic function of z and a smooth function of the time t, such that the particle velocity $u+\mathrm{i}v = W^*$ is given by

$$W = \chi_z, \quad W^* = (\chi_z)^* = (\chi^*)_{z^*}. \tag{2.3}$$

Here a suffix denotes partial differentiation, and an asterisk denotes the complex conjugate. The pressure p is then given by the Bernoulli equation in the form

$$p = \mathrm{Re}\,(gz - \chi_t - \tfrac{1}{2}\chi_z\chi_z^* + f), \tag{2.4}$$

where g denotes gravity and f is a function of t only.

To find an expression for $\mathrm{D}p/\mathrm{D}t$, where $\mathrm{D}/\mathrm{D}t$ denotes differentiation following a particle, we have first

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\partial p}{\partial t} + \left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right). \tag{2.5}$$

Now if $F(z, z^*)$ is any differentiable function of both z and z^* we have

$$F_x = (F_z + F_{z^*}), \quad F_y = \mathrm{i}(F_z - F_{z^*}). \tag{2.6}$$

Also

$$u = \tfrac{1}{2}(\chi_z + \chi_z^*), \quad v = \tfrac{1}{2}\mathrm{i}(\chi_z - \chi_z^*). \tag{2.7}$$

Therefore in general

$$\mathrm{D}F/\mathrm{D}t = F_t + (\chi_z^* F_z + \chi_z F_z^*). \tag{2.8}$$

Writing

$$F = -2p = (\chi_t + \chi_t^*) + \chi_z\chi_z^* - g(z+z^*) - 2f, \tag{2.9}$$

we find

$$\begin{aligned} -2\mathrm{D}p/\mathrm{D}t = {} & (\chi_{tt} + \chi_{tt}^*) + (\chi_{zt}\chi_z^* + \chi_{zt}^*\chi_z) - 2f_t \\ & + \chi_z^*(\chi_{zt} + \chi_{zz}\chi_z^* - g) \\ & + \chi_z(\chi_{zt}^* + \chi_{zz}^*\chi_z - g) \end{aligned} \tag{2.10}$$

and so

$$\mathrm{D}p/\mathrm{D}t = \mathrm{Re}\,[gW^* - (\chi_{tt} + 2W^*\chi_{zt} + W^{*2}\chi_{zz}) + f_t], \tag{2.11}$$

where

$$W^* = \chi_z^* = \mathrm{D}z/\mathrm{D}t. \tag{2.12}$$

The remarkably compact expression (2.11) seems not to have been recognized previously.

For steady flows or for motions such as progressive waves when referred to the appropriate coordinates, equation (2.11) reduces to

$$\mathrm{D}p/\mathrm{D}t = \mathrm{Re}\,[W^*(g - W^*\chi_{zz})]. \tag{2.13}$$

The first term on the right represents the pressure change due to vertical motion in the hydrostatic pressure field. The second term, involving χ_{zz}, represents the effect of the curvature of the streamlines.

3. χ AND t AS INDEPENDENT VARIABLES

For some problems (for example in waves of finite amplitude) it may be more convenient to take χ and t as independent variables, and to express the space coordinate z in terms of them. For steady waves (no time dependence) this method was suggested by Stokes (1880).

To find the corresponding expressions in the new variables, we have for example, by the method of differentials,

$$\left.\begin{aligned} \mathrm{d}z &= z_\chi \,\mathrm{d}\chi + z_t \,\mathrm{d}t, \\ \mathrm{d}\chi &= \chi_z \,\mathrm{d}z + \chi_t \,\mathrm{d}t. \end{aligned}\right\} \tag{3.1}$$

On eliminating $\mathrm{d}\chi$ from these equations and then equating coefficients of the independent increments $\mathrm{d}z$ and $\mathrm{d}t$ we obtain

$$\left.\begin{aligned} z_\chi \chi_z &= 1, \\ z_\chi \chi_t + z_t &= 0. \end{aligned}\right\} \tag{3.2}$$

Hence

$$\chi_z = 1/z_\chi = W \tag{3.3}$$

and

$$\chi_t = -z_t/z_\chi = -W z_t. \tag{3.4}$$

The expression (2.4) for the pressure then becomes

$$p = \operatorname{Re}(gz + W^* z_t - \tfrac{1}{2} W W^* + f). \tag{3.5}$$

To obtain the corresponding expression for $\mathrm{D}p/\mathrm{D}t$, we note that if $F(z,t)$ is any differentiable function which on substitution $z = z(\chi)$ becomes equal to $G(\chi, t)$ then, by the same argument,

$$F_z = G_\chi / z_\chi \tag{3.6}$$

and

$$F_t = G_t + \chi_t G_\chi, \tag{3.7}$$

so on substituting for χ_t from (3.4) we have

$$F_t = G_t - G_\chi z_t / z_\chi. \tag{3.8}$$

By repeated application of the results (3.6) and (3.8) we obtain

$$\left.\begin{aligned} \chi_{zz} &= -W^3 z_{\chi\chi}, \\ \chi_{zt} &= -W^2 z_{\chi t} + W^3 z_t z_{\chi\chi}, \\ \chi_{zz} &= -W z_{tt} - 2W^2 z_t z_{\chi t} - W^3 z_t^2 z_{\chi\chi}. \end{aligned}\right\} \tag{3.9}$$

Finally on substitution in (2.11) we find, after some rearrangement,

$$\mathrm{D}p/\mathrm{D}t = \operatorname{Re}[W(g + z_{tt} + 2H z_{\chi t} + H^2 z_{\chi\chi}) + f_t], \tag{3.10}$$

where

$$H = W(W^* - z_t). \tag{3.11}$$

Since from (2.8) the rate of change of χ following a particle is

$$\mathrm{D}\chi/\mathrm{D}t = \chi_t + \chi_z{}^* \chi_z = -Wz_t + WW^* \tag{3.12}$$

(from (3.3) and (3.4)) it follows also that

$$H = \mathrm{D}\chi/\mathrm{D}t, \tag{3.13}$$

showing the analogy between (2.11) and (3.10). However, since W $(= \chi_z = 1/z_\chi)$ is linear in χ, H is clearly not linear in z, showing that (2.11) is generally the simpler expression of the two.

In steady flows equation (3.10) reduces to

$$\mathrm{D}p/\mathrm{D}t = \mathrm{Re}\,[W(g + W^2 W^{*2} z_{\chi\chi})]. \tag{3.14}$$

The term due to curvature of the streamlines can also be written as

$$\mathrm{Re}\,(Wq^4 z_{\chi\chi}), \tag{3.15}$$

where q denotes the particle speed. In progressive gravity waves this term must exactly balance the hydrostatic term at the free surface itself.

4. χ AND z BOTH FUNCTIONS OF ω

The analysis of the two previous sections may be included in the general case when χ and z are each functions of a third complex variable ω, and of the time t, that is

$$\chi = X(\omega, t), \quad z = Z(\omega, t). \tag{4.1}$$

Thus §§2 and 3 correspond to the cases $Z \equiv \omega$ and $X \equiv \omega$ respectively. The form (4.1) also includes as a special case the formalism of John (1953) but is more general, since John assumed that ω was a constant following a particle, and moreover that ω was real at the free surface. This restricted the solution of flows of a special class. When we adopt the assumption (4.1), the equation of the free surface is given by a more general (but real) function of ω and ω^*.

Expressions for p and $\mathrm{D}p/\mathrm{D}t$ can be derived by an argument similar to that of §3. Thus we easily find

$$\chi_z = X_\omega / Z_\omega = W(\omega, t) \tag{4.2}$$

say, and

$$\chi_t = X_t - WZ_t \tag{4.3}$$

leading to

$$-2p = \tfrac{1}{2} WW^* + (X_t - WZ_t) - gZ - f + \text{c.c.}, \tag{4.4}$$

where c.c. stands for complex conjugate. The right hand side of equation (4.4) is a real symmetric function of ω, ω' and t.

Now if $F(z, z^*, t)$ is any real symmetric function of z and z^* which by the substitution $z = Z(\omega, t)$, $z^* = Z^*(\omega^*, t)$ becomes $G(\omega, \omega^*, t)$ it can easily be shown that

$$F_z = G_\omega / Z_\omega, \quad F_{z^*} = (F_z)^* = G_\omega{}^* / Z_\omega{}^* \tag{4.5}$$

and

$$F_t = G_t - G_\omega Z_t/Z_\omega - G_\omega^* Z_t^*/Z_\omega^*. \tag{4.6}$$

Therefore from (2.8) we have

$$\begin{aligned} DG/Dt = DF/Dt &= F_t + W^* F_z + W F_z^* \\ &= G_t + (W^* - Z_t) G_\omega/Z_\omega + (W - Z_t^*) G_\omega^*/Z_\omega^*. \end{aligned} \tag{4.7}$$

Applying this result to the function $-2p$ of (4.4) we obtain finally

$$\begin{aligned} -2Dp/Dt = (X_{tt} - W z_{tt}) &+ 2K(X_{\omega t} - W Z_{\omega t}) \\ &+ K^2(X_{\omega\omega} - W Z_{\omega\omega}) - gW - f_t, \end{aligned} \tag{4.8}$$

where

$$W = X_\omega/Z_\omega = Dz/Dt \tag{4.9}$$

and

$$K = (W^* - Z_t)/Z_\omega = D\omega/Dt. \tag{4.10}$$

It can be seen that when $Z \equiv \omega$ we have

$$Z_\omega = 1, \quad Z_t = 0; \quad Z_{\omega\omega} = Z_{\omega t} = Z_{tt} = 0, \tag{4.11}$$

so

$$W = X_\omega, \quad K = W^*, \tag{4.12}$$

and equation (4.8) reduces to (2.11). On the other hand when $X \equiv \omega$ we find

$$W = 1/Z_\omega, \quad K = W(W^* - Z_t) = H, \tag{4.13}$$

and so (4.8) reduces to (3.10).

In his formulation of the free-surface problem John (1953) defines a *Lagrangian* variable ω. This is equivalent to assuming initially that

$$D\omega/Dt \equiv 0, \tag{4.14}$$

and so

$$K = 0, \quad W^* = Z_t. \tag{4.15}$$

But we shall see that the description of many flows is in the end simpler if we do not make this assumption.

In order that the solution shall represent a permissible flow having a velocity field that is both finite and single-valued everywhere within the fluid, we must have

$$Z_\omega/X_\omega = 1/W \neq 0 \tag{4.16}$$

everywhere within the fluid. This implies, for example, that any zero of Z_ω within the relevant domain must coincide with a corresponding zero of X_ω.

5. Some exact solutions

We mention now some known exact (but non-trivial) solutions to the equations of § 4.

First is the *Dirichlet parabola* (John 1953; Longuet-Higgins 1976) in which

$$Z = \omega - \tfrac{1}{3}R/t^3, \quad X = \tfrac{1}{2}\omega^2/t, \tag{5.1}$$

R being an arbitrary real constant. For, from equations (4.9) and (4.10) we have

$$W = \omega/t, \quad K = \omega^*/t + R/t^4 \tag{5.2}$$

and so from (4.4), if $g \neq 0$,

$$p = \tfrac{1}{4}(\omega-\omega^*)^2/t^2 + \tfrac{1}{2}R(\omega+\omega^*)/t^5 + f, \tag{5.3}$$

while from (4.8) we find

$$\mathrm{D}p/\mathrm{D}t = -4p/t \tag{5.4}$$

provided $f = \frac{1}{4}R^2/t^8$. From (5.3) and (5.4) it is clear that both p and $\mathrm{D}p/\mathrm{D}t$ vanish on the free surface

$$(yt^3)^2 = R(xt^3 + \tfrac{1}{12}R) \tag{5.5}$$

which represents a parabola, expanding or contracting like t^{-3}.

Other self-similar flows, in which Z and X are both polynomials in ω, have been derived by Longuet-Higgins (1976).

Closely related to the flow given by equations (5.1) is the *Dirichlet ellipse* (Lamb 1932; Taylor 1960) corresponding to

$$Z = \omega, \quad X = \tfrac{1}{2}A\omega^2. \tag{5.6}$$

Here the function $A(t)$ is given inversely in terms of t by

$$t = \int_A^1 A^{-2}(1-A^4)^{-\frac{1}{2}}\,\mathrm{d}A, \tag{5.7}$$

and we have also $f \propto A^4$, $g = 0$.

Similarly we have the *Dirichlet hyperbola* in which

$$t = \int_A^\infty A^{-2}(1+A^4)^{-\frac{1}{2}}\,\mathrm{d}A. \tag{5.8}$$

This was discussed by Longuet-Higgins (1972, 1976). Both (5.5) and (5.6) have surfaces which are conics, with axes in *fixed* directions in space. However, some asymmetric and more general (irrotational) flows in which the free surface *rotates* about 0 with a non-zero angular velocity will be described in another paper (Longuet-Higgins 1980*a*).

The above-mentioned solutions are all gravity-free, and so their application is restricted to situations where the fluid is in free fall, or where the motion develops so rapidly that gravity can be neglected. Of great interest therefore is the *Stokes corner flow*, which in our formulation corresponds to

$$Z = \omega^2, \quad X = \tfrac{2}{3}\mathrm{i}g^{\frac{1}{2}}\omega^3. \tag{5.9}$$

The free surface is a pair of lines (or planes) inclined at angles $\pm\frac{1}{3}\pi$ to the vertical.

To see that this is a solution, we have from equations (4.9) and (4.10)

$$W = \mathrm{i}g^{\frac{1}{2}}\omega, \quad K = -\tfrac{1}{2}\mathrm{i}g^{\frac{1}{2}}(\omega^*/\omega), \tag{5.10}$$

and so from (4.4)

$$p = g(\omega^2 - \omega\omega^* + \omega^{*2}). \tag{5.11}$$

On the other hand from equation (4.8)

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{1}{4\mathrm{i}} g^{\frac{3}{2}} \frac{\omega^3 - 2\omega^2\omega^* + 2\omega\omega^{*2} - \omega^{*3}}{\omega\omega^*}. \tag{5.12}$$

Comparing equations (5.6) and (5.7) we see that

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{1}{4\mathrm{i}} g^{\frac{1}{2}} \frac{\omega - \omega^*}{\omega\omega^*} p, \tag{5.13}$$

from which it follows that $\mathrm{D}p/\mathrm{D}t$ vanishes everywhere on the surface $p = 0$, as required. Equation (5.11) shows that the free surface is in fact a degenerate conic consisting of the two straight lines

$$\arg\omega = \pm\tfrac{1}{6}\pi \tag{5.14}$$

through the origin. Since $z = \omega^2$, this implies $\arg z = \pm\frac{1}{3}\pi$.

The above method of solution is at first sight more complicated than Stokes's original method. However, as we shall see later (Longuet-Higgins 1979*b*), by merely adding to X a quadratic function of the variable ω, a simple but accurate solution may be obtained to the hitherto intractable problem of the flow in an overturning wave.

Meanwhile, in the next section of this paper, it will be convenient to derive some compact formulae relating to certain singularities in the pressure field. Not only will these be useful later, but they also illustrate the unexpected simplicity of the present analysis.

6. Singularities in the pressure field

The free surface, being a surface of constant pressure, can have a sharp corner only when the pressure gradient vanishes. Consider then the condition that the pressure gradient vanish while the pressure p remains a differentiable function of x and y, or equivalently of z and z^*.

From equation (2.9) and the general formulae (2.6) for differentiation of a smooth function we have

$$\left.\begin{aligned} -2p_x &= (\chi_{zz}\chi_z^* + \chi_{zz}^*\chi_z) + (\chi_{zt} + \chi_{zt}^*) - 2g, \\ -2p_y &= \mathrm{i}(\chi_{zz}\chi_z^* - \chi_{zz}^*\chi_z) + \mathrm{i}(\chi_{zt} - \chi_{zt}^*). \end{aligned}\right\} \tag{6.1}$$

At a stationary point, where $p_x = p_y = 0$, we then have

$$\left.\begin{aligned} \mathrm{Re}\,(\chi_{zz}\chi_z^* + \chi_{zt} - g) &= 0, \\ \mathrm{Im}\,(\chi_{zz}\chi_z^* + \chi_{zt} - g) &= 0, \end{aligned}\right\} \tag{6.2}$$

In other words the condition for a stationary value of the pressure p is simply

$$\chi_{zz}\chi_z^* + \chi_{zt} = g. \tag{6.3}$$

For a frame of reference in free fall, that is to say if the origin is accelerated downwards with the acceleration of gravity, we may set $g = 0$ in equation (6.3) so that the condition for a stationary value of the pressure reduces to

$$\chi_{zz}\chi_z^* + \chi_{zt} = 0. \tag{6.4}$$

Consider now the form of the contours $p =$ constant in the neighbourhood of a stationary point. Assuming the second derivatives of p to exist we have, to second order,

$$\mathrm{d}^2p = \tfrac{1}{2}(p_{xx}\,\mathrm{d}x^2 + 2p_{xy}\,\mathrm{d}x\,\mathrm{d}y + p_{yy}\,\mathrm{d}y^2). \tag{6.5}$$

We shall be concerned mainly with *saddle points* that is to say points at which two different contours $p =$ constant intersect one another. In that case the angle γ between the two contours is given by

$$\cos^2\gamma = \frac{(p_{xx}+p_{yy})^2}{(p_{xx}-p_{yy})^2 + 4p_{xy}^2}. \tag{6.6}$$

Now from (2.6) we have in general

$$\left.\begin{aligned} p_{xx} &= p_{zz} + 2p_{zz^*} + p_{z^*z^*}, \\ p_{xy} &= \mathrm{i}(p_{zz} - p_{z^*z^*}), \\ p_{yy} &= -p_{zz} + 2p_{zz^*} - p_{z^*z^*}, \end{aligned}\right\} \tag{6.7}$$

so that

$$\left.\begin{aligned} p_{xx} + p_{yy} &= 4p_{zz^*}, \\ p_{xx} - p_{yy} &= 2(p_{zz} + p_{z^*z^*}), \end{aligned}\right\} \tag{6.8}$$

and hence (6.6) becomes simply

$$\cos^2\gamma = \frac{p_{zz^*}^2}{p_{zz}p_{z^*z^*}}, \tag{6.9}$$

or since $p_{z^*z^*} = p_{zz}^*$ this is equivalent to

$$\cos\gamma = \left|\frac{p_{zz^*}}{p_{zz}}\right| \tag{6.10}$$

provided we choose γ so that $-\tfrac{1}{2}\pi \leqslant \gamma \leqslant \tfrac{1}{2}\pi$.

Now from equation (2.9) we see that in terms of the velocity potential χ.

$$\left.\begin{aligned} -2p_{zz} &= \chi_{zzt} + \chi_{zzz}\chi_z^*, \\ -2p_{zz^*} &= \chi_{zz}\chi_{zz}^*. \end{aligned}\right\} \tag{6.11}$$

Therefore on substitution in (6.10) we obtain

$$\cos\gamma = \frac{\chi_{zz}\chi_{zz}^*}{|\chi_{zzz}\chi_z^* + \chi_{zzt}|}. \tag{6.12}$$

The expressions for the mean curvature and the Gaussian curvature of the surface $p = p(x, y)$ are also very simple. In fact from (6.8) we have that

$$M = p_{xx} + p_{yy} = -2\chi_{zz}\chi_{zz}^*, \tag{6.13}$$

showing that *the mean curvature of p is in general negative*. Exceptionally where the *velocity gradient* vanishes, that is when

$$\chi_{zz}^* = \tfrac{1}{2}(u_x + \mathrm{i}v_x) = 0, \tag{6.14}$$

then the mean curvature will also vanish

The *Gaussian curvature*, from (6.8) is given by

$$\Omega = p_{xx}p_{yy} - p_{xy}^2 = 4(p_{zz^*}^2 - p_{zz}p_{zz}^*) \tag{6.15}$$

and from (6.11) this is

$$\Omega = (\chi_{zz}\chi_{zz}^*)^2 - |\chi_{zzz}\chi_z^* + \chi_{zzt}|^2, \tag{6.16}$$

which may be either positive or negative. Thus the pressure field can have both saddle points ($\Omega < 0$) or maxima ($\Omega > 0, M < 0$). On the other hand minima ($\Omega > 0, M > 0$) are ruled out by (6.13).

Lastly, the characteristic condition for a cusp in the free surface is that Ω shall vanish, that is to say

$$\chi_{zz}\chi_{zz}^* = |\chi_{zzz}\chi_z^* + \chi_{zzt}|. \tag{6.17}$$

7. Discussion

We have seen that the equations for time-dependent irrotational flow of a perfect fluid with a free surface become remarkably compact when formulated in terms of the complex variables $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$, instead of x and y respectively. This is because the velocity potential χ is a function of z and t only, independent of z^*, and because the pressure p is a real function of z, z^* and t which is symmetric in z and z^*. The kinematic boundary condition $\mathrm{D}p/\mathrm{D}t = 0$ is also much simpler than expected.

This simplicity is maintained when z and χ are each expressed in terms of a third complex variable ω, which need not be a Lagrangian coordinates, as in the formulation of the problem by F. John.

In all the exact solutions known at present it appears that p and $\mathrm{D}p/\mathrm{D}t$ are related by an equation of the form

$$\mathrm{D}p/\mathrm{D}t = \Phi(\omega, \omega^*, t)\,p, \tag{7.1}$$

where Φ is a real function, symmetric in ω and ω^*. In the Dirichlet conics, which are gravity-free, Φ is independent of ω and ω^*, and is a function of t only. In the Stokes corner flow, on the other hand, Φ is a function of ω and ω^* only, and independent of t, while in the self-similar solutions described by Longuet-Higgins (1976) χ, z and Φ all involve functions of a similarity variable ωt^λ, where λ is a constant.

However, equation (7.1) is not the most general relation which ensures the vanishing of $\mathrm{D}p/\mathrm{D}t$ when $p = 0$. For example, we may also have

$$\mathrm{D}p/\mathrm{D}t = \Phi_1 p + \Phi_2 p^2 + \dots. \tag{7.2}$$

Relations of this kind will be explored in the papers to follow.

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