Proc. R. Soc. Lond. A. **352**, 463–480 (1977)
Printed in Great Britain

The mean forces exerted by waves on floating or submerged bodies with applications to sand bars and wave power machines

BY M. S. LONGUET-HIGGINS, F.R.S.
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge and Institute of Oceanographic Sciences, Wormley, Surrey

(*Received* 2 *June* 1976)

[Plates 1–4]

Water waves transport both energy and momentum, and any solid body which absorbs or reflects wave energy must absorb or reflect horizontal momentum also. Hence the body is subject to a mean horizontal force. In low waves, the force may be calculated immediately when the incident, reflected and transmitted wave amplitudes are known. For wave power devices the mean force can be large, so that anchoring presents practical problems.

Experiments with models of the Cockerell wave-raft and the Salter 'duck' accurately confirm the predicted magnitude of the force at low wave amplitudes. For steeper waves, however, the magnitude of the force can be less than that given by linear theory. By experiments with submerged cylinders, it is shown that this is due partly to the presence of a free second harmonic on the down-wave side.

In breaking waves, it is confirmed that the mean force on submerged bodies is sometimes reduced, and even reversed. An explanation is suggested in terms of the 'wave set-up' produced by breaking waves. Submerged cylinders act as a kind of double beach. A negative mean force arises from an asymmetry in the breaking waves, associated with a time-delay in the response to the change in depth.

Similar arguments apply to submerged reefs and sand bars. Experiments with a model bar show that long low waves propel the bar towards the shore, whereas steep, breaking waves propel it seawards. This is similar to the observed behaviour of off-shore sand bars.

The existence of a horizontal momentum flux (or radiation stress) in water waves is demonstrated by using it to propel a small craft.

1. INTRODUCTION

Economically interesting methods of extracting power from sea waves have recently been proposed by Masuda (1972), Salter (1974), Woolley & Platts (1975) and others. Remarkably high efficiencies have been obtained in the laboratory. The present investigation was prompted by the realization that any device which extracts energy must on general grounds be subject to a mean horizontal force. Not only can this force be large, but it has a special practical significance in that its effect on an anchor cannot be reduced by any flexibility in the mooring cable.

Consider a two-dimensional irrotational wave train of amplitude a travelling with velocity c, in deep water. Owing to the mass transport velocity (see Lamb 1932, ch. 9) the waves have an average horizontal momentum I, which for low waves is simply proportional to the square of the wave amplitude:

$$I = \tfrac{1}{2}\rho g a^2/c, \tag{1.1}$$

where ρ is the density and g is gravity. Hence we expect a horizontal flux of momentum given by Ic_g, where c_g denotes the group-velocity. In deep water $c_g = \frac{1}{2}c$. So we expect a momentum flux

$$Ic_g = \tfrac{1}{4}\rho g a^2 \tag{1.2}$$

per unit distance across the waves. This flux is closely associated with the radiation stress (see Longuet-Higgins & Stewart 1964).

Suppose we have any wave power device acted on by the waves as in figure 1. If it absorbs all the wave energy then it must absorb the momentum also. Hence we expect that it will be subject to a mean horizontal force

$$F = \tfrac{1}{4}\rho g a^2 \tag{1.3}$$

per unit distance across the waves. If all the wave energy is reflected, then the momentum is all reversed, and the resulting force is just doubled. In general, we expect that the body will be subject to a force

$$F = (Ic_g)_{\text{in}} + (Ic_g)_{\text{ref}} - (Ic_g)_{\text{trans}} \tag{1.4}$$

where the terms on the right represent the momentum fluxes in the incident, reflected and transmitted waves respectively. In deep water this becomes

$$F = \tfrac{1}{4}\rho g(a^2 + a'^2 - b^2), \tag{1.5}$$

where a, a' and b are the respective wave amplitudes.

The maximum value of this expression $\frac{1}{2}\rho g a^2$ is equal to the horizontal stress acting on a dam, erected across a reservoir of depth a equal to the wave amplitude. If a is measured in metres, this force is $\frac{1}{2}a^2$ t/m (tonnes force per metre) measured along the dam. Thus waves of amplitude $a = 10$ m correspond to a maximum force of 50 t/m.

In water of finite depth h, the ratio c_g/c is more generally equal to $(\frac{1}{2} + kh/\sinh 2kh)$ where k is the wavenumber. This leads one to expect that in general

$$F = \tfrac{1}{4}\rho g(a^2 + a'^2 - b^2)\,(1 + 2kh/\sinh 2kh) \tag{1.6}$$

the last factor tending to unity as the depth tends to infinity.

In §2 of this paper we shall first establish equation (1.6) theoretically, under certain conditions. Thus the wave amplitude must be sufficiently small for the bilinear theory to apply, which excludes breaking waves, for example. Nevertheless under appropriate conditions equation (1.6) can be generalized so as to evaluate the total force on any number of floating or submerged bodies, and to the situation where the undisturbed depths of water on the up-wave and down-wave sides are unequal. Hence we can consider applying the result to submerged bodies and submarine reefs.

In § 3 we shall describe experiments which verify equation (1.6) experimentally for a Cockerell wave raft, and for a Salter 'duck' in waves of moderate amplitude.

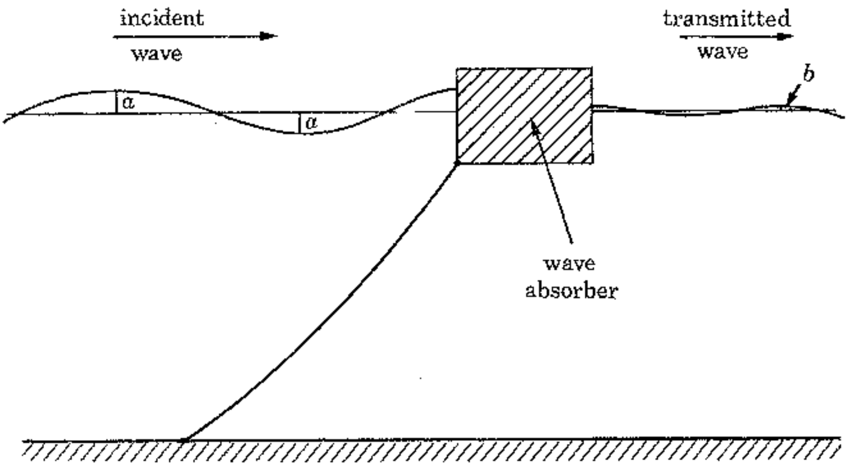


FIGURE 1. Schematic representation of a wave absorber situated in a train of waves.

For totally submerged bodies, however, it is found experimentally that the mean force can be less than expected, and in § 5 we show theoretically and experimentally that this is due partly to the presence of a second harmonic in the transmitted wave.

In breaking waves, Salter has found an even more drastic reduction, and even a reversal, of the mean force. This is discussed in § 6, and a qualitative explanation is put forward in terms of the wave set-up. In § 7 it is verified experimentally that a similar reversal can occur on submerged sand bars, long low waves driving the bars shorewards, but shorter, breaking waves driving them seawards.

Finally we discuss briefly the possibility of using wave momentum to propel a small craft. This is demonstrated by means of a model.

2. The bilinear theory

We shall first establish theoretically the results stated in the Introduction. The arguments, which are simple, depend solely on the conservation of the mean momentum.

Suppose that waves of low amplitude a approach from the left in water of undisturbed depth h. They are incident upon any number of floating or submerged bodies (which may be absorbing or generating wave energy at the same frequency) confined to a finite horizontal range. For simplicity, the mean depth on the right is assumed to be the same as that on the left, in the first place. The amplitudes of the reflected and transmitted waves are denoted by a' and b respectively, and we allow for a small, second-order displacement $\bar{\zeta}$ of the mean surface level on the left, and $(\bar{\zeta}+\Delta\bar{\zeta})$ on the right, due to the waves.

Horizontal and vertical coordinates are denoted by x and z, with x in the direction of the incident wave and z measured vertically upwards from the undisturbed mean water level. The horizontal and vertical components of the particle velocity are denoted by u and w.

A general expression for the flux of horizontal momentum across a vertical plane x = constant is

$$\int_{-h}^{\zeta} (p+\rho u^2)\,\mathrm{d}z,$$

where p is the pressure, ρ the density and $\zeta(x, t)$ the local elevation of the free surface. Subtracting the corresponding flux in the *absence* of the waves (which arises solely from the hydrostatic pressure $p_0 = -\rho g z$, and taking averages with respect to the time, we obtain the excess flux of momentum due to the waves as

$$S = \overline{\int_{-h}^{\zeta} (p+\rho u^2)\,\mathrm{d}z} - \int_{-h}^{0} p_0\,\mathrm{d}z. \tag{2.1}$$

In the case $a' = 0$, $\bar{\zeta} = 0$, this is just the radiation stress, which for waves of small amplitude has been evaluated by Longuet-Higgins & Stewart (1960, 1964). Generalizing their argument we note first that (2.1) may be written

$$S = \overline{\int_{0}^{\zeta} p\,\mathrm{d}z} + \int_{-h}^{0} \overline{(p-p_0+\rho u^2)}\,\mathrm{d}z \tag{2.2}$$

correct to second order. Now by conservation of *vertical* momentum of the fluid contained between (1) the free surface $z = \zeta$, (2) the horizontal plane z = constant and (3) any two vertical planes one wavelength apart, we have

$$\overline{p+\rho w^2} - \rho g(\bar{\zeta}-z) = p_0 + \rho g z,$$

where an overbar now denotes the double average over both a period and a wavelength. Therefore

$$\bar{p} - p_0 = \rho g \bar{\zeta} - \rho \overline{w^2} \tag{2.3}$$

and on substituting in (2.2) and taking averages over a wavelength we obtain

$$\bar{S} = \overline{\int_{0}^{\zeta} p\,\mathrm{d}z} + \int_{-h}^{0} \rho\overline{(u^2-w^2)}\,\mathrm{d}z + \rho g h \bar{\zeta}. \tag{2.4}$$

In the above integrals we may substitute the well-known first-order expressions for the pressure p and the orbital velocities u, w in Stokes waves of arbitrary depth $h \doteqdot h+\bar{\zeta}$, namely

$$u = \phi_x, \qquad w = \phi_z, \qquad p/\rho = \phi_t - gz, \qquad \zeta = g^{-1}(\phi_t)_{z=0},$$

where $$\phi = [a\cos(kx-\sigma t) + a'\cos(kx+\sigma t)]\frac{\sigma\cosh k(z+h)}{k\sinh kh}$$

and $$\sigma^2 = gk\tanh kh$$

(see, for example, Lamb 1932, ch. 9). On taking averages with respect to x and t, all terms proportional to the product aa' vanish, and we obtain

$$\overline{S} = \tfrac{1}{4}\rho g(a^2+a'^2)(1+4kh/\sinh 2kh)+\rho gh\overline{\zeta} \tag{2.5}$$

correct to second order. So the momentum fluxes in the incident and reflected waves are simply added. The last term $\rho gh\overline{\zeta}$ can be considered as the effect of an additional hydrostatic pressure $\rho g\overline{\zeta}$ exerted throughout the whole depth h.

Consider now the horizontal momentum of the fluid contained between two fixed vertical planes $x = x_1, x_2$, one far to the left and the other far to the right. If F denotes the sum of the mean horizontal forces exerted by the fluid on all the solid bodies contained between these two planes, then the flux of horizontal momentum from the bodies to the water is just $-F$. Assuming the mean horizontal momentum of the water to be conserved, we must therefore have

$$F = S_1 - S_2,$$

where S_1 and S_2 denote the fluxes of momentum across the two planes. In this equation we can take averages with respect to x_1 and x_2, each over one wavelength, with the result

$$F = \tfrac{1}{4}\rho g(a^2+a'^2-b^2)(1+4kh/\sinh 2kh)-\rho gh\Delta\overline{\zeta} \tag{2.6}$$

from (2.4).

To complete the calculation we must now evaluate the difference $\Delta\overline{\zeta}$ in the mean level on the two sides. To do this we introduce the further assumption (see Longuet-Higgins 1967) that the motion is irrotational to second order. Then we may use the Bernoulli integral

$$p/\rho + \tfrac{1}{2}(u^2+w^2)+gz+\partial\phi/\partial t = f(t)$$

and on taking both time and space averages we obtain

$$\overline{p}/\rho + \tfrac{1}{2}\overline{(u^2+w^2)}+gz = C(t),$$

where C is independent of both x and z. Combining this with equation (2.3) and writing $z = 0$, $p_0 = 0$ we have

$$g\overline{\zeta} = -\tfrac{1}{2}\overline{(u^2-w^2)}_{z=0}+C/\rho$$

and so $$g\Delta\overline{\zeta} = -\tfrac{1}{2}\overline{(u^2-w^2)}_{z=0}\Big|_{x=-\infty}^{x=\infty}. \tag{2.7}$$

Now substituting the first-order expression for the orbital velocities u and w we find

$$g\Delta\overline{\zeta} = \tfrac{1}{4}g(a^2+a'^2-b^2)\frac{2k}{\sinh 2kh}. \tag{2.8}$$

This combined with equation (2.5) gives us

$$F = \tfrac{1}{4}\rho g(a^2 + a'^2 - b^2)(1 + 2kh/\sinh 2kh), \tag{2.9}$$

the result to be proved.

When the depths of water on the two sides are unequal it is easily seen that the same arguments lead to

$$F = \tfrac{1}{4}\rho g(a^2 + a'^2)(1 + 2kh_1/\sinh 2kh_1) - \tfrac{1}{4}\rho g b^2(1 + 2kh_2/\sinh 2kh_2),$$

where h_1 and h_2 denote the mean depths on the two sides. That is to say

$$F = (Ic_g)_{\text{in}} + (Ic_g)_{\text{ref}} - (Ic_g)_{\text{trans}} \tag{2.10}$$

as expected.

We note that the assumption of a steady mean surface level $(\bar{\zeta} + \Delta\bar{\zeta})$ on the down-wave side may be appropriate only when there is a beach or other barrier to restrict the mean flow on the down-wave or the up-wave side. Otherwise, if the waves were started from rest, it would be difficult to achieve a steady state in a limited time.

The assumption that the flow is irrotational to second order also implies that the waves are not breaking.

3. Experimental verification

A Cockerell wave raft, consisting of six hinged floats each 12 in long × 23.5 in wide (see figures 8 and 9, plate 1) was placed in a wave tank of length 40 ft and width $W = 2$ ft. Periodic waves were generated by a plunger at one end of the tank, and absorbed by a sloping beach at the other. Power was extracted by a simple arrangement of pumps, generally two at each hinge, raising water to a height 1.6 m above the mean water level. The mean force on the float was measured with a spring balance.

Figure 2 shows a typical set of results for waves of period 1.0 s in water of depth $h = 0.36$ m. The measured force is plotted against the expression

$$WF = \tfrac{1}{4}\rho g W(a^2 + a'^2 - b^2)$$

at various wave amplitudes a. The broken line in figure 2 represents the force that would be exerted in deep water. The full line represents the theoretical force (1.6) after adjustment by the factor for finite depth, and it can be seen that the agreement is close. At higher values of the wave amplitude the accuracy of the measurements was reduced by a long-period seiche (about 10 s) which was set up in the tank and affected both the wave amplitude and the forces on the raft.

Further details of the experiments are given in table 1.

Salter, Jeffrey & Taylor (1976) have measured the mean forces on a nodding 'duck', which absorbs a high proportion of the incident wave energy. In figure 4 we have plotted their measured values against the theoretical value $\tfrac{1}{4}\rho g W a^2$ for low waves in deep water. Although there is greater scatter than in our measurements the

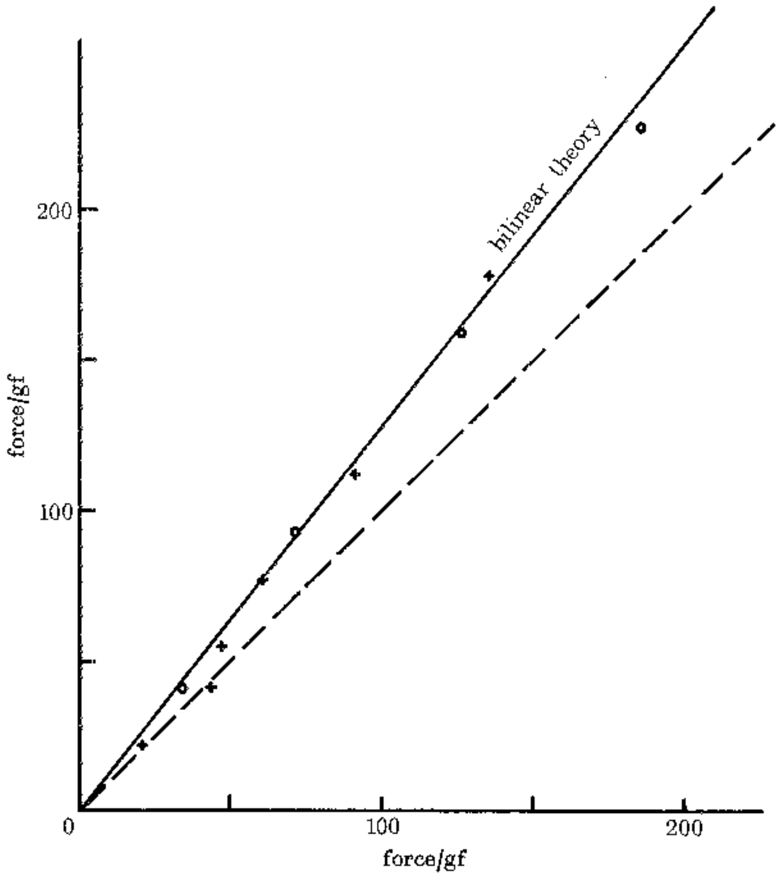


FIGURE 2. Mean horizontal forces on wave raft. Wave period $T = 1.0$ s; mean depth $h = 0.36$ m.

TABLE 1. PARAMETERS FOR THE DATA OF FIGURE 2

run	period/s	a/cm	$(a'/a)^2$	$(b/a)^2$	efficiency	loss
A1	1.00	1.71	.11	.34	.10	.44
A2	1.02	2.36	.07	.24	.14	.55
A3	1.04	3.03	.07	.17	.12	.64
A4	1.05	3.65	.07	.16	.15	.62
B1	1.01	1.31	.11	.32	.13	.44
B2	1.01	1.88	.11	.28	.16	.45
B3	1.02	1.88	.13	.25	.20	.42
B4	1.03	2.18	.05	.21	.19	.55
B5	1.04	2.68	.06	.22	.18	.54
B6	1.05	3.15	.08	.18	.17	.57

observed values in figure 4 are in fair agreement at low wave amplitudes. The points on the right of the figure are for breaking waves. It is not surprising that the agreement is less good. Nevertheless the reduction in force is interesting, and reasons for it will be discussed below.

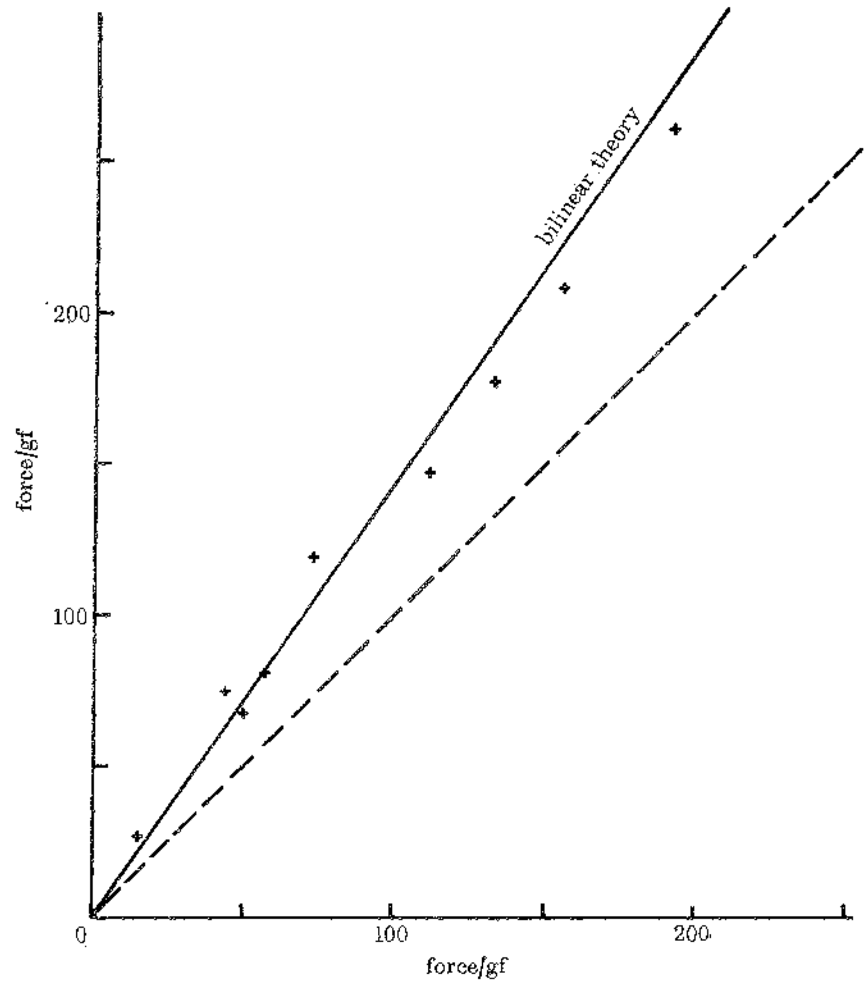


FIGURE 3. Mean horizontal forces on wave raft. $T = 1.0$ s, $h = 0.25$ m.

4. EXPERIMENTS WITH A SUBMERGED CIRCULAR CYLINDER

Salter *et al.* (1976) also measured the forces on a circular cylinder, held with axis horizontal so as to be completely submerged in still water. For non-breaking waves the mean horizontal force was found to be quite small, as would be expected from equation (1.5) since a submerged circular cylinder has in fact the remarkable property that its transmission coefficient is unity and its reflexion coefficient is zero, according to linearized non-viscous theory (Dean 1948; Ursell 1950; Ogilvie 1963). Thus in equation (1.5) we should have $a' = 0$, $b = a$.

At higher wave amplitudes, however, the horizontal force was observed to change sign, i.e. the mean force was found to be *towards* the wavemaker. How is this to be explained?

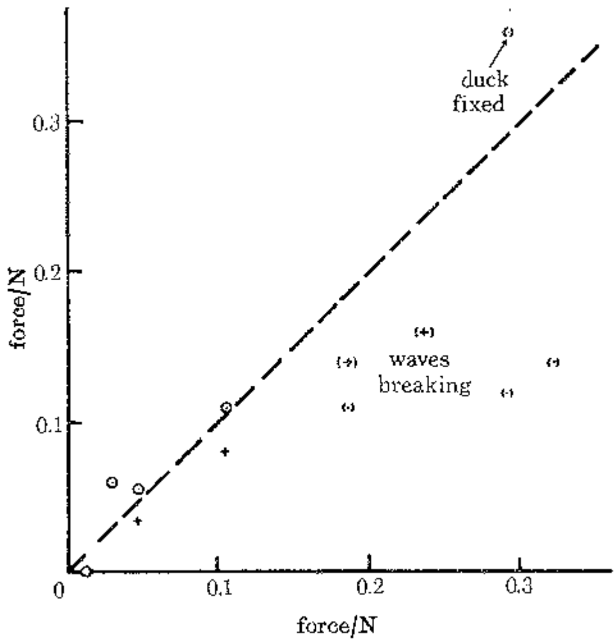


FIGURE 4. Mean horizontal forces acting on a Salter 'duck'.

The present author carried out a somewhat similar experiment in which a submerged cylinder of diameter 15 cm was suspended below the surface by a vertical arm, free to swing about a pivot above the surface (see figures 10 and 11, plate 2). In this way the cylinder was constrained vertically but was free to make small oscillations in a horizontal direction. Being flooded internally, the mean density only slightly exceeded that of the water, and the period of free oscillation (about 10 s) was long compared to the wave period.

Now the amplitude of the waves reflected from a submerged, neutrally buoyant cylinder, constrained vertically but free horizontally, may be shown (see the appendix) to be given by

$$a'/a = \sin(\psi_1 - \psi_2), \tag{4.1}$$

where ψ_1 is the phase-lag of the force on a *fixed* cylinder (relative to the force on a fluid particle on the axis in the absence of the cylinder), and ψ_2 is the phase-lag of the displacement of a *completely free* cylinder (relative to the displacement of a particle on the axis, in the absence of the cylinder). The amplitude of the transmitted wave is then

$$b'/a = \cos(\psi_1 - \psi_2). \tag{4.2}$$

The angles ψ_1 and ψ_2 have been computed by Ogilvie (1963), and with the parameters of the experiment it is found that a'/a is fairly small, lying between 0.25 and 0.35. For incident waves of moderate amplitude we therefore expect a small mean force on the cylinder, directed down-wave.

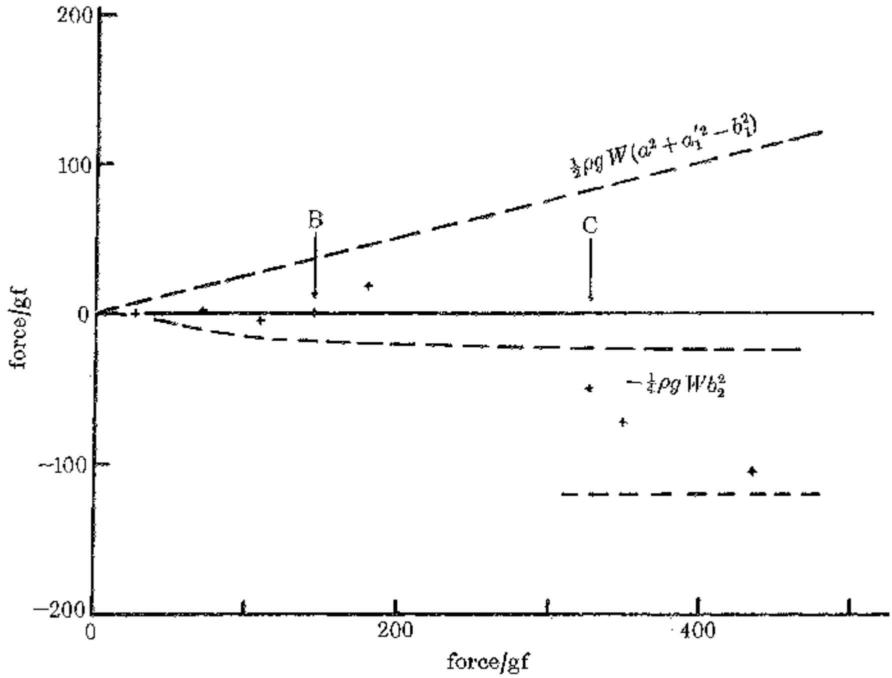


FIGURE 5. Mean horizontal forces on a submerged circular cylinder. Wave period 1.0 s, mean depth 0.52 m: depth of immersion 5.0 cm.

Figure 5 shows a typical set of measurements when the depth d of immersion (of the uppermost part of the cylinder below the still-water level) was 5.0 cm. The horizontal axis corresponds to the value of $\frac{1}{4}\rho g W a^2$, and is a measure of the incident momentum flux. The broken line represents the measured value of $\frac{1}{4}\rho g W(a^2 + a_1'^2 - b_1^2)$, where a, a_1' and b_1 are the measured amplitudes of the incident, reflected and transmitted fundamental frequencies. The ratios a_1'/a and b_1/a were found to be in fair agreement with equations (4.1) and (4.2). The measured mean forces are shown in figure 5 by crosses. Up to point B, where the waves were first observed to be breaking, the force was smaller than expected, though generally positive. After point C it had definitely reversed sign. Under these conditions the cylinder tended to be deflected strongly towards the wave-maker (see figure 11, plate 2). The effect was obviously similar to the one reported by Salter. We consider now some possible explanations.

5. The effect of higher harmonics

Waves in the presence of submerged bodies tend to behave quite non-linearly, and with a submerged cylinder it is easy to detect visually the presence of an appreciable second harmonic (twice the fundamental frequency) in the transmitted waves. The probable reason for this is that the wave amplitude above the cylinder quickly grows to a significant fraction of the local depth d. Also, the horizontal fluid velocity is of order $(gd)^{\frac{1}{2}}$ or greater. Both these facts imply strong nonlinearity, and the production of higher harmonics. This is even without the occurrence of wave breaking.

Consider the effect of a second harmonic in the transmitted wave. Since the second harmonics have a frequency double that of the first harmonics, their group velocity in deep water is only $\frac{1}{2}c_g$. So their ratio of momentum flux to energy flux is doubled. Denoting the first and second harmonics in the reflected and transmitted waves by $a_1', a_2'; b_1', b_2'$ respectively, we have by conservation of energy

$$a^2 = (a_1'^2 + b_1^2) + \tfrac{1}{2}(a_2'^2 + b_2^2) + \epsilon, \tag{5.1}$$

where ϵ is a positive term representing the dissipation or extraction of energy. But by conservation of momentum

$$F = \tfrac{1}{4}\rho g[(a^2 + a_1'^2 - b_1^2) + (a_2'^2 - b_2^2)]. \tag{5.2}$$

If the reflected waves are small we may ignore $a_1'^2$ and $a_2'^2$ compared to the other terms, and on substituting for a^2 we have

$$F = \tfrac{1}{4}\rho g(\epsilon - \tfrac{1}{2}b_2^2). \tag{5.3}$$

Thus the sign of the force depends on a balance between the dissipation term and the amplitude of the transmitted second harmonic. When the latter is larger, the radiation stress in the second harmonic reverses the sign of F.

To measure the second harmonic b_2 in the transmitted wave, the waves were abruptly shut off by lowering a gate into the water down-wave from the cylinder. The waves continued to be recorded at a fixed distance down-wave of the gate. The rear of the fundamental wave-train, of amplitude b_1, passed first, with group velocity c_g; then the rear of the second harmonic, travelling with velocity $\frac{1}{2}c_g$. Between the two times of arrival, the amplitude b_2 of the second harmonic could be measured.

The two upper curves in figure 5 show the measured values of $\frac{1}{4}\rho g W(a^2 + a_1'^2 - b_1^2)$ and $-\frac{1}{4}\rho g W b_2^2$. These represent the observed mean forces associated with the fundamental wave and with the transmitted second harmonic, respectively. The former, though not accurately measured, is necessarily positive. The latter is negative, and is of the same order as the measured force, but is limited in magnitude. (The lowest broken line corresponds to the force that would be exerted by a second harmonic of limiting steepness, in otherwise still water.)

We may conclude that the second harmonic contributes an appreciable part, but not all, of the observed negative force.

The explanation for the remainder of the force may lie partly in the existence of harmonics higher than the second, which are effectively damped before reaching the recording point (2.5 m from the cylinder). However, in breaking-wave conditions we must in all probability go beyond the range of small-amplitude, irrotational theory, as follows.

6. Breaking waves

We suggest an explanation for the negative forces in breaking waves by analogy with the situation when waves approach a simple beach. The waves cause a change in the local mean water level $\bar{\zeta}$, called the wave 'set-up', which was studied experimentally by Saville (1961) and explained quantitatively by Longuet-Higgins & Stewart (1963, 1964). On entering shallow water the wave amplitude, after an initial decrease, begins to increase sharply. This produces an increase in the radiation stress (the momentum flux due to the waves) which has to be offset by a *decrease* in the hydrostatic pressure. The mean level therefore falls, and there is a wave 'set-down'. The set-down increases almost till the breaking point, when the waves begin to lose height and the radiation stress diminishes. The static pressure must now *in*crease, and there is a dramatic rise in mean level, producing the much larger wave 'set-up'. Assuming that the breaker height is proportional to the local depth of water then it can be shown that the surface tilt is just proportional to the local slope s of the bottom ($\partial\bar{\zeta}/\partial s \doteqdot 0.2s$). This result has been rather accurately confirmed by Bowen, Inman & Simmons (1968).

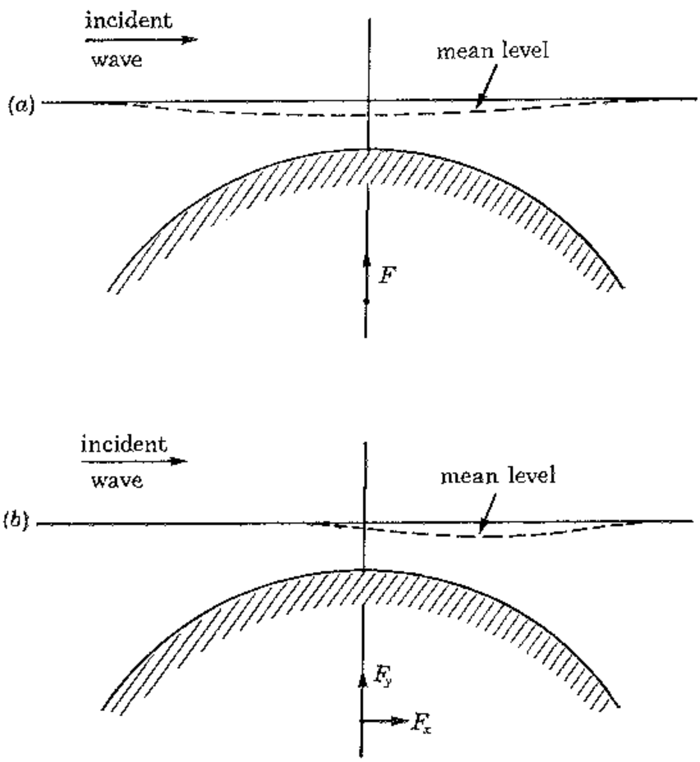


Figure 6. Schematic picture of the changes in mean sea level of waves in the presence of a submerged cylinder, if the waves are not breaking: (*a*) symmetrical; (*b*) unsymmetrical.

We may think of the set-up as being due to the waves shooting their horizontal momentum horizontally at the beach. The resulting pile-up is not statically maintained, but is balanced by the momentum flux term $(\rho u) u$, where u is the horizontal velocity of the particles.

Now we can think of a submerged circular cylinder as two beaches, back-to-back (see figures 6 and 7). Suppose first that there is no breaking (figure 6). Then there will be

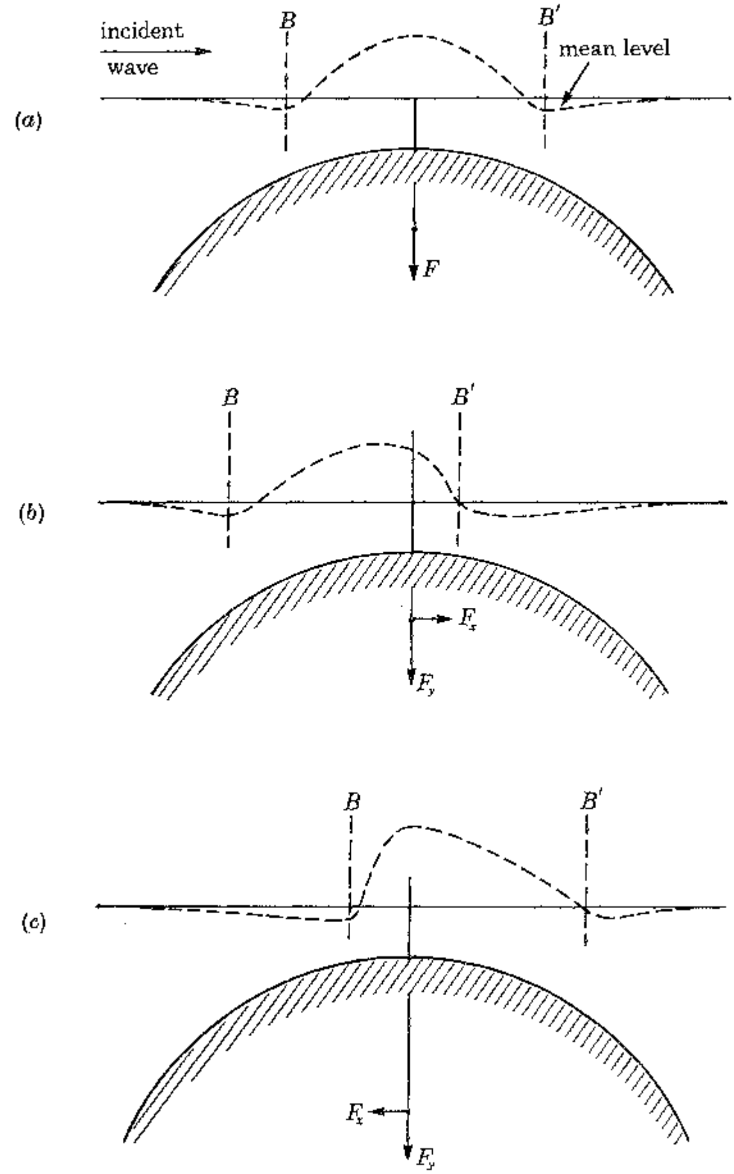


FIGURE 7. Schematic picture of the changes in mean level of waves in the presence of a submerged cylinder (*a*) symmetrical, (*b*) unsymmetrical, wavelength large compared to that of curvature (*c*) unsymmetrical, wavelength not large compared to radius of curvature.

a small wave set-down, but no set-up. If the set-down is symmetrical (figure 6*a*) there will be no mean horizontal force on the cylinder, but a small vertical force, directed upwards. This is in fact the situation for waves of *small* amplitude impinging on a submerged circular cylinder, either fixed or neutrally buoyant and free to move (see Ogilvie 1963). According to the linearized theory there is no reflexion. Hence the wave velocities may be all simply reversed in time, the incident and reflected waves having their rôles exchanged. The mean level must therefore be symmetric about the mid-point. For larger waves, however, both non-linearity and viscosity may make the mean level unsymmetrical, as in figure 6*b*. Then there can be a small horizontal force as well as a mean vertical force.

Suppose now that the waves are breaking as in figure 7. The points where the waves begin and end their breaking are shown by B and B' respectively. Outside these limits, approximately, there is a wave set-down. But inside, there is a much larger wave set-up, to balance the loss of horizontal momentum flux. If the set-up is symmetrical, as in figure 7*a*, there will be no mean horizontal force, but simply a large downwards force on the cylinder. If the set-up is unsymmetrical as in figure 7*b*, there will be a net horizontal force to the right. This is the situation we might expect if the wavelength is short compared to the diameter of the cylinder. For then the change in depth above the cylinder will be relatively slow and the breaker-height will have time to adjust to the local depth of water above the cylinder. To the right of the mid-point, when the depth begins to increase, breaking will soon cease, because the waves will no longer be forced to try to become steeper.

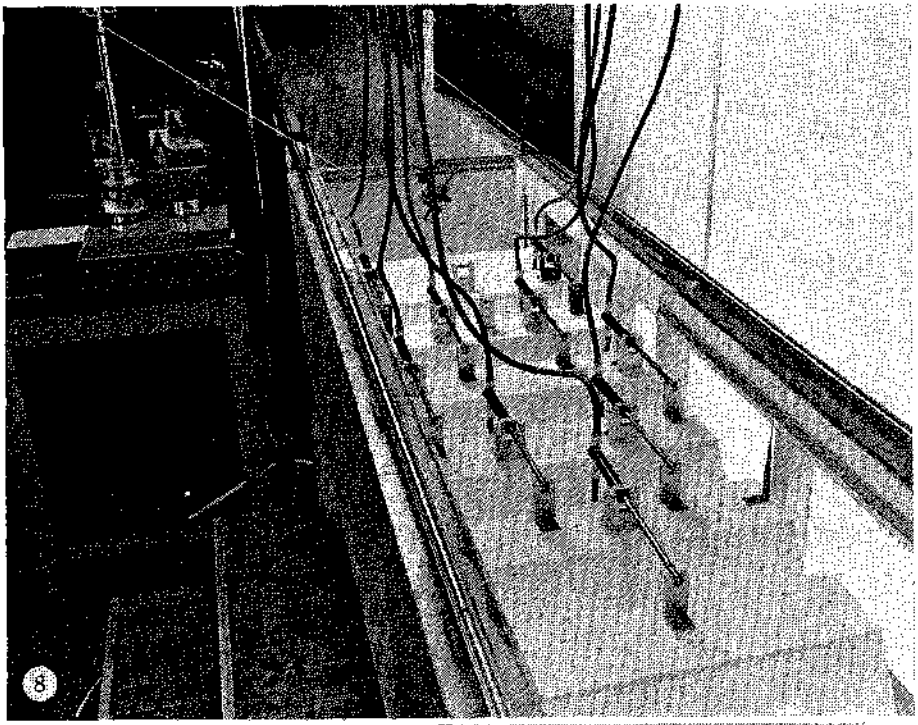
In the present experiments, however, the wavelength is not small compared to the diameter of the cylinder. The waves are forced to break, from their point of view, without much warning, and there is a delay in the onset of breaking until near the point of minimum depth. Moreover, breaking continues until some time after the depth begins to increase again. Hence the wave set-up is unsymmetrical as in figure 7*c*, with most of the set-up occurring on the right. This produces a net force to the left, as shown.

7. Experiments with submerged bars

If our reasoning is correct, a similar reversal of the mean force is to be expected when breaking waves impinge on a sand-bar or on any other submerged body resting on the bottom.

The author carried out exploratory tests with an artificial sand-bar mounted on wheels (see, figures 12 and 13, plate 3) which was free to move horizontally in either direction. When subjected to long, low waves (period $T = 1.15$ s, amplitude $a = 1.0$ cm) from the left (see plate 5) the mean force on the bar was positive. Thus the forces corresponding to the reflected wave predominated. If left entirely free, the bar tended to move towards the beach, with a mean speed of 0.95 cm/s.

When on the other hand the bar was subjected to short, steep waves ($T = 0.75$ s, $a = 4.0$ cm) the waves broke on the far side of the bar (see figure 13) and the mean



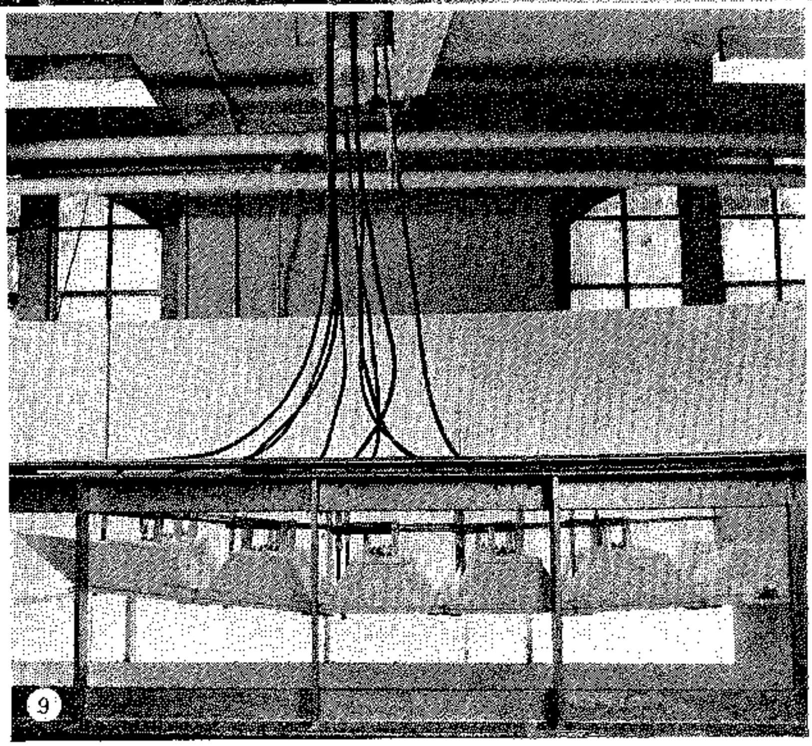
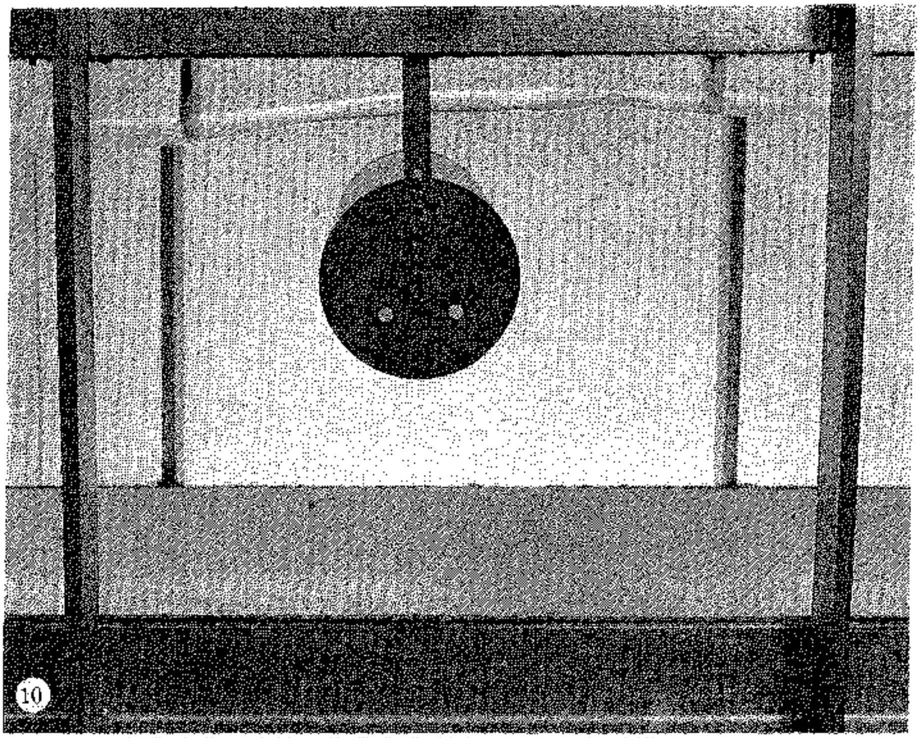


FIGURE 8. The wave raft in position, facing incident waves. On the left is the spring balance for measuring the mean horizontal force. Width of tank = 2 ft.

FIGURE 9. The raft under the action of waves: $T = 1.0$ s, $a = 2.0$ cm.

(*Facing p.* 476)



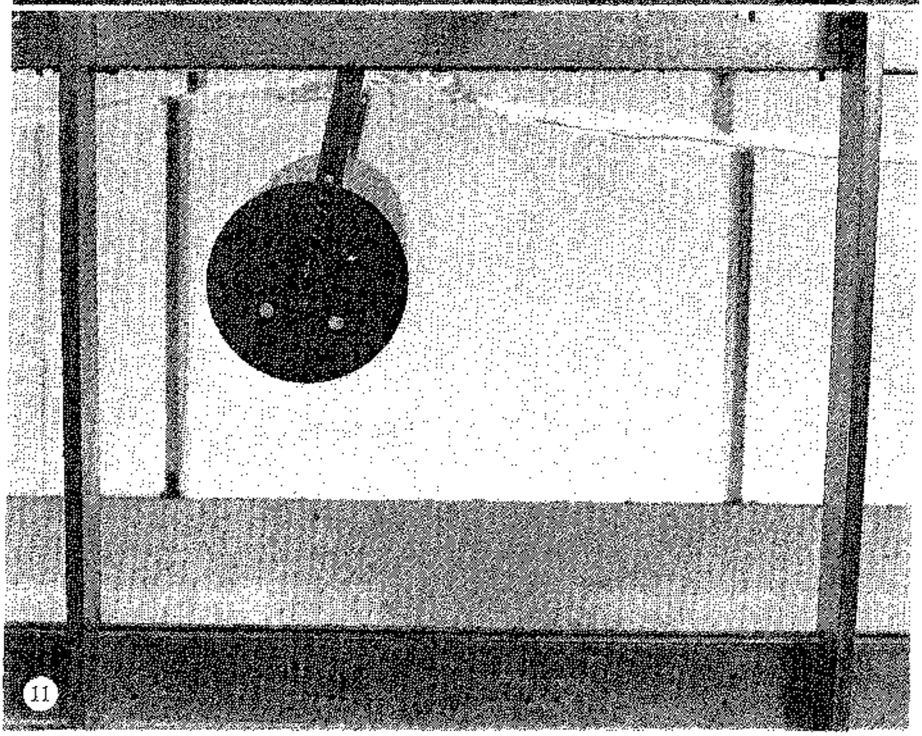
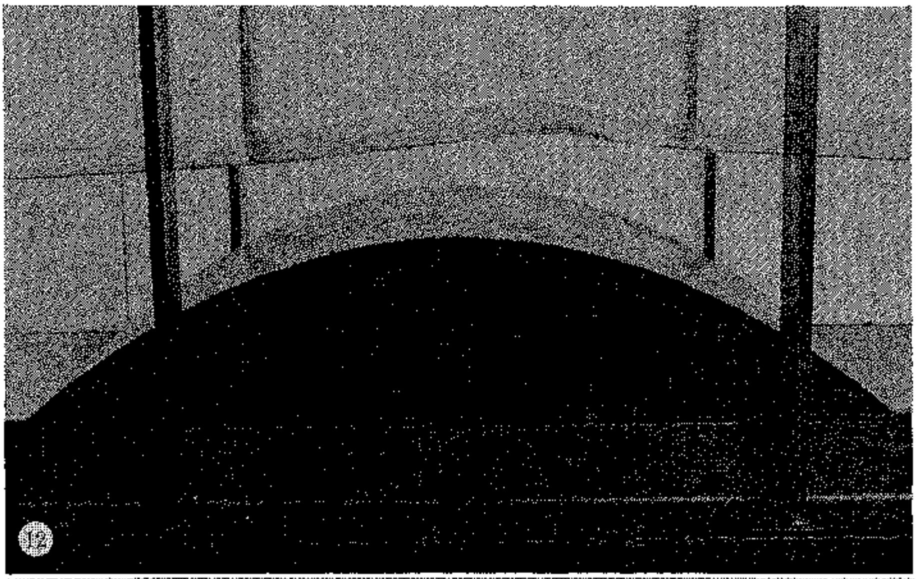


FIGURE 10. Submerged cylinder in low waves incident from the left: d = 3.5 cm, T = 1.0 s, a = 1.5 cm.

FIGURE 11. Submerged cylinder in breaking waves incident from the left: d = 3.5 cm, T = 1.0 s, a = 4.0 cm.



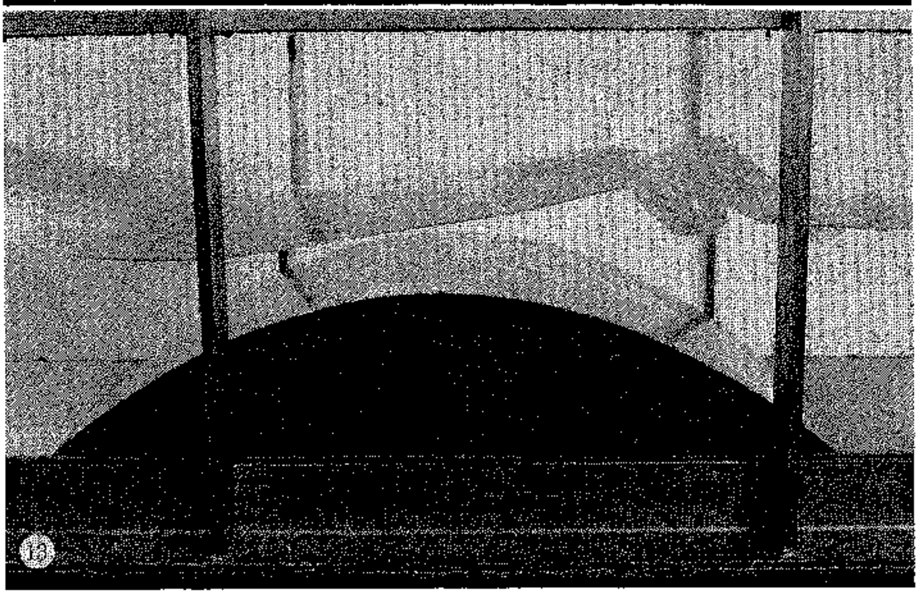
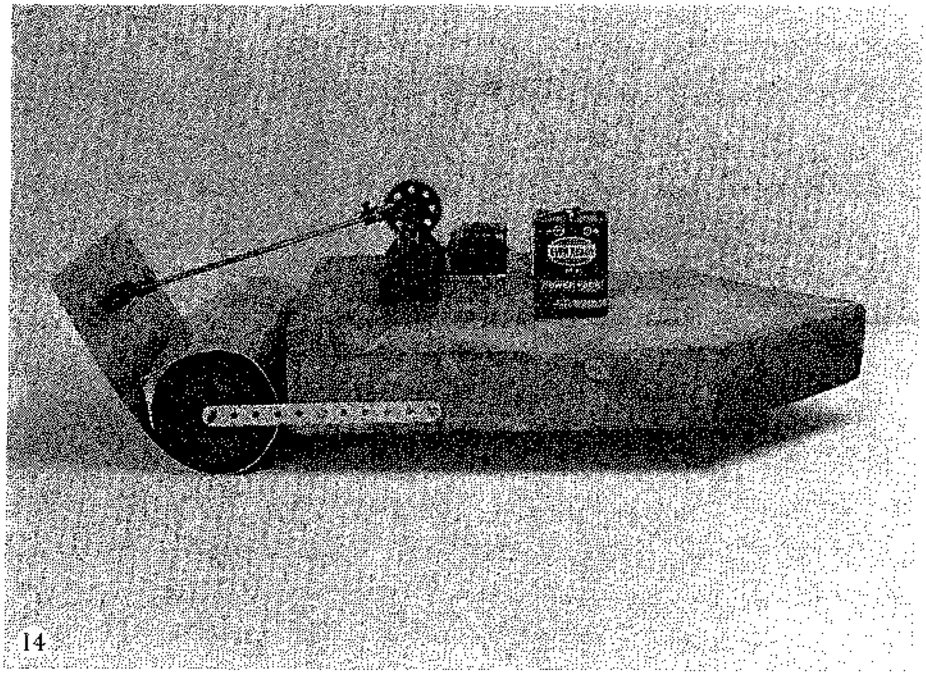
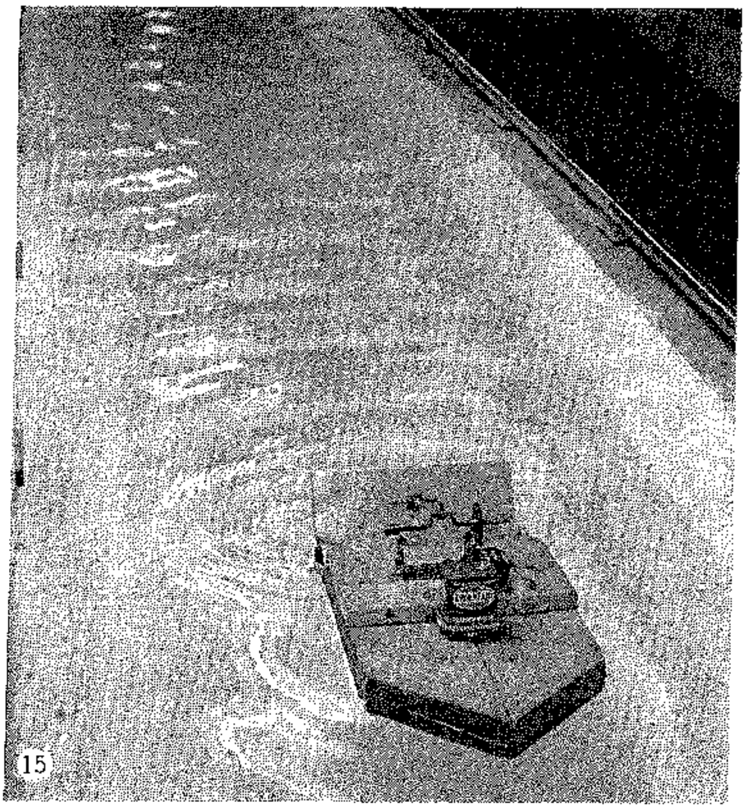


FIGURE 12. Artificial sand-bar in low waves incident from the left. $d = 7.0$ cm, $T = 1.15$ s, $a = 1.0$ cm. Mean motion of bar = 0.95 cm/s to the right.

FIGURE 13. Artificial sand-bar in steep waves incident from the left. $d = 7.0$ cm, $T = 0.75$ s, $a = 4.0$ cm. Mean motion of bar ≈ 1.2 cm/s to the left.



14



15

FIGURE 14. Model boat (side view) with drive and Salter cam at stern.

FIGURE 15. Boat propelled by wave momentum flux. Forward speed 12 cm/s.

force was reversed. If left to itself the bar now tended to move in the reverse direction, with a mean speed -1.2 cm/s.

This behaviour is qualitatively similar to the well-known behaviour of offshore sand-bars; long, low waves tend to move them beachwards, but short, steep waves tend to move them seawards (Sheppard & LaFond 1940).

It is generally supposed that most of the sediment transport due to waves is either 'bed-load', taking place in a thin mobile layer close to the bottom, or else 'suspended load', i.e. carried at a higher level. However, preliminary observations by the author have suggested that it is also possible for the motion of the sand to penetrate much deeper, presumably in response to the horizontal pressure gradients in the waves above. In that case shear takes place at a lower level, and the mass above moves more nearly as a solid body.

It seems desirable to determine the importance of this effect by further experiments.

8. Use of the radiation stress for propulsion

The momentum flux in waves will necessarily produce a mean reaction on any wave-maker. Thus the radiation stress may be actually set up to use for propelling a small craft (see, figure 14, plate 4). In this model, a Salter cam is attached to the stern and is made to oscillate by attachment to a crank driven by a small electric motor. At wave-maker frequencies of 3 s^{-1} the boat is propelled along at speeds of 10–15 cm/s. The ratio of thrust to power expended on the waves is quite advantageous. For we have

$$I = E/c,$$

where E denotes the energy density $\frac{1}{2}\rho g a^2$ and c is the phase velocity. Hence the ratio of the thrust to the power expended is given by

$$F/Ec_g = Ic_g/Ec_g = 1/c.$$

This is larger than for some conventional propellors. The total thrust, at given frequency is however limited by the maximum steepness of the waves, and the need to avoid cross-waves, which only generate wave momentum in a transverse direction.

By designing a wave-power device in conjunction with a wave-maker on the down-wave side which generates waves at a higher frequency, it should be possible, on the basis of equation (5.3) to design a wave-powered craft which can advance against the waves. However, it is necessary for both the reflected wave amplitudes and the energy dissipated to be sufficiently small. In practice this requirement is quite stringent.

Appendix. Proof of equations (4.1) and (4.2)

Consider a submerged circular cylinder of radius R, either fixed or making small oscillations, with its axis horizontal and at a mean distance $(R+d)$ below the free surface. Let (x, y) be horizontal and vertical coordinates, with $x = 0$ as the plane of symmetry, and y vertically upwards, and let (ξ, η) denote the instantaneous displacements of the axis from its mean position. We consider two-dimensional motions, with waves approaching from, or diverging towards, $x = \infty$.

We know the following:

Theorem A. When the cylinder is fixed, then the coefficient of reflexion vanishes (Dean 1948; Ursell 1950).

Theorem B. When the cylinder is free and neutrally buoyant, then the coefficient of reflexion also vanishes (Ogilvie 1963).

Theorem C. When the centre describes a small circle, it generates or absorbs waves travelling only in the direction of the motion of the cylinder at the top of its orbit (Ogilvie 1963).

In general, let ζ_- and ζ_+ denote the free surface displacements as $x \to -\infty$ and $+\infty$ respectively. Consider the following situations.

(1) The cylinder generates waves by making small vertical oscillations:

$$\xi = 0, \quad \eta = \mathrm{i}s\mathrm{e}^{-\mathrm{i}\sigma t},$$

where s and σ are constants and t is the time. By symmetry about the plane $x = 0$ we have

$$\left.\begin{aligned} \zeta_+ &= a\mathrm{e}^{\mathrm{i}(kx-\sigma t+\alpha)}, \\ \zeta_- &= -a\mathrm{e}^{\mathrm{i}(-kx-\sigma t+\alpha)}, \end{aligned}\right\}$$

where a and α are amplitude and phase angles.

(2) The cylinder generates waves by making small horizontal oscillations:

$$\xi = s\mathrm{e}^{-\mathrm{i}\sigma t}, \quad \eta = 0.$$

Because the motion is antisymmetric about $x = 0$,

$$\left.\begin{aligned} \zeta_+ &= \bar{a}\mathrm{e}^{\mathrm{i}(kx-\sigma t+\bar{\alpha})}, \\ \zeta_- &= -\bar{a}\mathrm{e}^{\mathrm{i}(-kx-\sigma t+\bar{\alpha})}, \end{aligned}\right\}$$

where $\bar{a}$ and $\bar{\alpha}$ are new constants.

(3) The cylinder generates waves by making small circular motions in a clockwise sense:

$$\xi = s\mathrm{e}^{-\mathrm{i}\sigma t}, \quad \eta = -\mathrm{i}s\mathrm{e}^{-\mathrm{i}\sigma t}.$$

We simply subtract (1) from (2). But by theorem C, ζ_- vanishes. Hence $\bar{a} = a$, $\bar{\alpha} = \alpha$ and we have

$$\left.\begin{aligned} \zeta_+ &= 2a\mathrm{e}^{\mathrm{i}(kx-\sigma t+\alpha)}, \\ \zeta_- &= 0. \end{aligned}\right\}$$

(4) The cylinder absorbs waves coming from $x = -\infty$. In (3), reverse the signs of x, t and ξ. Taking complex conjugate expressions, and adding a phase π we get

$$\xi = s\mathrm{e}^{-\mathrm{i}\sigma t}, \quad \eta = -\mathrm{i}s\mathrm{e}^{-\mathrm{i}\sigma t}$$

and
$$\left.\begin{aligned}\zeta_+ &= 0,\\ \zeta_- &= -2a\mathrm{e}^{\mathrm{i}(kx-\sigma t-\alpha)}.\end{aligned}\right\}$$

(5) The cylinder is fixed and subject to waves incident from $x = -\infty$. Taking $\frac{1}{2}[(3)-(4)]$ we have
$$\xi = 0, \quad \eta = 0$$
and
$$\left.\begin{aligned}\zeta_+ &= a\mathrm{e}^{\mathrm{i}(kx-\sigma t+\alpha)},\\ \zeta_- &= a\mathrm{e}^{\mathrm{i}(kx-\sigma t+\alpha)}.\end{aligned}\right\}$$

This incidentally proves theorem A.

Consider now the mean forces (X, Y) on the cylinder. In cases (1) and (2) these are, by symmetry,
$$X = 0, \qquad Y = Q\mathrm{e}^{\mathrm{i}(-\sigma t+\delta)}$$
and
$$X = P\mathrm{e}^{\mathrm{i}(-\sigma t+\gamma)}, \quad Y = 0,$$
where P, Q and γ, δ are real constants. Hence in case (5) we obtain
$$X = \mathrm{i}P\sin\gamma\,\mathrm{e}^{-\mathrm{i}\sigma t}, \quad Y = -Q\sin\delta\,\mathrm{e}^{-\mathrm{i}\sigma t}.$$
The phase-lag ψ_1 of the force Y behind the vertical acceleration $\partial^2\zeta/\partial t^2$ at $x = 0$ in the incident wave is therefore
$$\psi_1 = \pi+\alpha. \tag{A 1}$$

(6) The cylinder responds freely to waves incident from $x = -\infty$. By theorem B we know there is no reflected wave, so the motion has the form
$$(6) = \tfrac{1}{2}[(3)\,\mathrm{e}^{\mathrm{i}\epsilon}+(4)\,\mathrm{e}^{-\mathrm{i}\epsilon}],$$
where ϵ is a constant phase. Thus we have
$$\xi = s\cos\epsilon\,\mathrm{e}^{-\mathrm{i}\sigma t}, \quad \eta = -\mathrm{i}s\cos\epsilon\,\mathrm{e}^{-\mathrm{i}\sigma t}$$
and
$$\left.\begin{aligned}\zeta_+ &= a\mathrm{e}^{\mathrm{i}(kx-\sigma t+\alpha+\epsilon)},\\ \zeta_- &= -a\mathrm{e}^{\mathrm{i}(-kx-\sigma t-\alpha-\epsilon)}.\end{aligned}\right\}$$
The phase-lag ψ_2 of the vertical displacement η behind the vertical displacement at $x = 0$ in the incident wave is therefore
$$\psi_2 = -\tfrac{1}{2}\pi-(-\alpha-\epsilon-\pi) = \tfrac{1}{2}\pi+\alpha-\epsilon. \tag{A 2}$$
It will not be necessary to determine ϵ, although this may easily be done from the condition that $X = M\partial^2\xi/\partial t^2$ where M is the mass of the cylinder. Finally

(7) The cylinder is constrained vertically and free horizontally. We simply modify (6) by subtracting a fraction of the forced motion (1) to cancel the vertical displacement. Thus taking $(7) = (6)-(1)\cos\epsilon$, we have
$$\xi = s\cos\epsilon\,\mathrm{e}^{-\mathrm{i}\sigma t}, \quad \eta = 0.$$
The addition of the forced motion involved no extra displacement or force in the x-direction; hence the freedom of the motion in the horizontal is unaffected. Now the amplitudes of the incident and reflected waves are equal to a and $a\cos\epsilon$ respectively. Therefore the coefficient of reflexion is
$$\cos\epsilon = \cos(\psi_1-\psi_2)$$

from equations (A 1) and (A 2). This proves equation (4.1). Since by conservation of energy $a'^2 + b^2 = a^2$, equation (4.2) follows immediately.

Equations (1.1)–(1.5) were pointed out by the present author in a memorandum to a meeting on wave power at the C.E.G.B. Headquarters on 17 March 1975, under the chairmanship of Dr D. T. Swift-Hook. A qualitative confirmation of the radiation stress was reported to the author on a subsequent visit to British Hovercraft Corporation, Isle of Wight at the invitation of Sir Christopher Cockerell and Mr Peter Crewe. The measurements by Salter *et al.*, made originally at the author's prompting, are here quoted by kind permission of Mr Salter. The author has had many interesting discussions with those mentioned and also with Mr J. Platts and Mr I. Glendenning. The contents of the present paper were outlined at a discussion meeting at the Society for Underwater Technology on 10 March 1976, and at the annual meeting of the British Theoretical Mechanics Colloquium in Edinburgh.

References

Bowen, A. T., Inman, D. L. & Simmons, V. P. 1968 Wave 'set-down' and wave set-up. *J. geophys. Res.* **73**, 2569–2577.

Dean, W. R. 1948 On the reflection of surface waves by a submerged cylinder. *Proc. Camb. Phil. Soc.* **44**, 483–491.

Lamb, H. 1932 *Hydrodynamics*, 6th ed., ch. 9. Cambridge University Press.

Longuet-Higgins, M. S. 1967 On the wave-induced difference in mean sea level between the two sides of a submerged breakwater. *J. mar. Res.* **25**, 148–153.

Longuet-Higgins, M. S. & Stewart, R. W. 1960 Changes in the form of short gravity waves on long waves and tidal currents. *J. Fluid Mech.* **8**, 565–583.

Longuet-Higgins, M. S. & Stewart, R. W. 1963 A note on wave set-up. *J. mar. Res.* **21**, 4–10.

Longuet-Higgins, M. S. & Stewart, R. W. 1964 Radiation stresses in water waves; a physical discussion, with applications. *Deep-Sea Res.* **11**, 529–562.

Masuda, Y. 1972 Study of wave activated generator and future view as island power source. 2nd International Ocean Development Conference Preprints **2**, 2074.

Ogilvie, T. F. 1963 First- and second-order forces on a cylinder submerged under a free surface. *J. Fluid Mech.* **16**, 451–472.

Salter, S. H. 1974 Wave power. *Nature, Lond.* **249**, 720–724.

Salter, S. H., Jeffrey, D. C. & Taylor, J. R. M. 1976 The architecture of nodding duck wave power generators. *The Naval Architect*, Jan. 1976, pp. 21–24.

Saville, T. 1961 Experimental determination of wave set-up. *Proc. 2nd Tech. Conf. on Hurricanes*, Miami Beach, Florida, pp. 242–252. U.S. Dept. of Commerce.

Sheppard, F. P. & LaFond, E. C. 1940 Sand movements along the Scripps Institution Pier, California. *Am. J. Sci.* **238**, 272–285.

Ursell, F. 1950 Surface waves on deep water in the presence of a submerged circular cylinder. I. *Proc. Camb. Phil. Soc.* **49**, 141–152.

Woolley, M. & Platts, J. 1975 Energy on the crest of a wave. *New Scientist*, 1 May 1975, pp. 241–243.