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On the nonlinear transfer of energy in the peak of a gravity-wave spectrum: a simplified model

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An equation given by Davey & Stewartson (1974) for the evolution of wave packets in three dimensions is employed to discuss the resonant transfer of energy within the peak of a narrow spectrum of gravity waves. It is shown that the coupling coefficient $G(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ between four nearly equal wavenumbers $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4$ is not zero (as had been speculated) but is equal to 4π. This implies that the exchange of energy within the peak itself is of dominant importance, and leads to a simplified discussion of the energy transfer.

1. INTRODUCTION

The gradual transfer of energy between gravity waves of different wavelengths and directions, which was suggested on theoretical grounds by Phillips (1960) and Hasselmann (1962, 1963), was confirmed in special situations by laboratory experiments (Longuet-Higgins 1962; Longuet-Higgins & Smith 1966; McGoldrick, Phillips, Huang & Hodgson 1966). An interpretation of recent wave measurements in the North Sea (Hasselmann *et al.* 1973) strongly suggests that these nonlinear transfers of energy play an essential role in the development of the wave spectrum in wave-fields where the fetch is limited, particularly in the growth of the wave energy at low frequencies.

Nevertheless the present state of the theory is in some respects unsatisfying. The expressions for the energy transfer are necessarily of the form

$$\frac{\partial n_1}{\partial t} = \int \ldots \int G(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)\left[(n_1+n_2)\, n_3 n_4 - (n_3+n_4)\, n_1 n_2\right] \\ \times \delta(\sigma_1+\sigma_2-\sigma_3-\sigma_4)\, \delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3\, \mathrm{d}\boldsymbol{k}_4, \quad (1)$$

where n_i denotes the action-density at wavenumber $\boldsymbol{k}_i$ and frequency $\sigma_i = \sqrt{(g|\boldsymbol{k}_i|)}$, and where the delta-functions express conditions for resonance between the waves $i = 1, 2, 3, 4$. However, in the theory as given by Hasselmann (1962, 1963) the coupling coefficient $G(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ is exceedingly complicated, and the algebraic manipulations required to obtain it are both lengthy and heavy. No physical interpretation for the precise form of G has been given. The evaluation of G, and of the multiple integral in (1) has been attempted only on a computer, and with seemingly conflicting results. Thus, in the first computations by Hasselmann (1963) using the Pierson–Moskowitz form of the wave spectrum, it appeared that the rates of energy transfer would tend to augment the peak value of the spectral density, at the expense of the moderately higher frequencies. Later computations using the narrower spectra measured in the Jonswap field experiments (Hasselmann *et al.* 1973) indicated a different behaviour, namely that the energy flow was such as to shift the energy peak towards lower frequencies.

Moreover, at present it is necessary to rely for these critical conclusions upon the numerical computation of certain complicated multiple integrals, which cannot be accurately evaluated. There is obviously a need for a much simpler approach, more amenable to physical interpretation.

An attempt to discuss the energy transfer to wavenumbers well outside the main peak in the spectrum was recently made by Webb (1974), with some success. However, it is clear from equation (1) that the rate of flux $\partial n/\partial t$ depends in general on the *third power* of the energy density. Hence the rates of transfer in or near the peak of the spectrum are likely to be the most important in order of magnitude, provided only that when $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$, $\boldsymbol{k}_4$ are all nearly equal, the coupling coefficient $G(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4)$ does not become small or zero. Significantly, G has never been evaluated, even in this important case, because of the complexity of the algebra involved. Even numerically this was not possible, because the expressions given by Hasselmann contain singularities, which can only be overcome by passage to the limit in an appropriate manner. The question whether or not G vanishes in the limit is clearly crucial for a discussion of the dominant energy flux.

An opportunity to study the problem from a different point of view came with a recent paper by Davey & Stewartson (1974) who considered the evolution of the form of three-dimensional packets of gravity waves, on the surface of water of uniform depth. They discussed the deterministic problem of the change in form of the wave envelope, when it is assumed that the 'width' of the Fourier transform is of the same order of smallness (ϵ) as the maximum slope of the waves. They then derived a remarkably simple equation (equation (2.4) below) for the rate of change of the envelope function, one, moreover, for which a simple physical interpretation can be given.

It should be noted that an equivalent but more complicated system of coupled equations was previously derived by Benney & Roskes (1969). Chu & Mei (1970, 1971) gave coupled equations for the two-dimensional case ($\partial/\partial\eta = 0$) and for two dimensions the compact form of equation (2.4) was derived by Hasimoto & Ono (1972). All these authors considered the case of arbitrary uniform depth. Equation (2.4) is the form to which the equation of Davey & Stewartson is reduced when the ratio of the depth to the wavelength tends to infinity (see footnote on p. 314).

The two dimensional form of equation (2.4) is in fact a special case of the wave-modulation equation discussed for example by Karpman & Krushkal (1969) and by Zakharov & Shabat (1972). This equation has many interesting properties, including the existence of exact solutions in which the envelope has the form of a solitary wave (soliton). The initial-value problem is soluble, and it is possible also to discuss the stability of the soliton solution. However, the added introduction of variations in the horizontal direction (η) *transverse* to the direction of propagation is essential in our problem, as we shall see.

The object of this note is to show how the simple evolution equation (2.5) can be used to study the energy transfer in a gravity wave spectrum, when all the wave-numbers $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4$ are near the peak. In § 2 we state the equation and give a simple interpretation of it. In § 3 we derive the resonance conditions for four wavenumbers, and in § 4 we calculate the coupling coefficient. This is shown to be non-zero, thereby answering one leading question, and simplifying the discussion of the energy transfer. A preliminary discussion of the consequences is given in §§ 5–7. It is shown that the energy from an isolated peak in the spectrum tends to spread outwards along two characteristic lines, making angles $\pm \arctan(1/\sqrt{2})$ with the mean direction. The relation of this to the Benjamin–Feir instability is discussed in § 8, and in § 9 we present some computations of the energy transfer in a symmetric normal spectrum. The conclusions are summarized in § 10.

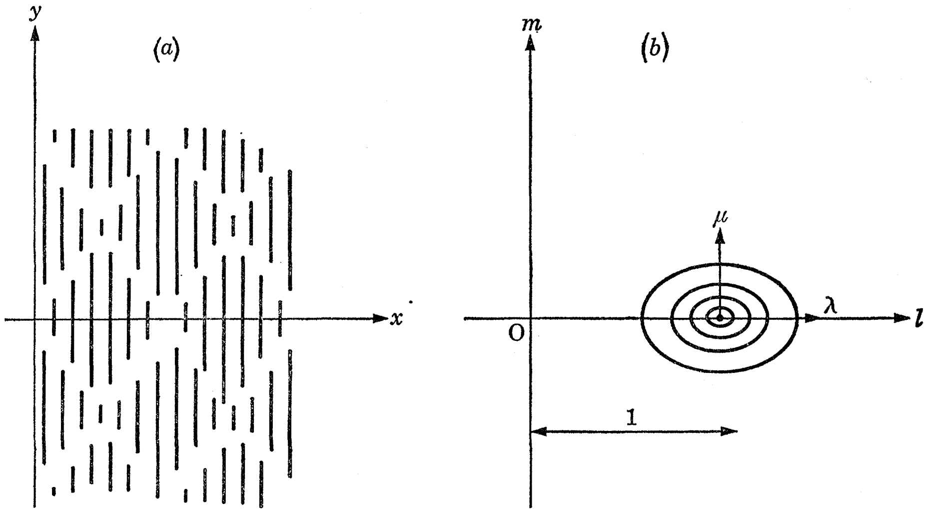

FIGURE 1. (*a*) Schematic representation of a surface with a narrow spectrum, and (*b*) the corresponding representation in the wavenumber plane.

2. The equation for the wave envelope

We consider free, irrotational gravity waves, on the surface of an inviscid, incompressible fluid of infinite depth. Let rectangular coordinates (x, y, z) be taken with the origin in the undisturbed free surface, the z-axis vertically upwards and the x-axis in the direction of propagation. In a wave train of small surface slope but slowly modulated amplitude and phase (see figure 1) the vertical displacement ζ of the free surface can be expressed in the form

$$\zeta = \mathscr{R}\epsilon A \, \mathrm{e}^{\mathrm{i}(\bar{l}x+\bar{m}y-\bar{\sigma}t)}, \tag{2.1}$$

where the exponential term represents a carrier wave of wavenumber $\bar{\boldsymbol{k}} = (\bar{l}, \bar{m})$ and radian frequency σ satisfying the linear dispersion relation for waves in deep water, namely

$$\bar{\sigma}^2 = g\bar{k}, \quad (\bar{k} = |\bar{\boldsymbol{k}}|) \tag{2.2}$$

and A represents a slowly changing amplitude and phase. ϵ is a small parameter which may be made to tend to zero. It is convenient to choose units of length and time so that

$$g = 1, \quad \bar{\boldsymbol{k}} = (1, 0), \quad \bar{\sigma} = 1. \tag{2.3}$$

It is well known that in linear theory the wave envelope is propagated with approximately the group-velocity $c' = \partial\sigma/\partial k = \frac{1}{2}$, that is to say A is nearly a function of $(x-c't)$ alone. More accurately, it can be shown (see Davey & Stewartson 1974) that the evolution of A is governed by the partial differential equation

$$\boxed{2\mathrm{i}A_\tau = \tfrac{1}{4}(A_{\xi\xi} - 2A_{\eta\eta}) + (AA^*)A} \tag{2.4}$$

where ξ, η and τ denote the scaled variables

$$\xi = \epsilon(x - c't), \quad \eta = \epsilon y, \quad \tau = \epsilon^2 t. \tag{2.5}$$

and the suffixes in (2.4) denote partial differentiation.†

The form of the scaling (2.5) implies an assumption that the length scales for the variation of A in any horizontal direction is of order ϵ^{-1}; in other words the spectral width is of order ϵ. From the governing equations it then follows that time-scale for the evolution of the envelope is of order ϵ^{-2}.

Equation (2.4) has simple sine wave solutions of the form

$$A = a\,\mathrm{e}^{\mathrm{i}(\lambda\xi + \mu\eta - \omega\tau)}, \tag{2.6}$$

provided that

$$\omega = -\tfrac{1}{8}(\lambda^2 - 2\mu^2) + \tfrac{1}{2}a^2. \tag{2.7}$$

The interpretation of this dispersion relation is very simple. From (2.1) and (2.6) λ, μ and ω are related to the physical wavenumber $\boldsymbol{k} = (l, m)$ and to the physical frequency σ by

$$(lx + my - \sigma t) = (\lambda\xi + \mu\eta - \omega\tau) + (\bar{l}x + \bar{m}y - \bar{\sigma}t).$$

Substituting from equations (2.3) and (2.5) we have

$$l = 1 + \epsilon\lambda, \quad m = \epsilon\mu, \quad \sigma = 1 + \tfrac{1}{2}\epsilon\lambda + \epsilon^2\omega.$$

Substituting in the dispersion relation

$$\sigma^2 = k(1 + \epsilon a^2 k^2), \quad k^2 = l^2 + m^2, \tag{2.8}$$

(Lamb 1932, C. 9) we find that the terms in ϵ^0 and ϵ^1 are already in agreement, while those in ϵ^2 give precisely equation (2.7). The first two terms on the right hand side of (2.7) arise essentially from the curvature of the linearized dispersion relation; the last term is the well-known correction to the wave speed arising from the finite amplitude.

† From (1) it will be noted that our definition of A differs from that of Davey & Stewartson (1974) by a factor of 2. Davey & Stewartson state that (2.4) is valid asymptotically as $kh \to \infty$ (h = the depth), provided $\epsilon kh \to 0$, but further investigation shows that the troublesome terms in the third approximation are cancelled out in the next approximation. Thus (2.4) is uniformly valid as $kh \to \infty$.

Physically, the term $-\frac{1}{8}\lambda^2$ on the right of (2.7) is associated with the fact that the dispersion curve $\sigma^2 = gl$ for long-crested waves ($m = 0$) is convex (see figure 2*a*). This implies that the envelope of two sine waves with different wavenumbers $(1 \pm \epsilon\lambda)$, travels slightly *slower* than the speed corresponding to the mean wavenumber; for, the midpoint of the chord joining two points on the dispersion curve lies slightly below the curve.

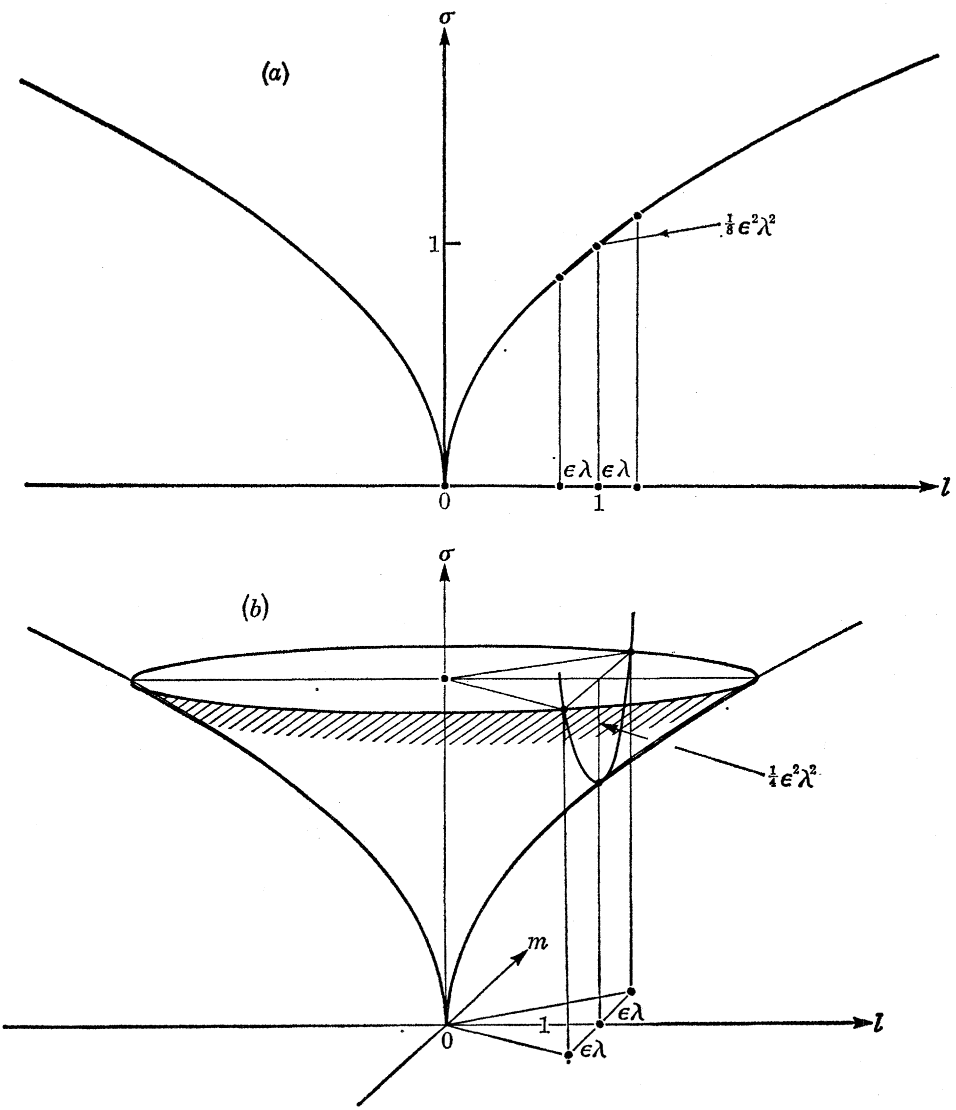

FIGURE 2. Plane sections of the linear dispersion surface $\sigma^2 = g(l^2+m^2)^{\frac{1}{2}}$ by (*a*) the axial plane $m = 0$, (*b*) the transverse plane $l = 1$. In case (*a*) the mean frequency corresponding to adjacent wavenumber lies below the surface; in (*b*) the mean frequency lies above the surface.

The positive term $\frac{1}{4}\mu^2$, on the other hand arises from the fact that the dispersion surface $\sigma^2 = gk$ (where $k^2 = l^2+m^2$) has axial symmetry. This, together with the fact that σ increases with k, implies that a transverse section of the surface is concave upwards (see figure 2*b*). Hence two waves of the same wavelength but travelling in slightly different directions combine to form an envelope which travels *faster* than either wave separately.

The hyperbolic form of the terms $(\lambda^2-2\mu^2)$ corresponds to the fact that the dispersion surface has principal curvatures of opposite sign. Hence any plane close and parallel to a tangent plane intersects the surface in a hyperbola, not an ellipse. The tangent plane itself intersects the surface in two lines which are asymptotes to the hyperbola.

3. Conditions for resonance

Consider four elementary waves

$$A_n = a_n \mathrm{e}^{\mathrm{i}(\lambda_n\xi+\mu_n\eta-\omega_n\tau)} \quad (n = 1, \ldots, 4) \tag{3.1}$$

each satisfying a dispersion relation similar to (2.7), that is

$$\omega_n = -\tfrac{1}{8}(\lambda_n^2-2\mu_n^2)+2M, \tag{3.2}$$

where M is a constant, independent of the suffix n. (Later, M will be identified with the total action density; see appendix A.) The conditions for resonance are then

$$\omega_1+\omega_2 = \omega_3+\omega_4 \tag{3.3}$$

and

$$\kappa_1+\kappa_2 = \kappa_3+\kappa_4, \tag{3.4}$$

where $\kappa_n = (\lambda_n, \mu_n)$. From (3.2) and (3.3) we have at once

$$(\lambda_1^2-2\mu_1^2)+(\lambda_2^2-2\mu_2^2) = (\lambda_3^2-2\mu_3^2)+(\lambda_4^2-2\mu_4^2). \tag{3.5}$$

Now (3.4) implies that κ_1, κ_2 and κ_3, κ_4 are opposite vertices of a parallelogram (see figure 3). Denoting the centre of this parallelogram by $\bar{\kappa}$, we may write

$$\left.\begin{aligned} \kappa_1 &= \bar{\kappa}-\kappa', \quad \kappa_2 = \bar{\kappa}+\kappa', \\ \kappa_3 &= \bar{\kappa}-\kappa'', \quad \kappa_4 = \bar{\kappa}+\kappa'', \end{aligned}\right\} \tag{3.6}$$

where $\bar{\kappa} = (\bar{\lambda}, \bar{\mu})$, $\kappa' = (\lambda', \mu')$, $\kappa'' = (\lambda'', \mu'')$. On substitution in (3.5) we find simply

$$\lambda'^2-2\mu'^2 = \lambda''^2-2\mu''^2. \tag{3.7}$$

This shows that κ_1, κ_2, κ_3 and κ_4 all lie on the same hyperbola, with centre $\bar{\kappa}$ and with asymptotes making an angle $\arctan(1/\sqrt{2})$ with the ξ-axis (see figure 3). For fixed κ_1 but variable κ_2, κ_3 and κ_4, the centre of the hyperbola is arbitrary and also the length of one principal axis, but the directions of the asymptotes are fixed.

The asymptotes can be seen to correspond to the tangents to the figure-of-eight curve derived by Phillips (1960) just at the central point $\bar{\kappa}$. The family of hyperbolae corresponds to the local aspect of the family of curves shown by Hasselmann (1963, figure 6) in the neighbourhood of the central point.

4. The transfer of energy

Let the envelope function $A(\xi, \eta, \tau)$ be represented in the form

$$A = \sum_n a_n(\tau)\, e^{i(\lambda_n \xi + \mu_n \eta - \omega_n \tau)}, \tag{4.1}$$

where ω_n is given by (3.2) and $a_n(\tau)$ is a relatively slowly varying function of τ only. a_n is complex in general, with slowly varying amplitude and phase. We shall suppose that the magnitudes of the a_n are small and their phases uncorrelated, to first order, and moreover that in the limit they become densely distributed in wavenumber space in such a way that when summed over an element $d\boldsymbol{\kappa}$

$$\sum_n \tfrac{1}{2} a_n a_n^* \sim N(\boldsymbol{\kappa})\, d\boldsymbol{\kappa}. \tag{4.2}$$

The function $N(\boldsymbol{\kappa})$ corresponds to the local action-density.

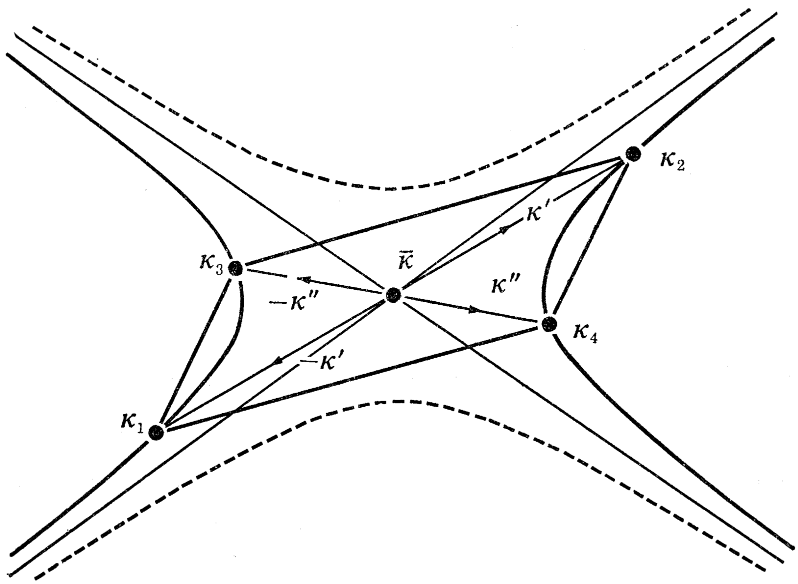

Figure 3. Illustrating the conditions for resonant interaction between four wavenumbers.

Substituting for A from (4.1) into the differential equation (2.4) we see that the derivatives with respect to ξ and η both cancel. Then on equating coefficients of $e^{i(\lambda_n \xi + \mu_n \eta)}$ on each side we obtain simply

$$2i \frac{da_n}{d\tau} = \sum_{p,q,r} a_p a_q a_r^* \, e^{-i(\omega_p + \omega_q - \omega_r - \omega_n)\tau}\, \delta(\boldsymbol{\kappa}_p + \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_r - \boldsymbol{\kappa}_n) \tag{4.3}$$

the delta function ensuring that in the triple sum only those terms are retained for which the exponential factors agree. Multiplying (4.3) by $a_n^*/2i$ and adding the complex conjugate we obtain

$$\frac{d}{d\tau}(a_n a_n^*) = \mathscr{R} \frac{1}{i} \sum_{p,q,r} a_p a_q a_r^* a_n^* \, e^{-i(\omega_p + \omega_q - \omega_r - \omega_n)\tau}\, \delta(\boldsymbol{\kappa}_p + \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_r - \boldsymbol{\kappa}_n). \tag{4.4}$$

Now since the phases of the a_n are assumed to be nearly uncorrelated, the product $a_p a_q a_r^* a_n^*$ will, on the average, be negligible except when either $p = n$ and $q = r$, or else $q = n$ and $p = r$. Hence the sum reduces to

$$2a_n a_n^* \sum_r a_r a_r^*$$

which is real. Hence the right hand side of (4.4) vanishes, to order ϵ^4. To calculate higher order terms, we first differentiate the product $a_p a_q a_r^* a_n^*$ with respect to τ. Then substituting from (4.3) we have

$$\begin{aligned} 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}(a_p a_q a_r^* a_n^*) = \quad & a_q a_r^* a_n^* \sum_{u,v,w} a_u a_v a_w^* \,\mathrm{e}^{-\mathrm{i}(\omega_u+\omega_v-\omega_w-\omega_p)\tau}\,\delta(\boldsymbol{\kappa}_u+\boldsymbol{\kappa}_v-\boldsymbol{\kappa}_w-\boldsymbol{\kappa}_p) \\ & + a_p a_r^* a_n^* \sum_{u,v,w} a_u a_v a_w^* \,\mathrm{e}^{-\mathrm{i}(\omega_u+\omega_v-\omega_w-\omega_q)\tau}\,\delta(\boldsymbol{\kappa}_u+\boldsymbol{\kappa}_v-\boldsymbol{\kappa}_w-\boldsymbol{\kappa}_q) \\ & - a_p a_q a_n^* \sum_{u,v,w} a_u^* a_v^* a_w \,\mathrm{e}^{\mathrm{i}(\omega_u+\omega_v-\omega_w-\omega_r)\tau}\,\delta(\boldsymbol{\kappa}_u+\boldsymbol{\kappa}_v-\boldsymbol{\kappa}_w-\boldsymbol{\kappa}_r) \\ & - a_p a_q a_r^* \sum_{u,v,w} a_u^* a_v^* a_w \,\mathrm{e}^{\mathrm{i}(\omega_u+\omega_v-\omega_w-\omega_n)\tau}\,\delta(\boldsymbol{\kappa}_u+\boldsymbol{\kappa}_v-\boldsymbol{\kappa}_w-\boldsymbol{\kappa}_n). \end{aligned} \tag{4.5}$$

Now in the first summation we must have either $(u, v, w) = (r, n, q)$ or else $(u, v, w) = (n, r, q)$. So taking averages in (4.5) and writing

$$\overline{a_n a_n^*} = C_n \tag{4.6}$$

we find altogether

$$\begin{aligned} 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\overline{(a_p a_q a_r^* a_n^*)} &= 2(C_q C_r C_n + C_p C_r C_n - C_p C_q C_n - C_p C_q C_r) \\ &\qquad \times \mathrm{e}^{\mathrm{i}(\omega_p+\omega_p-\omega_r-\omega_n)\tau}\,\delta(\boldsymbol{\kappa}_p+\boldsymbol{\kappa}_q-\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_n). \end{aligned} \tag{4.7}$$

Now on integrating with respect to τ from $-\infty$ (where the correlations are assumed negligible) up to τ and then substituting into (4.4) we obtain

$$\begin{aligned} \frac{\mathrm{d}C_n}{\mathrm{d}\tau} = \mathscr{R} \sum_{p,q,r} \int_{-\infty}^{\tau} & [(C_r + C_n) C_p C_q - (C_p + C_q) C_r C_n] \\ & \times \mathrm{e}^{\mathrm{i}(\omega_p+\omega_q-\omega_r-\omega_n)(\tau'-\tau)}\,\delta(\boldsymbol{\kappa}_p+\boldsymbol{\kappa}_q-\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_n). \end{aligned} \tag{4.8}$$

The factor in square brackets may be assumed to vary only slowly with τ' compared to the rest of the integrand, so that it can be taken outside the integral. We use the result that if ω has a small negative imaginary part, then

$$\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s = \frac{1}{\mathrm{i}\omega} + \pi\delta(\omega), \tag{4.9}$$

where $\delta(\omega)$ denotes the Dirac delta function. If we introduce a small artificial damping into equation (2.4) to indicate the direction of time, then ω_p, ω_q will have a small negative imaginary part. In equation (4.8) $(\omega_p+\omega_q-\omega_r-\omega_n)$ must be replaced by $(\omega_p+\omega_q-\omega_r^*-\omega_n^*)$, which will have a small negative imaginary part also, so that (4.8) becomes simply

$$\frac{\mathrm{d}C_n}{\mathrm{d}\tau} = \pi \sum_{p,q,r} [(C_r+C_n)C_pC_q-(C_p+C_q)C_rC_n] \times \delta(\omega_p+\omega_q-\omega_r^*-\omega_n^*)\,.\,\delta(\boldsymbol{\kappa}_p+\boldsymbol{\kappa}_q-\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_n). \tag{4.10}$$

Finally, in view of the relation (4.2) we see that (4.10) is equivalent to

$$\boxed{\frac{\mathrm{d}N_1}{\mathrm{d}\tau} = 4\pi \int\ldots\int [(N_1+N_2)N_3N_4-(N_3+N_4)N_1N_2] \times \delta(\omega_1+\omega_2-\omega_3-\omega_4)\,\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_4)\,\mathrm{d}\boldsymbol{\kappa}_2\,\mathrm{d}\boldsymbol{\kappa}_3\,\mathrm{d}\boldsymbol{\kappa}_4} \tag{4.11}$$

where $N_i = N(\boldsymbol{\kappa}_i)$.

Since $N_i\,\mathrm{d}\boldsymbol{\kappa}_i$ is equivalent to $n_i\,\mathrm{d}\boldsymbol{k}_i$ where $\boldsymbol{k}_i = (\mathbf{1}+\epsilon^2\boldsymbol{\kappa}_i)$ and $\mathrm{d}\boldsymbol{k}_i = \epsilon^2\,\mathrm{d}\boldsymbol{\kappa}_i$, we see that equation (4.11) is of the same form as (1.1). Moreover, when $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$, $\boldsymbol{k}_4$ are equal, the coupling coefficient does *not* vanish. On the contrary, we have, in dimensionless units,

$$G(1,1,1,1) = 4\pi. \tag{4.12}$$

One would expect the above value of the coupling coefficient to agree with the value derived from the lengthy expressions given by Hasselmann (1962, pp. 490, 491 and 1963, p. 276) on passing to the limit in a suitable manner, and it can indeed be verified that this is so. The details need not be given here.

5. Properties of the energy flux

From (4.11) we can deduce immediately certain properties of the energy transfer within the spectral peak. First, there will be a tendency for energy to move towards regions of low density. Consider for instance the extreme case shown in figure 4, where $N(\boldsymbol{\kappa})$ vanishes to the left of a given line in the wavenumber plane, and is positive to the right of it. If more than one of the $\boldsymbol{\kappa}_i$ lie to the left of the line, as in figure 4*a*, then clearly the integrand in (4.11) vanishes and there is zero contribution to the energy flux. If $\boldsymbol{\kappa}_1$ lies to the left of the line and $\boldsymbol{\kappa}_2$, $\boldsymbol{\kappa}_3$, $\boldsymbol{\kappa}_4$ lie to the right, as in figure 4*b*, then the contribution to $\partial N_1/\partial\tau$ is positive, and is proportional in fact to $N_2N_3N_4$. Altogether, since there will be some appropriate hyperbolae through $\boldsymbol{\kappa}_1$ intersecting the boundary, there will be a positive flow of energy into wavenumber $\boldsymbol{\kappa}_1$.

In general we may expect that the transfer of energy will tend to reduce any asymmetry in the spectrum.

Suppose next that the spectrum is symmetrical, as in figure 5, but the energy is confined mainly to a limited region surrounding the peak density. There will be a tendency for energy to flow outwards from the peak. For, consider any wavenumber $\boldsymbol{\kappa}_1$ outside the peak. Provided that $\boldsymbol{\kappa}_1$ lies close to an asymptote, it may be possible to find an opposite wavenumber $\boldsymbol{\kappa}_2$ and two companion wavenumbers

κ_3, κ_4 within the peak itself, such that all lie on the same hyperbola. Since N_1 and N_2 are small compared to N_3 and N_4, the important terms in (4.11) are $(N_1+N_2)N_3N_4$, the terms $(N_3+N_4)N_1N_2$ being relatively small.

If, however, κ_1 lies outside the peak but *not* near one of the asymptotes $\lambda^2 = 2\mu^2$ it will be impossible to find any allowable hyperbola intersecting the peak, on which κ_3, κ_4 can lie. So there will be a negligible flux of energy into κ_1.

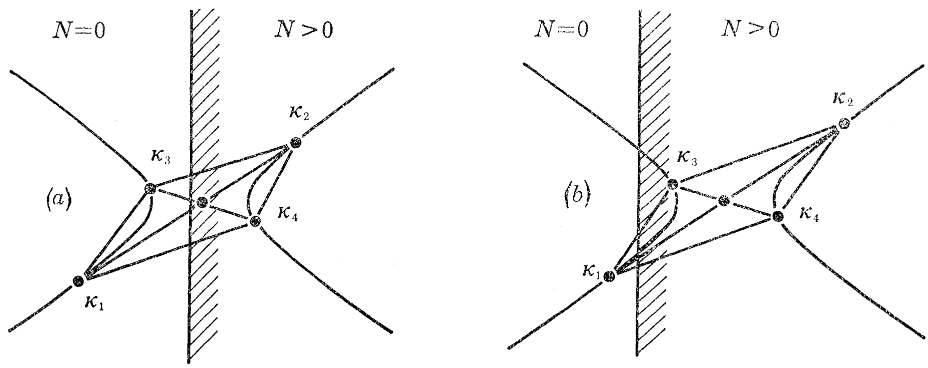

FIGURE 4. Curves for resonant interaction in an asymmetric spectrum.

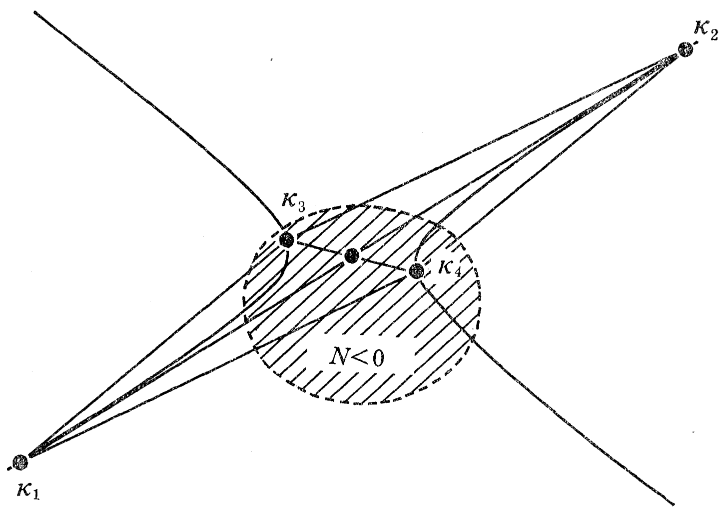

FIGURE 5. Curves for resonant transfer of energy to wavenumbers outside the spectral peak.

It follows that energy will tend to stream outwards from the peak in the directions $d\lambda = \pm\sqrt{2}\,d\mu$. Subsequently, the energy will tend to diffuse from the neighbourhood of the asymptotes into other parts of the plane.

In short, we may consider the directions defined by $d\lambda/d\mu = \pm\sqrt{2}$ as defining characteristics, in a certain sense, for the flow of energy outwards from the peak.

6. Conservation laws

The combined flow of energy to different parts of the plane is, however, governed by certain conservation laws. Thus from (4.11) we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\iint N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = 4\pi\int\ldots\int f(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)\,\mathrm{d}\boldsymbol{\kappa}_1\,\mathrm{d}\boldsymbol{\kappa}_2\,\mathrm{d}\boldsymbol{\kappa}_3\,\mathrm{d}\boldsymbol{\kappa}_4,$$

where f is a function which is symmetric in $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ or $\boldsymbol{\kappa}_3$, $\boldsymbol{\kappa}_4$ but antisymmetric in $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_3$. Thus, by interchanging $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_3$, say, and adding, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\iint N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = 0.$$

So
$$\iint N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = M, \quad \text{constant.} \tag{6.1}$$

From (4.1) we have also

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\iint \boldsymbol{\kappa}N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = 4\pi\int\ldots\int \boldsymbol{\kappa}_1 f(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)\,\mathrm{d}\boldsymbol{\kappa}_1\,\mathrm{d}\boldsymbol{\kappa}_2\,\mathrm{d}\boldsymbol{\kappa}_3\,\mathrm{d}\boldsymbol{\kappa}_4$$

so that by cyclic permutation of the $\boldsymbol{\kappa}_i$, and adding, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\iint \boldsymbol{\kappa}N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = \pi\int\ldots\int (\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_4)f(\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_4)\,\mathrm{d}\boldsymbol{\kappa}_1\ldots,\mathrm{d}\boldsymbol{\kappa}_4.$$

But since f contains as a factor the delta-function $\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_4)$ the integral on the right vanishes. Hence

$$\iint \boldsymbol{\kappa}N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = \boldsymbol{I}, \quad \text{constant.} \tag{6.2}$$

Similarly
$$\iint \omega(\boldsymbol{\kappa})\,N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = E, \quad \text{constant.} \tag{6.3}$$

Equations (6.1) and (6.2) together give

$$\iint (1+\epsilon\boldsymbol{\kappa})\,N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = \iint \boldsymbol{k}n(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} = \text{constant,} \tag{6.4}$$

and similarly equations (6.1) and (6.3) give

$$\iint (1+\epsilon^2\omega)\,N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = \iint \sigma(\boldsymbol{k})\,n(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} = \text{constant.} \tag{6.5}$$

Equation (6.1) expresses the constancy of the total action, and (6.4) and (6.5) express the constancy of the total momentum and energy, respectively (see also appendix A). This is always assuming that the contribution to the integrals from those parts of the spectrum outside the peak can be neglected.

The constancy of the three spectral moments in (6.1), (6.2) and (6.3) imposes certain constraints on the exchange of energy within the peak. Thus if there is a flux of energy towards lower wavenumbers $\boldsymbol{\kappa}$ (as there may be in a typical wind-wave spectrum) there must also be compensating flux towards much higher wavenumbers, to maintain the mean wavenumber constant. And if there is some flux into the zones where $\omega(\boldsymbol{\kappa}) > \omega_0$ there must be a compensating flux into the zones where $\omega(\boldsymbol{\kappa}) < \omega_0$.

7. Equilibrium spectra

Does an equilibrium spectrum exist? The argument of § 5 makes it appear unlikely that there could be a continuous spectrum, with finite total energy, for which $N > 0$ and $\partial N/\partial t = 0$ everywhere. For if the total energy is finite, the density must be low in certain parts of the $\boldsymbol{\kappa}$-plane, especially at large distances. Hence energy will tend to flow into these areas, particularly in the direction of the characteristics $\mathrm{d}\lambda/\mathrm{d}\mu = \pm\sqrt{2}$.

The equilibrium solutions quoted by Hasselmann (1963) in which

$$\frac{1}{N_1}+\frac{1}{N_2}-\frac{1}{N_3}-\frac{1}{N_4} = 0 \tag{7.1}$$

for all tetrads satisfying the resonance conditions, imply that

$$\frac{1}{N(\boldsymbol{\kappa})} = \boldsymbol{P}.\boldsymbol{\kappa}+Q\omega+R, \tag{7.2}$$

where $\boldsymbol{P}$, Q and R are constants. If either $\boldsymbol{P}$ or Q is not zero, then $N(\boldsymbol{\kappa})$ becomes negative for some $\boldsymbol{\kappa}$. If $\boldsymbol{P}$ and Q both vanish, but not R, then the density is uniform and the total energy is infinite. In neither case does the solution have any physical significance.

The nearest approach to an equilibrium spectrum appears to be when the energy is concentrated in the form of isolated delta functions, or along a pair of characteristic lines. Both forms, however, would appear to be unstable with respect to the addition of small amounts of energy in other parts of the plane.

8. Side-band instabilities

The stability of the pure sine wave (2.11), considered as a wave of finite amplitude, may be discussed by means of equation (2.4) itself (see Davey & Stewartson 1974). Slightly generalizing their result, we set

$$A = a(1+a')\,\mathrm{e}^{\mathrm{i}(\lambda\xi+\mu\eta-\omega\tau+\theta')}, \tag{8.1}$$

where a' and θ' are real functions representing small perturbations of the solution (2.11). Substituting in equation (2.4) and retaining only first order terms in a' and θ' we obtain, from the real and imaginary parts, the two equations

$$\left.\begin{aligned}\left[\frac{1}{8}\left(\frac{\partial^2}{\partial\xi^2}-2\frac{\partial^2}{\partial\eta^2}\right)+a^2\right]a'+\left[\frac{\partial}{\partial\tau}-\frac{1}{4}\left(\lambda\frac{\partial}{\partial\xi}-2\mu\frac{\partial}{\partial\eta}\right)\right]\theta' = 0,\\ \left[\frac{\partial}{\partial\tau}-\frac{1}{4}\left(\lambda\frac{\partial}{\partial\xi}-2\mu\frac{\partial}{\partial\eta}\right)\right]a'-\frac{1}{8}\left(\frac{\partial^2}{\partial\xi^2}-2\frac{\partial^2}{\partial\eta^2}\right)\theta' = 0.\end{aligned}\right\} \tag{8.2}$$

Then a', θ' will have solutions

$$\left.\begin{aligned} a' &= \mathscr{R} a_1\, e^{i(\lambda'\xi+\mu'\eta-\omega'\tau)}, \\ \theta' &= \mathscr{R} \theta_1\, e^{i(\lambda'\xi+\mu'\eta-\omega'\tau)}, \end{aligned}\right\} \tag{8.3}$$

provided that

$$[\omega' + \tfrac{1}{4}(\lambda\lambda' - 2\mu\mu')]^2 = \tfrac{1}{8}(\lambda'^2 - 2\mu'^2)\,[\tfrac{1}{8}(\lambda'^2 - 2\mu'^2) - a^2]. \tag{8.4}$$

The imaginary part of ω' will be non-zero, and hence there may be instability, if the right hand side is negative, that is to say if

$$0 < \lambda'^2 - 2\mu'^2 < 8a^2. \tag{8.5}$$

Hence for instability, (λ', μ') lies in a zone bounded by the hyperbola $(\lambda'^2 - 2\mu'^2) = 8a^2$ and the two asymptotes $\lambda'^2 - 2\mu'^2 = 0$. The fastest-growing instabilities are such that the right hand side of (8.4) is a minimum, that is when

$$\lambda'^2 - 2\mu'^2 = 4a^2. \tag{8.6}$$

Then (λ', μ') lies on a hyperbola centre $(0, 0)$ and with vertices $(\pm 2a, 0)$.

For waves of small amplitude, the instabilities will all tend to lie close to the asymptotes $\mu'/\lambda' = \pm\sqrt{2}$. This suggests a close relation between this type of instability and the type described in § 5.

However, this side-band instability, which is essentially an extension of the one-dimensional instability discovered by Benjamin & Feir (1962), is essentially dependent on the initial *phases*. In this respect it is quite different from the phase-averaged exchange of energy discussed in §§ 4 and 5. In the side-band instability the flow of energy can theoretically be reversed; in the phase-averaged theory the flow is irreversible.

9. Rate of change of the peak density

Finally, let us calculate the flow of energy away from the peak of the symmetric normal spectrum

$$N(\boldsymbol{\kappa}) = N_0\, e^{-\frac{1}{2}P\lambda^2 - Q\mu^2}, \tag{9.1}$$

where N_0, P, Q are constants. We use the form of the flux equation (4.11) derived in appendix B (equations (B 8) and (B 10)). At the peak itself, where $\boldsymbol{\kappa}_1 = (0, 0)$ so $\bar{\boldsymbol{\kappa}} = \boldsymbol{\kappa}'$, the expression for F contains four exponentials, each with a factor α. The integrations with respect to α may be carried out immediately, giving

$$\frac{\partial N_1}{\partial \tau} = 16\pi N_0^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(P, Q, \theta', \theta'')\, d\theta'\, d\theta'', \tag{9.2}$$

where Φ is a rational function of P, Q and of $\cosh\theta'$, $\sinh\theta'$, $\cosh\theta''$, $\sinh\theta''$. Also Φ is symmetric in P and Q. The integral in (9.2) is easily evaluated numerically, giving

$$\frac{\partial N_1}{\partial \tau} = -\frac{N_0^3}{(PQ)^{\frac{1}{2}}} H(P/Q), \tag{9.3}$$

where H is the function shown in table 1. It can be seen that H is a maximum when $P/Q = 1$ and that even over so wide a range as $0.2 < P/Q < 5.0$ the value of H differs from $H(1)$ by less than 2 %. Thus for all practical purposes we may take

$$H(P/Q) = H(1) = 32.9. \tag{9.4}$$

To express the result in terms of convenient parameters, let $\bar{a}$ denote the r.m.s. wave amplitude defined by

$$\bar{a}^2 = \sum_n a_n a_n^* = 2\iint N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} = \frac{4\pi N_0}{(2PQ)^{\frac{1}{2}}} \tag{9.5}$$

and let Δ_1, Δ_2 denote the width of the spectrum, in the ξ and η directions, at half the peak amplitude, so

$$\Delta_1 = 2\sqrt{\left(\frac{\ln 4}{P}\right)}, \quad \Delta_2 = 2\sqrt{\left(\frac{\ln 4}{2Q}\right)} \tag{9.6}$$

and $(2PQ)^{\frac{1}{2}} = 4\ln 4/(\Delta_1 \Delta_2)$. Then (9.3) may be written

$$\frac{\partial N_1}{\partial \tau} \doteqdot -0.72\frac{\bar{a}^6}{\Delta_1^2 \Delta_2^2}. \tag{9.7}$$

In other words, the peak density tends to diminish at a rate proportional to the sixth power of the mean-square amplitude, and inversely as the square of the band-width in each direction; and almost independently of the shape of the peak.

TABLE 1. CALCULATED VALUES OF $H(P/Q)$

$\frac{P}{Q}$ or $\frac{Q}{P}$	H	$\frac{P}{Q}$ or $\frac{Q}{P}$	H
1.0	32.91	0.08	30.31
0.5	32.82	0.06	29.41
0.4	32.75	0.04	27.88
0.3	32.59	0.02	24.74
0.2	32.19	0.01	21.33
0.1	30.91	0.00	0.00

10. CONCLUSIONS

We have shown that the assumption of a narrow spectrum enables the transfer of energy within the spectral peak to be discussed in much simpler terms. The coupling coefficient has been evaluated explicitly, and is positive, not zero. In the wavenumber plane, the curves for resonantly interacting waves are simple hyperbolae and there is a tendency for energy to spread outwards from the peak in directions making angles arctan $(1/\sqrt{2})$ with the mean direction.

The possibility of a steady, continuous spectrum, with finite total energy, is ruled out. For a normal spectrum, the energy density at the peak frequency tends always to diminish, though at a rate critically dependent on the mean-square amplitude and on the spectral width (equation (9.7)).

A further discussion of the effects of asymmetry in the spectrum, and of the relative importance of contributions from outside the spectral peak, will be given in a second paper.

I am indebted to Mr M. J. H. Fox for confirming the calculations in §9.

Appendix A. Integrals of the envelope equation

Starting from equation (2.4), let us multiply each side by A^*. Then subtracting the complex conjugate we obtain

$$2\mathrm{i}(AA^*)_\tau = \tfrac{1}{4}[(A_\xi A^* - AA^*_\xi)_\xi - 2(A_\eta A^* - AA^*_\eta)_\eta]. \tag{A 1}$$

Suppose A is statistically uniform over the (ξ,η)-plane. On integrating over a large area of linear dimensions L, the left hand side is $O(L^2)$ while the right hand side, by the Green theorem, reduces to a line integral and so is $O(L)$. Hence dividing by L and letting $L\to\infty$ we get

$$(\overline{AA^*})_\tau = 0, \tag{A 2}$$

where an overbar denotes the average value with respect to (ξ,η). So

$$\tfrac{1}{2}\overline{AA^*} = M, \tag{A 3}$$

a constant. Similarly we find

$$\left.\begin{aligned} 2\mathrm{i}(A_\xi A^* - AA^*_\xi)_\tau &= X_\xi + Y_\eta,\\ 2\mathrm{i}(A_\eta A^* - AA^*_\eta)_\tau &= X_\eta - \tfrac{1}{2}Y_\xi,\end{aligned}\right\} \tag{A 4}$$

where $$X = (AA^*)^2, \quad Y = \mathscr{R}(A_\xi A^*_\eta - AA^*_{\xi\eta}). \tag{A 5}$$

This leads to

$$\left.\begin{aligned} \frac{1}{2\mathrm{i}}\overline{(A_\xi A^* - AA^*_\xi)} &= I_1,\\ \frac{1}{2\mathrm{i}}\overline{(A_\eta A^* - AA^*_\eta)} &= I_2,\end{aligned}\right\} \tag{A 6}$$

where I_1 and I_2 are real constants. Thirdly, we find

$$2\mathrm{i}[A_\xi A^*_\xi - 2A_\eta A^*_\eta + 4(AA^*)^2]_\tau = U_\xi + V_\eta, \tag{A 7}$$

where $$\left.\begin{aligned} U &= \tfrac{1}{4}(A_{\xi\xi}A^*_\xi - A_\xi A^*_{\xi\xi}) - \tfrac{1}{2}(A_{\xi\eta}A^*_\eta - A_\eta A^*_{\xi\eta}) + AA^*(A_\xi A^* - AA^*_\xi),\\ V &= -\tfrac{1}{2}(A_{\xi\eta}A^*_\xi - A_\xi A^*_{\xi\eta}) + (A_{\eta\eta}A^*_\eta - A_\eta A^*_{\eta\eta}) - 2AA^*(A_\eta A^* - AA^*_\eta).\end{aligned}\right\} \tag{A 8}$$

Hence $$\tfrac{1}{8}(\overline{A_\xi A^*_\xi - 2A_\eta A^*_\eta}) + \tfrac{1}{2}(\overline{AA^*})^2 = 2E \tag{A 9}$$

say. The relations (A 3), (A 6) and (A 9) correspond respectively to the conservation of action, momentum and energy.

If in equation (A 3) we substitute the gaussian form (4.1) we find on taking averages that

$$M = \tfrac{1}{2}\sum_n \overline{a_n a_n^*} = \iint N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa}. \tag{A 10}$$

Similarly from (A 6)

$$(I_1, I_2) = \iint \boldsymbol{\kappa} N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa}. \tag{A 11}$$

The expression for $(\overline{AA^*})^2$ is a quadruple sum, which on taking average reduces to

$$2\sum_n \overline{a_n a_n^*}\sum_m \overline{a_m a_m^*} - \sum_n (\overline{a_n a_n^*})^2 + O(a_n^6). \tag{A 12}$$

As the number of terms in the summation increases, the second group becomes negligible compared to the first. So to order ϵ^4 we have

$$(\overline{AA^*})^2 = 8M^2. \tag{A 13}$$

Hence (A 9) becomes

$$2E = \iint [-\tfrac{1}{8}(\lambda^2 - 2\mu^2) + 2M]\, 2N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa}, \tag{A 14}$$

that is

$$E = \iint \omega N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa}, \tag{A 15}$$

where

$$\omega = -\tfrac{1}{8}(\lambda^2 - 2\mu^2) + 2M \tag{A 16}$$

in accordance with the dispersion relation (3.2).

From the *two*-dimensional form of equation (2.4), without the term in $A_{\eta\eta}$, Zakharov & Shabat have derived an infinite sequence of conservation laws, and it should be possible to do the same for equation (2.4) in the general case also. However, this sequence of laws involves successively higher powers of ϵ (the expressions in (A 3), (A 6) and (A 9) are proportional to ϵ^2, ϵ^3 and ϵ^4 respectively. Since the original equation (2.4) is correct only to order ϵ^3, the relations beyond $n = 4$ will have no physical meaning in the present context.

Appendix B. Transformation of equation (4.11)

To calculate the energy transfer in particular cases it is convenient to transform equation (4.11) as follows. Because of the delta-function $\delta(\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3 - \boldsymbol{\kappa}_4)$ in the integrand, we can integrate with respect to $\boldsymbol{\kappa}_3$ by writing

$$\boldsymbol{\kappa}_3 = \boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_4 = \boldsymbol{\kappa}_1 + \boldsymbol{\kappa}' - \boldsymbol{\kappa}''.$$

Then, since $\boldsymbol{\kappa}_2 = \boldsymbol{\kappa}_1 + 2\boldsymbol{\kappa}'$ and $\boldsymbol{\kappa}_4 = \boldsymbol{\kappa}_1 + \boldsymbol{\kappa}' + \boldsymbol{\kappa}''$ we have

$$\frac{\partial(\boldsymbol{\kappa}_2, \boldsymbol{\kappa}_4)}{\partial(\boldsymbol{\kappa}', \boldsymbol{\kappa}'')} = 2. \tag{B 1}$$

So (4.11) becomes

$$\frac{\partial N_1}{\partial \tau} = 8\pi \int \ldots \int F(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')\,\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)\,\mathrm{d}\boldsymbol{\kappa}'\,\mathrm{d}\boldsymbol{\kappa}'', \tag{B 2}$$

where F stands for the square bracket in equation (4.11). To eliminate the remaining delta-function in equation (B 2) we now write

$$\alpha' = \lambda'^2 - 2\mu'^2, \quad \alpha'' = \lambda''^2 - 2\mu''^2, \tag{B 3}$$

so that
$$\delta(\omega_1+\omega_2-\omega_3-\omega_4) = \delta[\tfrac{1}{4}(\alpha''-\alpha')] = 4\delta(\alpha''-\alpha') \tag{B 4}$$

and if β', β'' are arbitrary functions of (λ', μ') and (λ'', μ'') then

$$J' = \frac{\partial(\alpha', \beta')}{\partial(\lambda', \mu')} = 2\lambda'\frac{\partial\beta'}{\partial\mu'} + 4\mu'\frac{\partial\beta'}{\partial\lambda'}, \tag{B 5}$$

with a similar expression for J''. Hence (B 2) becomes

$$\frac{\partial N_1}{\partial\tau} = 32\pi\iiint\frac{F\,\mathrm{d}\alpha\,\mathrm{d}\beta'\,\mathrm{d}\beta''}{|J'J''|} \tag{B 6}$$

in which α', α'' are both set equal to α.

When $\alpha > 0$ we may take simply $\beta' = \mu'$ and $\beta'' = \mu''$ so $J' = 2\lambda'$ and $J'' = 2\mu''$. Then substituting

$$\left.\begin{aligned}\lambda' &= \pm\alpha^{\frac{1}{2}}\cosh\theta', \quad \surd 2\,\mu' = \alpha^{\frac{1}{2}}\sinh\theta',\\ \lambda'' &= \pm\alpha^{\frac{1}{2}}\cosh\theta'', \quad \surd 2\,\mu'' = \alpha^{\frac{1}{2}}\sinh\theta'',\end{aligned}\right\} \tag{B 7}$$

we find for the contribution to $\partial N_1/\partial\tau$

$$4\pi\sum\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty F\,\mathrm{d}\alpha\,\mathrm{d}\theta'\,\mathrm{d}\theta'' \tag{B 8}$$

the sum being taken over the four combinations of signs for λ' and λ''.

When $\alpha < 0$ we make precisely similar substitutions except that α is replaced by $-\alpha$.

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