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Integral properties of periodic gravity waves of finite amplitude

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A number of exact relations are proved for periodic water waves of finite amplitude in water of uniform depth. Thus in deep water the mean fluxes of mass, momentum and energy are shown to be equal to $2T/c$, $(4T-3V)$ and $(3T-2V)c$ respectively, where T and V denote the kinetic and potential energies and c is the phase velocity. Some parametric properties of the solitary wave are here generalized, and some particularly simple relations are proved for variations of the Lagrangian $(T-V)$. The integral properties of the wave are related to the constants Q, R and S which occur in cnoidal wave theory.

The speed, momentum and energy of deep-water waves are calculated numerically by a method employing a new expansion parameter. With the aid of Padé approximants, convergence is obtained for waves having amplitudes up to and including the highest. For the highest wave, the computed speed and amplitude are in agreement with independent calculations by Yamada and Schwartz. At the same time the computations suggest that the speed and energy, for waves of a given length, are greatest when the height is less than the maximum. In this respect the present results tend to confirm previous computations on solitary waves.

1. INTRODUCTION

In a recent paper (Longuet-Higgins 1974, to be referred to as (I)) some new relations were found between certain fundamental integral properties of solitary waves in water. These relations were used to obtain simple approximations to the profile of the solitary wave of maximum amplitude and, in a second paper (Longuet-Higgins & Fenton 1974) to assist in the accurate calculation of the speed, energy, momentum, etc., of solitary waves of arbitrary amplitude.

One of the unexpected findings of the second paper (II) was that the speed and energy of solitary waves attain maxima for waves of *less* than the maximum amplitude. This property has possible implications for the manner in which waves break in shallow water.

There is naturally some interest in the question whether all gravity waves of finite amplitude have a similar property. The present paper was stimulated by an attempt to answer this question for periodic gravity waves and in particular for gravity waves in deep water. The answer, as we shall see, is in the affirmative.

In the first part of the paper we deduce a number of *exact* relations for periodic waves in water of uniform depth (and in particular when the depth is large compared to a wavelength). The analysis for waves of finite length differs in some significant respects from that for solitary waves, and some new definitions are required.

Let rectangular coordinates (x, y) be chosen with the x-axis horizontal and the y-axis vertically upwards. Let the equations of the free surface and the bottom be $y = \eta$ and $y = -h$ respectively. The velocity (u, v) is assumed irrotational $(= \nabla\phi)$ and periodic in x with wavelength λ.

We choose axes so that the mean elevation $\overline{\eta}$, given by

$$\lambda\overline{\eta} = \int_0^\lambda \eta \,\mathrm{d}x = M \tag{1.1}$$

is zero. So the origin lies in the mean surface level, and h equals the mean depth (or the mass per unit horizontal distance).

Similarly by choosing axes moving with the required horizontal velocity, we may make the mean velocity $\overline{u}$, defined by

$$\lambda\overline{u} = \int_0^\lambda u \,\mathrm{d}x = [\phi]_{x=0}^{\lambda} = C \tag{1.2}$$

vanish at one particular level and hence (since the motion is irrotational) at all levels within the fluid.

The wavelength λ being finite, the vanishing of $\overline{\eta}$ and $\overline{u}$ implies that both M and C must vanish, whereas for the solitary wave both these quantities are positive (see (I)).

We define the mean wave momentum or *impulse* per unit horizontal distance

$$I = \overline{\int_{-h}^{\eta} u \,\mathrm{d}y} \tag{1.3}$$

(an overbar denotes the average over one wavelength or period), the mean kinetic energy

$$T = \overline{\int_{-h}^{\eta} \tfrac{1}{2}(u^2 + v^2) \,\mathrm{d}y}, \tag{1.4}$$

and the mean potential energy

$$V = \overline{\int_0^{\eta} gy \,\mathrm{d}y}. \tag{1.5}$$

We also define the *radiation stress* (the excess flux of momentum due to the waves)

$$S_{xx} = \overline{\int_{-h}^{\eta} (p + u^2) \,\mathrm{d}y - \tfrac{1}{2}gh^2}, \tag{1.6}$$

and the mean energy flux

$$F = \overline{\int_{-h}^{\eta} [p + \tfrac{1}{2}(u^2 + v^2) + gy]\, u \,\mathrm{d}y}. \tag{1.7}$$

Between these quantities some simple relations have been established. Thus Levi-Cività (1924) showed that

$$2T = cI, \tag{B}$$

where c is the phase velocity. A short proof is given in § 2 of this paper, where we also deduce some alternative expressions for the kinetic energy T. Next in § 3 we prove that

$$S_{xx} = 4T - 3V + h\overline{u_{\mathrm{b}}^2}, \tag{C}$$

where $\overline{u_{\mathrm{b}}^2}$ denotes the mean velocity on the bottom. We show that (C) is another form of a relation due to Starr (1947). It becomes particularly simple in deep water, when u_{b} tends to zero. Similarly we prove that in general

$$F = (3T - 2V)c + \tfrac{1}{2}(ch + I)\,\overline{u_{\mathrm{b}}^2},$$

a relation not found previously. In deep water this also simplifies, so that the three fluxes of mass, momentum and energy are given respectively by

$$I = 2T/c, \quad S_{xx} = 4T - 3V, \quad F = (3T - 2V)c,$$

(the first relation remains true in water of any depth).

In § 4 we deduce some new parametric relations between these integral properties, which govern the rates of increase of T, V and c when the wave amplitude is allowed to vary, with the depth and wavelength fixed. For instance we show that

$$\mathrm{d}(T/c^2) = (1/c^2)\,\mathrm{d}V, \tag{D}$$

which generalizes a result proved previously for the solitary wave. (Equations (B), and (C) and (D) are analogues of the corresponding equations of paper (I).) from (D) can be deduced a number of interesting relations, particularly some involving the Lagrangian $(T - V)$.

Finally in § 5 these integral properties of the waves will be related to the quantities Q, R and S introduced into shallow-water theory by Benjamin & Lighthill (1954). It is noted that the integral quantities I, T and V may have some advantages as parameters of the wave motion, particularly in deep water.

We emphasize that all the relations just mentioned are exact, and do not in any way depend upon the approximations either of small-amplitude wave theory or of cnoidal wave theory.

In the second part of the paper we use some of the above relations to calculate the values of I, T, V and c for waves in deep water. The small-amplitude expansions for this case are quite different from those used previously for solitary waves. Nevertheless a similar result appears, namely that the speed, momentum and energy all appear to attain their maximum values for wave amplitudes *less* than the limiting amplitude (for which the crest angle is 120°). Moreover, the calculated values of the speed and height of waves of limiting amplitude are in very good agreement with the independent calculations of Yamada (1957) and Schwartz (1974). Further discussion of these results is given in § 7.

2. Momentum and kinetic energy

Let $-Q$ be the mass flux in the steady flow relative to an observer moving with the phase velocity c, that is let

$$\int_{-h}^{\eta} (u-c)\,\mathrm{d}y = -Q. \tag{2.1}$$

On integrating each side with respect to x between 0 and λ we have

$$\lambda I - c(\lambda h + M) = -\lambda Q, \tag{2.2}$$

where M is given by (1.1). Since M is taken to be zero this gives

$$ch - I = Q, \tag{A}$$

a relation analogous to equation (A) of (I).

Now to prove equation (B) of § 1 we have from (1.4)

$$\begin{aligned} 2\lambda T &= \int_0^{\lambda}\int_{-h}^{\eta} \{[(u-c)+c]^2 + v^2\}\,\mathrm{d}y\,\mathrm{d}x \\ &= \iint [(u-c)^2+v^2]\,\mathrm{d}x\,\mathrm{d}y + 2c\iint (u-c)\,\mathrm{d}x\,\mathrm{d}y + c^2\iint \mathrm{d}x\,\mathrm{d}y \\ &= \iint \mathrm{d}\Phi\,\mathrm{d}\Psi - 2c\lambda Q + c^2(\lambda h + M), \end{aligned}$$

where Φ and Ψ denote the velocity potential and stream function of the steady flow, since $\partial(\Phi, \Psi)/\partial(x,y) = [(u-c)^2+v^2]$. But

$$[\Phi]_{x=0}^{\lambda} = C - \lambda c, \qquad [\Psi]_{y=-h}^{\eta} = -Q.$$

Hence

$$\begin{aligned} 2\lambda T &= -Q(C-\lambda c) - 2c\lambda Q + c^2(\lambda h + M) \\ &= -QC - c\lambda Q + c^2(\lambda h + M) \\ &= -QC + c\lambda I, \end{aligned} \tag{2.3}$$

by (2.2). Taking $C = 0$, we arrive at equation (B).

It is worth noting that in some physical circumstances it is appropriate to assume $I = 0$ (the total horizontal flux is zero) rather than $C = 0$. Then equation (2.3) leads to

$$2T = -QC/\lambda, \tag{2.4}$$

a different result in general.

Equations (A) and (B) have some further consequences. Since $(x+\mathrm{i}y)$ is an analytic function of $(\Phi + \mathrm{i}\Psi)$ we have, by the Cauchy–Riemann relations

$$\iint \frac{\partial y}{\partial \Psi}\mathrm{d}\Phi\,\mathrm{d}\Psi = \iint \frac{\partial x}{\partial \Phi}\mathrm{d}\Phi\,\mathrm{d}\Psi,$$

where the integral may be taken over one wavelength, and from bottom to free surface. Hence

$$\int (h+\eta)\,\mathrm{d}\Phi = -\lambda Q.$$

Since $\Phi = \phi - cx$ and $\mathrm{d}\Phi = \mathrm{d}\phi - c\,\mathrm{d}x$ we have

by (A). Hence
$$\int (h+\eta)\,\mathrm{d}\phi = \lambda(ch-Q) = \lambda I$$

$$\int \eta\,\mathrm{d}\phi = \lambda I - hC. \tag{2.5}$$

Taking $C = 0$, equation (B) now gives

$$2T = (c/\lambda)\int \eta\,\mathrm{d}\phi. \tag{2.6}$$

This is analogous to equation (2.2) of (I).

Further we have

$$\mathrm{d}\phi = \mathrm{d}\Phi + c\,\mathrm{d}x = -q\,\mathrm{d}s + c\,\mathrm{d}x, \tag{2.7}$$

where q is the speed at the free surface in the steady motion. Writing

$$p + \tfrac{1}{2}[(u-c)^2 + v^2] + g(y+h) = R, \tag{2.8}$$

for the total head we have

$$q^2 = 2R - 2g(h+\eta).$$

Hence altogether (2.6) yields

$$2T = (c/\lambda)\int_0^\lambda \{c - (1+\eta'^2)^{\frac{1}{2}}[2R - 2g(h+\eta)]^{\frac{1}{2}}\}\,\mathrm{d}x, \tag{E}$$

where $\eta' = \mathrm{d}\eta/\mathrm{d}x$. This expresses the kinetic energy as an integral involving only the surface elevation η and other constants of the motion.

3. Momentum flux and potential energy

To prove relation (C), let us assume that M and C both vanish, and consider Bernoulli's equation in the form

$$[p + (u-c)^2] + (gy - c^2) + v^2 + (p + gy) = 2B, \tag{3.1}$$

where clearly we have from equation (2.8)

$$B = R - gh - \tfrac{1}{2}c^2. \tag{3.2}$$

Now by the equation of vertical momentum

$$(y+h)\left[\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{\partial}{\partial y}(p+gy)\right] = 0. \tag{3.3}$$

Adding (3.1) and (3.3) and rearranging terms we have

$$[p + (u-c)^2 + (gy - c^2)] + \frac{\mathrm{D}}{\mathrm{D}t}[(y+h)v] + \frac{\partial}{\partial y}[(y+h)(p+gy)] = 2B. \tag{3.4}$$

On integrating the first group of terms on the left of (3.4) we get

$$\int_0^\lambda \int_{-h}^{\eta} [p + (u-c)^2 + (gy - c^2)]\,\mathrm{d}y\,\mathrm{d}x = \lambda(S_{xx} - 2cI + V).$$

When integrated similarly, the second group of terms in (3.4) vanishes, while the third group yields (since p vanishes at the free surface)

$$\int_0^\lambda (h+\eta)\,g\eta\,\mathrm{d}x = ghM + 2\lambda V = 2\lambda V.$$

Altogether we find $$\lambda(S_{xx} - 2cI + 3V) = 2\lambda Bh,$$

and so $$S_{xx} = 4T - 3V + 2Bh. \tag{3.5}$$

A simple expression for B can be found as follows. Consider the total vertical momentum of the fluid over one wavelength, between $x = 0$ and $x = \lambda$, say. Its time-rate of change is zero. Since this rate of change is the result of the external forces acting on it we must have simply

$$\int_0^\lambda p_\mathrm{b}\,\mathrm{d}x = \lambda gh,$$

where p_b is the pressure on the bottom. The fluxes of vertical momentum across the two planes $x = 0, \lambda$ just cancel, by periodicity. So on integrating both sides of (3.1) over one wavelength at $y = -h$ we obtain

$$2B = \frac{1}{\lambda}\int_0^\lambda u_\mathrm{b}^2\,\mathrm{d}x = \overline{u_\mathrm{b}^2}, \tag{3.6}$$

where u_b denotes the velocity on the bottom.

On substituting for B into equation (3.5) we obtain the relation (C).

In deep water it can be shown (see, for example, Longuet-Higgins 1953) that (u^2+v^2) decreases exponentially with y, in any motion which is irrotational, incompressible and periodic in x, and such that $\overline{u} = 0$. Hence as $h \to \infty$

$$Bh \to 0,$$

and (C) reduces to $$S_{xx} = 4T - 3V. \tag{C$'$}$$

Note that for waves of infinitesimal amplitude $T \doteqdot V \doteqdot \frac{1}{2}E$, where E is the total density of energy per unit horizontal area. Equation (C$'$) then reduces to

$$S_{xx} \doteqdot \tfrac{1}{2}E,$$

the well-known relation for waves of small amplitude (Longuet-Higgins & Stewart 1960).

It can be shown that equation (C) is related to equation (4.5) of Starr (1947), which involves the difference in kinetic energies of the horizontal and vertical motions.

The mean energy flux F defined in § 1 can be expressed in terms of the other quantities. For from (1.7) and (3.1) we have

$$F = \overline{\int_{-h}^{\eta} (B+cu)\,u\,\mathrm{d}y} = BI + c\overline{\int_{-h}^{\eta} u^2\,\mathrm{d}y} \tag{3.7}$$

and from (1.6)

$$S_{xx}+V = \overline{\int_{-h}^{\eta}(p+gy)\,\mathrm{d}y}+\overline{\int_{-h}^{\eta}u^2\,\mathrm{d}y}. \tag{3.8}$$

But from (3.1)
$$B = (p+gy)+\tfrac{1}{2}(u^2+v^2)-cu.$$

So on integrating with respect to y and taking mean values,

$$Bh = \overline{\int_{-h}^{\eta}(p+gy)\,\mathrm{d}y}+T-cI.$$

Using equation (B) we have then

$$\overline{\int_{-h}^{\eta}(p+gy)\,\mathrm{d}y} = T+Bh. \tag{3.9}$$

From (3.7), (3.8) and (3.9) it follows that

$$F = BI+c(S_{xx}+V-T-Bh),$$

and so on substituting for S_{xx} from (3.5) we find

$$F = (3T-2V)c+(I+ch)B. \tag{3.10}$$

In deep water, when $Bh\to 0$, we have simply

$$F = (3T-2V)c. \tag{3.11}$$

For waves of small amplitude this reduces to

$$F \doteqdot \tfrac{1}{2}Ec,$$

the usual formula, since $\frac{1}{2}c$ equals the group velocity.†

4. Differential relations

We shall now prove some differential relations between I, T, V and c by using a variational method analogous to that used in paper (I) (see also Luke (1967) and Benjamin (1973)).

The total energy λE over one wavelength may be written

$$\lambda E = \int_0^{\lambda}\int_{-h}^{\eta}\tfrac{1}{2}(\phi_x^2+\phi_y^2)\,\mathrm{d}y\,\mathrm{d}x+\int_0^{\lambda}\tfrac{1}{2}g\eta^2\,\mathrm{d}x, \tag{4.1}$$

where ϕ is the velocity potential. The motion being progressive, the kinematical and dynamical conditions at the free surface can be written

$$\left.\begin{aligned}\phi_x\eta_x-\phi_y &= c\eta_x,\\ \tfrac{1}{2}(\phi_x^2+\phi_y^2)+g\eta &= c\phi_x+\text{constant.}\end{aligned}\right\} \tag{4.2}$$

† The relations (3.10) and (3.11) are not among those given by Starr & Platzman (1948). Equation (32.48) of Wehausen & Laitone (1960) appears to be true only if $B = 0$ and $c = 1$.

Now let $\delta\phi$ and $\delta\eta$ denote arbitrary small variations of the velocity potential and surface elevation such that $\delta\phi_x$ and $\delta\eta$ are *periodic* with period λ. It follows that

$$[\delta\phi]_{\mathbf{x}}^{\mathbf{x}+\lambda} = \delta C, \tag{4.3}$$

independently of the point $\mathbf{x} = (x, y)$. Assuming that $\delta\bar{\eta}] = 0$ we have

$$\begin{aligned}\lambda\,\delta E &= \int_0^\lambda \int_{-h}^\eta (\phi_x\,\delta\phi_x + \phi_y\,\delta\phi_y)\,\mathrm{d}y\,\mathrm{d}x + \int_0^\lambda [\tfrac{1}{2}(\phi_x^2+\phi_y^2)+g\eta]_s\,\delta\eta\,\mathrm{d}x \\ &= \int (\delta\phi)_s \frac{\partial\phi}{\partial n}\mathrm{d}s + \int_{-h}^\eta [\delta\phi]\,\phi_x\,\mathrm{d}y + \int_0^\lambda c(\phi_x)_s\,\delta\eta\,\mathrm{d}x\end{aligned}$$

by Green's theorem and (4.2), where a suffix s denotes the value at the free surface, and

$$\frac{\partial\phi}{\partial n} = (-\phi_x\eta_x + \phi_y)\frac{\mathrm{d}x}{\mathrm{d}s} = -c\eta_x\frac{\mathrm{d}x}{\mathrm{d}s},$$

by (4.2). Hence

$$\lambda\,\delta E = c\int_0^\lambda [(\phi_x)_s\,\delta\eta - (\delta\phi)_s\,\eta_x]\,\mathrm{d}x + \delta C\int_{-h}^\eta \phi_x\,\mathrm{d}y. \tag{4.4}$$

Also, starting from
$$\lambda I = \int_0^\lambda \int_{-h}^\eta \phi_x\,\mathrm{d}y\,\mathrm{d}x,$$

we find
$$\lambda\,\delta I = \int_0^\lambda \int_{-h}^\eta \delta\phi_x\,\mathrm{d}y\,\mathrm{d}x + \int_0^\lambda (\phi_x)_s\,\delta\eta\,\mathrm{d}x, \tag{4.5}$$

and since
$$\frac{\partial}{\partial x}\int_{-h}^\eta \delta\phi\,\mathrm{d}y = \int_{-h}^\eta \delta\phi_x\,\mathrm{d}y + \eta_x\,(\delta\phi)_s,$$

we have on integration with respect to x,

$$(h+\eta)\,\delta C = \int_0^\lambda \int_{-h}^\eta \delta\phi_x\,\mathrm{d}y\,\mathrm{d}x + \int_0^\lambda \eta_x(\delta\phi)_s\,\mathrm{d}x. \tag{4.6}$$

From (4.4), (4.5) and (4.6) it follows that

$$\lambda(\delta E - c\,\delta I) = \left[\int_{-h}^\eta \phi_x\,\mathrm{d}y - c(h+\eta)\right]\delta C = -Q\,\delta C. \tag{4.7}$$

Taking $\delta C = 0$ as before we have

$$\delta E = c\,\delta I. \tag{4.8}$$

Suppose now that the variations $\delta\phi$ and $\delta\eta$ are specialized so as to describe the growth of a wave of fixed wavelength λ but variable amplitude. Then we may write

$$\mathrm{d}(T+V) = c\,\mathrm{d}I. \tag{4.9}$$

But in addition equation (B) will apply, so that we have also

$$2\,\mathrm{d}T = c\,\mathrm{d}I + I\,\mathrm{d}c. \tag{4.10}$$

Subtracting, we get
$$\mathrm{d}(T-V) = I\,\mathrm{d}c. \tag{4.11}$$

These relations are identical with those for solitary waves (see (I) § 3) except that now I, T and V denote densities of momentum and energy. It follows immediately that

$$\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{T}{c^2}\right) = \frac{1}{c^2}\frac{\mathrm{d}V}{\mathrm{d}c}, \tag{D}$$

$$T = c^2\int\frac{\mathrm{d}V}{c^2} = V - c^2\int V\,\mathrm{d}\left(\frac{1}{c^2}\right), \tag{D$'$}$$

and if $L = T - V$ denotes the Lagrangian, then

$$\frac{\mathrm{d}L}{\mathrm{d}c} = \frac{2T}{c} = I, \tag{D$''$}$$

$$\frac{\mathrm{d}(L/c)}{\mathrm{d}(1/c)} = -E, \tag{D$'''$}$$

$$L = \int I\,\mathrm{d}c = \int T\frac{\mathrm{d}c^2}{c^2} = -c^2\int V\,\mathrm{d}\left(\frac{1}{c^2}\right) = -c\int E\,\mathrm{d}\left(\frac{1}{c}\right). \tag{D$''''$}$$

Equation (4.9) may be further generalized to include variations in the wavelength λ and depth h by the following argument.

If in (4.1) we allow a change δh in the depth but keep λ constant, then an extra term

$$\int_0^\lambda \tfrac{1}{2}u_{\mathrm{b}}^2\,\mathrm{d}x\,\delta h = \lambda B\,\delta h,$$

must be added to the right-hand side of (4.4), and an extra term

$$\int_0^\lambda u_{\mathrm{b}}\,\mathrm{d}x\,\delta h = C\,\delta h,$$

must be added to (4.5). Since $C = 0$ this means that equation (4.8) becomes in general

$$\delta E = c\,\delta I + B\,\delta h. \tag{4.12}$$

Hence for the special variations appropriate to free waves we have in place of (4.11) the more general relation

$$\mathrm{d}L = I\,\mathrm{d}c - B\,\mathrm{d}h. \tag{4.13}$$

If we now include variations in the wavelength λ, equation (4.13) becomes

$$\mathrm{d}L = I\,\mathrm{d}c - B\,\mathrm{d}h + \kappa\,\mathrm{d}\lambda, \tag{4.14}$$

where the coefficient κ remains to be determined. Consider a variation which simply enlarges the scale of the wave, keeping the shape constant, then since λ is proportional to h and since $L \propto h^2$ and $c \propto h^{\frac{1}{2}}$ we have from (4.14)

$$2L = \tfrac{1}{2}cI - Bh + \kappa\lambda, \tag{4.15}$$

or on using equation (B)

$$\kappa = (T - 2V + Bh)/\lambda. \tag{4.16}$$

Altogether then we have

$$\mathrm{d}L = 2T\,\mathrm{d}c/c + (T - 2V + Bh)\,\mathrm{d}\lambda/\lambda - B\,\mathrm{d}h. \tag{4.17}$$

5. Relation to Q, R and S

We now seek to express the above relations in terms of the constants Q, R and S used by Benjamin & Lighthill (1954) in their approximate theory of cnoidal waves.

Q and R have already been defined in (2.1) and (2.8) respectively. The third constant S is the momentum flux in the steady motion:

$$S = \int_{-h}^{\eta} [p + (u-c)^2]\, dy. \tag{5.1}$$

On expanding the right-hand side and taking mean values it is readily seen that

$$S = S_{xx} - 2cI + h(c^2 + \tfrac{1}{2}gh). \tag{5.2}$$

To express I, T and V in terms of Q, R and S we have, first, from equation (A),

$$I = ch - Q, \tag{5.3}$$

then from equation (B), § 1,
$$2T = c(ch - Q), \tag{5.4}$$

and on eliminating $(S_{xx} - 2cI)$ and B from (3.2), (3.5) and (5.2),

$$3V = 2hR - S - \tfrac{3}{2}gh^2. \tag{5.5}$$

The radiation stress, from (5.2) and (5.3), is given by

$$S_{xx} = S - 2cQ + h(c^2 - \tfrac{1}{2}gh), \tag{5.6}$$

while from (3.9) we find after some reduction

$$F = Q(gh - c^2 - R) + \tfrac{2}{3}c(Rh + S) + ch(\tfrac{1}{2}c^2 - gh). \tag{5.7}$$

From § 4 we can also write down differential relations for Q, R, S and c, but these appear less simple.

We note that apart from an arbitrary phase constant a gravity wave is uniquely defined by three parameters, for instance the wavelength, wave height and the mean depth. The three quantities Q, R and S are particularly suitable for use in shallow water, when $h \ll \lambda$. However, in deep water when $h/\lambda \to \infty$ we have

$$\left.\begin{aligned} Q &\sim ch - I_\infty, \\ R &\sim gh + \tfrac{1}{2}c^2, \\ S &\sim \tfrac{1}{2}gh^2 + c^2h - 3V_\infty, \end{aligned}\right\} \tag{5.8}$$

where the suffix ∞ denotes the limiting value as $h/\lambda \to \infty$. Hence Q, R and S all tend to infinity, for fixed λ.

On the other hand all the integral quantities I, T, V, S_{xx} and F remain finite as $h/\lambda \to \infty$. Any three of these quantities, within certain ranges, would serve to define the motion. However no exact differential equation for the wave profile, with these quantities as parameters, has yet been given.

6. Calculation of c, T and V for waves in deep water

We shall now apply some of the foregoing analysis to the calculation of the speed and energy of finite-amplitude waves in deep water.

Following Stokes (1880) we consider the motion in a frame of reference moving with the phase velocity, and take the velocity potential Φ and stream-function Ψ as independent variables. Let

$$\exp\left[\mathrm{i}(\Phi+\mathrm{i}\Psi)/c\right] = W \tag{6.1}$$

say. On the free surface we have

$$\Psi = 0, \quad W = \mathrm{e}^{\mathrm{i}\Phi/c}, \tag{6.2}$$

and as $y \to -\infty$, so

$$\Psi \sim -cy, \quad W \to 0. \tag{6.3}$$

In general $(x+\mathrm{i}y)$ is expanded in the Fourier series

$$(x+\mathrm{i}y) = \mathrm{i}\left(\ln W + a_0 + a_1 W + \tfrac{1}{2}a_2 W^2 + \ldots + \frac{1}{n}a_n W^n + \ldots\right) \tag{6.4}$$

(the wavelength being normalized to 2π) and hence the particle velocity (U, v) is given by

$$\frac{1}{U-\mathrm{i}v} = \frac{\mathrm{d}(x+\mathrm{i}y)}{\mathrm{d}(\Phi+\mathrm{i}\Psi)} = (1 + a_1 W + a_2 W^2 + \ldots + a_n W^n + \ldots). \tag{6.5}$$

The coefficients $a_1, a_2, \ldots$, which are all real, are determined from the constant pressure condition at the free surface, which can be written

$$|U-\mathrm{i}v|^2 + 2g(y-a_0) = K \quad \text{on} \quad \Psi = 0, \tag{6.6}$$

the constant a_0 being at our disposal. Equating coefficients of $\cos(n\Phi/c)$ in this equation we get a sequence of relations to be satisfied by $a_1, a_2, a_3, \ldots, c^2$ and K (see, for example, Schwartz 1974).

Schwartz assumes expansions in powers of a small parameter ϵ, dependent on the wave amplitude, in the form

$$\left.\begin{aligned} a_j &= \sum_{k=0}^{\infty} \alpha_{jk}\epsilon^{j+2k}, \\ c^2 &= \sum_{k=0}^{\infty} \gamma_k \epsilon^{2k}, \\ K &= \sum_{k=0}^{\infty} \delta_k \epsilon^{2k}, \end{aligned}\right\} \tag{6.7}$$

and gives algorithms for the successive calculation of the coefficients α_{jk}, γ_k, δ_k. As expansion parameter Stokes (1880) chose $\epsilon = a_1$. Schwartz (1974) shows that a_1 is not a monotonically increasing function of the wave height, and prefers instead $\epsilon = a$, where a is the wave amplitude, defined by

$$a = \tfrac{1}{2}(\eta_{\text{crest}} - \eta_{\text{trough}}). \tag{6.8}$$

Thus for instance it is found that

$$\left.\begin{aligned} c^2 &= 1+a^2+\tfrac{1}{2}a^4+\tfrac{1}{4}a^6-\tfrac{22}{45}a^8-\ldots, \\ K &= 1+2a^2-\tfrac{1}{2}a^4-\tfrac{5}{6}a^6-\tfrac{301}{80}a^8-\ldots. \end{aligned}\right\} \tag{6.9}$$

From these expressions we can calculate not only c but also I and T in powers of a, for from equation (2.5) we have

$$I = \frac{1}{\lambda}\int \eta \,\mathrm{d}\phi = \frac{1}{\lambda}\int \eta(c\,\mathrm{d}x+\mathrm{d}\Phi) = c(\overline{\eta}-a_0), \tag{6.10}$$

while on letting $y \to -\infty$ in Bernoulli's theorem we find from (6.6) that

$$K = c^2+2g(\overline{\eta}-a_0). \tag{6.11}$$

With $g = 1$ the last two equations yields

$$I = \tfrac{1}{2}c(K-c^2), \tag{6.12}$$

and so

$$T = \tfrac{1}{2}cI = \tfrac{1}{4}c^2(K-c^2). \tag{6.13}$$

In calculating the potential energy V from equation (1.5) we must be careful to choose a_0 so that the mean level $\overline{\eta}$ vanishes. Alternatively we can let $a_0 = 0$, as in Schwartz (1974), and then use the more general relation

$$V = \tfrac{1}{2}g(\overline{\eta^2}-\overline{\eta}^2). \tag{6.14}$$

In terms of $a_1, a_2, \ldots$ we have, if $a_0 = 0$,

$$\overline{\eta^2} = \frac{1}{2\pi c}\int_0^{2\pi c}\left(y^2\frac{\partial x}{\partial \Phi}\right)_{\Psi=0}\mathrm{d}\Phi = \sum_{n=2}^{\infty}\sum_{l+m=n}\frac{3a_l a_m a_n}{4lm}+\sum_{m=1}^{\infty}\frac{a_m^2}{2m^2},$$

and

$$\overline{\eta} = \frac{1}{2\pi c}\int_0^{2\pi c}\left(y\frac{\partial x}{\partial \Phi}\right)_{\Psi=0}\mathrm{d}\Phi = \sum_{m=1}^{\infty}\frac{a_m^2}{2m}.$$

Using the expansions of the coefficients a_n in powers of a as given by Schwartz we find that

$$\left.\begin{aligned} T &= \tfrac{1}{4}a^2 \qquad\quad -\tfrac{19}{48}a^6-\tfrac{3317}{2880}a^8-\ldots, \\ V &= \tfrac{1}{4}a^2-\tfrac{1}{8}a^4-\tfrac{19}{48}a^6-\tfrac{3077}{2880}a^8-\ldots, \end{aligned}\right\} \tag{6.15}$$

and

$$(T-V) = \tfrac{1}{8}a^4-\tfrac{1}{12}a^8-\tfrac{1017}{2880}a^{10}-\ldots. \tag{6.16}$$

These expansions confirm those obtained by Platzman (1947), who expanded T and V in powers of $\beta = (c^2-1)^{\frac{1}{2}}$.

However, for waves of large amplitude a, terms of much higher order are required. Nor is it known *a priori* what is the maximum value of a, expected to correspond to the wave of limiting amplitude. For this purpose we introduce a new parameter ω defined by

$$\omega = 1-\frac{q_{\text{crest}}^2\, q_{\text{trough}}^2}{c^2 c_0^2}, \tag{6.17}$$

where q_{crest} and q_{trough} denote the particle speeds at the wave crest and wave trough respectively, and c_0 is the speed of waves of infinitesimal amplitude. This parameter has the following convenient properties:

(1) For waves of small amplitude, $\omega \ll 1$, while for waves with a sharp-angled crest, at which q_{crest} vanishes, we have $\omega = 1$. Thus the limiting value of ω is accurately known.

(2) Since from equation (6.5)

$$\left.\begin{aligned} 1/q_{\text{crest}} &= (1+a_1+a_2+a_3+\ldots)/c, \\ 1/q_{\text{trough}} &= (1-a_1+a_2-a_3+\ldots)/c, \end{aligned}\right\} \tag{6.18}$$

and a_n contains only odd or even powers of a according as n is odd or even, it follows that ω can be expanded in powers of a^{2n}. Thus

$$\omega = 5a^2 - \tfrac{17}{6}a^4 + \tfrac{931}{180}a^6 + \ldots. \tag{6.19}$$

Reverting, we have

$$a^2 = \tfrac{1}{5}\omega + \tfrac{17}{75}\omega^2 - \tfrac{353}{112500}\omega^3 - \ldots, \tag{6.20}$$

and c^2, T and V can also be expanded in powers of ω. This not only reduces drastically the length of the computation at high orders, but also improves considerably the rate of numerical convergence.

(3) It will be noticed that for solitary waves the 'trough' occurs at $x = \pm\infty$, so that $q_{\text{trough}} = c$. Hence ω becomes simply $(1 - q^2_{\text{crest}}/c_0^2)$, which is the parameter used previously by Longuet-Higgins & Fenton (1974).

Schwartz (1974) calculated the coefficients α_{jk}, γ_k and δ_k as far as the terms in a^{117}. The numerical values of these coefficients were kindly supplied to the present author by E. D. Cokelet, who had recalculated them on the I.B.M. 370 in quadruple precision (about 32 decimal places). The present author then inverted the series for ω and expressed c^2, T and V as power series in ω. Calculations were carried to order ω^{40} (that is to a^{80}) beyond which point rounding errors prevent any further gain in accuracy.

As an important check on the algebra, the coefficients of ω^n on each side of the identity

$$T - V = \int T \frac{\mathrm{d}c^2}{c^2}$$

(see § 4) were compared and were found to be in agreement to one part in 10^p, where p ranged from 32 when $n = 4$, down to 21 when $n = 40$.

The series for α^2, c^2, I, T and V were then summed by $[N, N]$ Padé approximants. Surprisingly good convergence was obtained right up to the limiting value $\omega = 1$ (see, for example, table 1). The final values as determined in this way are given in table 2, and also shown graphically in figures 1 and 2.

With the alternative expansion parameter

$$\omega' = 1 - q^2_{\text{crest}}\, q^2_{\text{trough}}/c^4,$$

the rate of convergence was less rapid, but the limiting values were consistent with those in table 2.

TABLE 1. SUCCESSIVE PADÉ APPROXIMANTS $[N, N]$ TO THE SQUARE OF THE WAVE SPEED c^2, FOR VALUES OF ω CLOSE TO THE MAXIMUM

N	$\omega = 0.96$	$\omega = 0.97$	$\omega = 0.98$	$\omega = 0.99$	$\omega = 1.00$
10	1.194453	1.19417	1.19370	1.19315	1.1927
11	1.194454	1.19417	1.19371	1.19316	1.1927
12	1.194458	1.19418	1.19374	1.19324	1.1930
13	1.194458	1.19419	1.19374	1.19325	1.1931
14	1.194458	1.19419	1.19374	1.19325	1.1931
15	1.194458	1.19419	1.19374	1.19326	1.1931
16	1.194458	1.19419	1.19374	1.19326	1.1931

TABLE 2. CALCULATED VALUES OF I, T, V, c AND a/π AS FUNCTIONS OF ω

ω	I	T	V	c	a/π
.10	.010003	.005052	.005001	1.01016	.045266
.20	.019961	.010187	.009978	1.02065	.064351
.30	.029778	.015357	.014880	1.03143	.079187
.40	.039315	.020492	.019633	1.04247	.091809
.50	.048374	.025485	.024135	1.05366	.102959
.55	.052638	.027879	.026245	1.05926	.108093
.60	.056664	.030169	.028231	1.06482	.112962
.65	.060391	.032318	.030061	1.07029	.117572
.70	.063746	.034282	.031696	1.07558	.121921
.75	.066633	.036002	.033089	1.08059	.125993
.80	.068936	.037403	.034180	1.08516	.129760
.85	.070507	.038392	.034899	1.08904	.133178
.90	.071176	.038856	.035165	1.09184	.136178
.91	.071188	.038876	.035158	1.09222	.136723
.92	.071157	.038871	.035130	1.09252	.137249
.93	.071085	.038839	.035082	1.09275	.137755
.94	.070972	.038783	.035016	1.09290	.138242
.95	.070823	.038703	.034932	1.09295	.138712
.96	.070644	.038604	.034836	1.09291	.13917
.97	.070449	.038493	.034734	1.09279	.13961
.98	.070260	.038382	.034638	1.09258	.14006
.99	.07012	.03830	.03457	1.0924	.14053
.00	.0701	.0383	.0346	1.0923	.1411

7. DISCUSSION

It will be seen from figure 1 and table 2 that whereas the steepness (a/π) is a monotonic function of ω, both I, T, V and c^2 apparently have maxima *before* the highest wave is reached. A similar property was found for the solitary wave by Longuet-Higgins & Fenton (1974). Both sets of calculations relied on Padé approximants, but the series expansions in the two cases were quite different. In the previous paper a physical discussion was given of the possibility of a maximum phase speed within the range of ω, and it was seen that apparently such a maximum was not unreasonable. Maxima in the magnitudes of the Fourier coefficients $a_1, a_2, \ldots$ were previously found by Schwartz (1974).

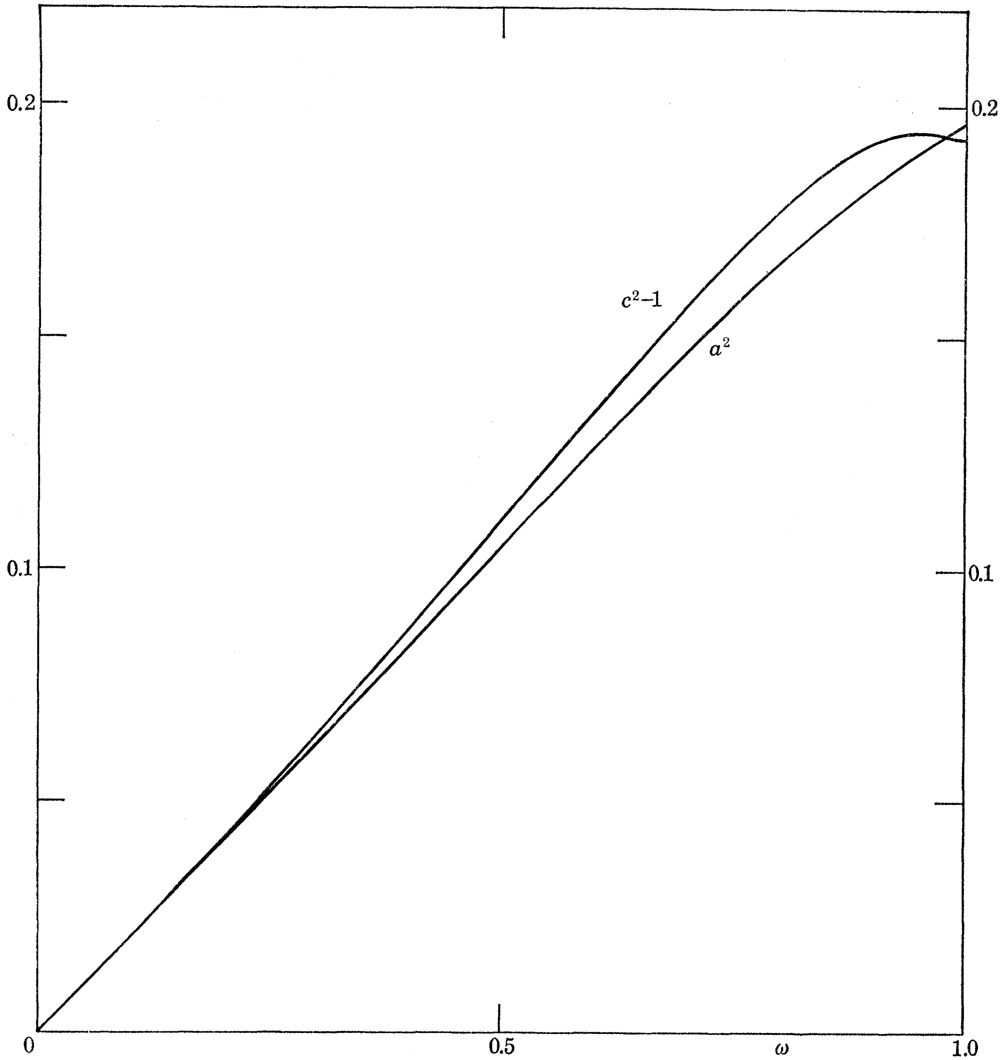

FIGURE 1. The square of the wave amplitude a and wave speed c as functions of the parameter ω.

As a check on the calculations we may compare the values of c and a in the extreme case $\omega = 1$ with the limiting values of the speed and wave height as found by previous authors.

Yamada (1957) calculated the profile of the highest wave by fitting a Fourier series at 12 points on the profile, so as to satisfy the constant pressure condition numerically. His result $c^2 = 1.1932$ is close to the value $c^2 = 1.1931$ in our table 1. Schwartz (1974) calculated $c^2 = 1.1930$ at a wave steepness $a/\pi = 0.14$ very near to the limiting wave and suggested that the speed had a stationary value (maximum) in the limit. (Our calculations suggest that if there is such a stationary value, it is a minimum.)

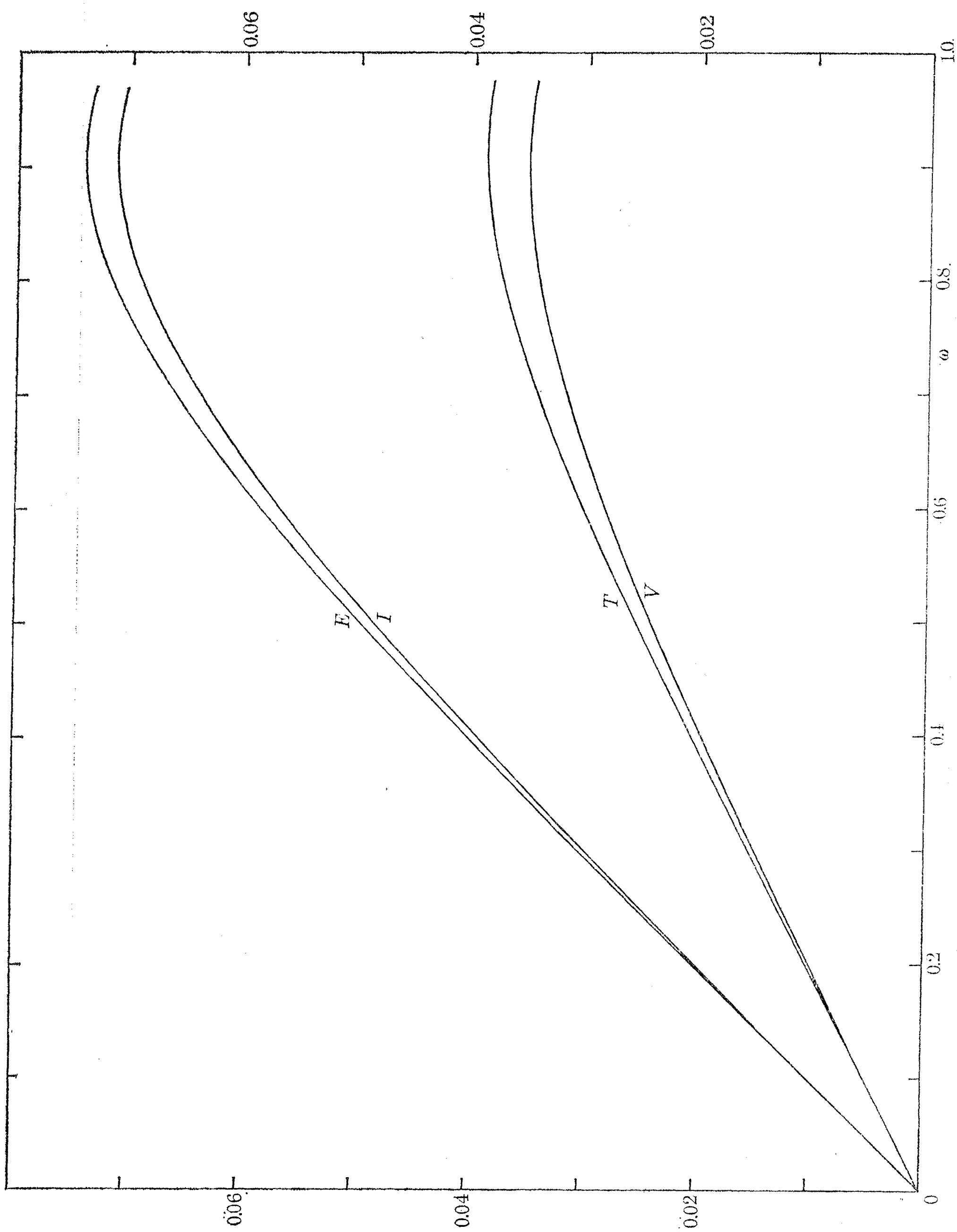

FIGURE 2. The densities of momentum I, total energy E and of the kinetic and pontential energies T and V, as functions of the parameter ω.

As regards the limiting wave amplitude, Yamada (1957) found $a/\pi = 0.0412$, and Schwartz the same value, to four places of decimals.† This is close to the limiting value 0.1411 found by the present method. Thus it would appear that the values in table 2 are not seriously in error, at least for the limiting wave.

One point at issue in the previous calculations for the solitary wave (II) was whether the small-amplitude expansion used in that paper was truly convergent, or only asymptotically valid as the amplitude tended to zero. In the present situation of waves in deep water it is known that the small-amplitude expansions in powers of h or ω are convergent for sufficiently small wave amplitude, as was proved by Levi-Cività (1925). Numerically, the radius of convergence in powers of ω is found to be close to unity, and there is no reason to doubt the completeness and convergence of the series up to and including the highest wave.

As pointed out by Longuet-Higgins & Fenton (1974), a maximum in the total energy $(T+V)$ would have some implications for wave breaking, and may contribute to the intermittency of spilling breakers as observed for example by Longuet-Higgins & Turner (1974). However the energy maximum has no apparent connexion with the instability of gravity waves in deep water which was discovered by Benjamin & Feir (1967). This latter instability can occur at much lower wave amplitudes.

References

Benjamin, T. B. 1973 Lectures on nonlinear wave motion. *Fluid Mech. Res. Inst., Univ. Essex*, Rep. no. 44.

Benjamin, T. B. & Lighthill, M. J. 1954 On cnoidal waves and bores. *Proc. R. Soc. Lond.* A **224**, 448–460.

Benjamin, T. B. & Feir, J. E. 1967 The disintegration of wave trains on deep water. Part I. Theory. *J. Fluid Mech.* **27**, 417–430.

Davies, T. V. 1951 Theory of symmetrical gravity waves of finite amplitude. *Proc. R. Soc. Lond.* A **208**, 475–486.

Grant, M. A. 1973 The singularity at the crest of a finite amplitude progressive Stokes wave. *J. Fluid Mech.* **59**, 257–262.

Havelock, T. H. 1918 Periodic irrotational waves of finite height. *Proc. R. Soc. Lond.* A **95**, 38–51.

Jeffreys, H. 1951 On the highest gravity waves on deep water. *Q. Jl Mech. appl. Math.* **4**, 385–387.

Levi-Cività, T. 1924 *Questioni di meccanica classica e relativista*, II. Bologna: Zanichelli.

Levi-Cività, T. 1925 Determination rigoureuse des ondes permanentes d'ampleur finie. *Math. Ann.* **93**, 264–314.

Longuet-Higgins, M. S. 1953 On the decrease of velocity with depth in an irrotational water wave. *Proc. Camb. Phil. Soc.* **46**, 552–560.

Longuet-Higgins, M. S. 1974 On the mass, momentum, energy and circulation of a solitary wave. *Proc. R. Soc. Lond.* A **337**, 1–13.

Longuet-Higgins, M. S. & Fenton, J. D. 1974 On the mass, momentum, energy and circulation of a solitary wave. II. *Proc. R. Soc. Lond.* A **340**, 471–493.

Longuet-Higgins, M. S. & Stewart, R. W. 1960 Changes in the form of short gravity waves on long waves and tidal currents. *J. Fluid Mech.* **8**, 565–583.

† Slightly different values were obtained by Havelock (1918) (revised by Jeffreys (1951) and Schwartz (1974)) but these were based on only four terms of an expansion which may not converge (see Grant 1973). Davies (1951) gives only three terms of a different expansion.

Longuet-Higgins, M. S. & Turner, J. S. 1974 An 'entraining-plume' model of a spilling breaker. *J. Fluid Mech.* **63**, 1–20.

Luke, J. C. 1967 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

Platzman, G. W. 1947 The partition of energy in periodic irrotational waves on the surface of deep water. *J. mar. Res.* **6**, 194–202.

Schwartz, L. W. 1974 Computer extension and analytic continuation of Stokes's expansion for gravity waves. *J. Fluid Mech.* **62**, 553–578.

Starr, V. P. 1947 Momentum and energy integrals for gravity waves of finite height. *J. mar. Res.* **6**, 175–193.

Starr, V. P. & Platzman, G. W. 1948 The transmission of energy by gravity waves of finite height. *J. mar. Res.* **7**, 229–238.

Stokes, G. G. 1880 Supplement to a paper on the theory of oscillatory waves. *Mathematical and physical papers*, pp. 314–326. Cambridge University Press.

Wehausen, J. V. & Laitone, E. V. 1960 Surface waves. In *Encyclopaedia of physics* (ed. S. Flugge), **9**, 446–778. Berlin: Springer-Verlag.

Yamada, H. 1957 Highest waves of permanent type on the surface of deep water. *Rep. Res. Inst. Appl. Mech. Kyusha Univ.* **5**, 37–57.