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On the mass, momentum, energy and circulation of a solitary wave

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Some new relations are given between the kinetic and potential energies of a solitary wave. In particular, the Lagrangian is expressed as an integral involving the total energy. Alternative proofs are also given of two other exact relations satisfied by the profile of the wave. These relations are used to derive some very simple but close approximations to the form of the wave of maximum amplitude.

1. INTRODUCTION

For many purposes, particularly for calculations of the dynamics of waves breaking in shallow water (Longuet-Higgins & Turner 1974), it is desirable to know not only the profile of a solitary wave, and how this depends on the wave amplitude, but also certain integral properties of the motion such as the total energy, mass and momentum. In particular we are interested in the wave of maximum amplitude, which as shown by Stokes (1880) develops a sharp corner at the wave crest, with an interior angle of 120°.

Despite recent calculations by Lenau (1966), Byatt-Smith (1970), Strelkoff (1971), Fenton (1972) and others, there are still some discrepancies in the determination of the wave profile, especially for high waves, nor have any reliable limits been set to the accuracy of the various determinations.

Special interest therefore attaches to the existence of certain *exact* relations satisfied by the solitary wave, some of which have been known previously, others being given in the present paper. It will be shown how these relations can be used to check the accuracy of previous calculations and to provide a simple approximation with a high degree of accuracy.

Consider a solitary wave, of arbitrary amplitude a, propagated with velocity c in water of undisturbed depth h, as in figure 1. We may define the Froude number:

$$F = c/\sqrt{(gh)},$$

where g denotes the acceleration due to gravity. Taking axes as in figure 1, with the origin in the mean level and the x-axis horizontal, let u and v denote the horizontal and vertical components of the particle velocity, and $y = \eta$ denote the surface elevation. η is assumed to tend to zero at infinity.

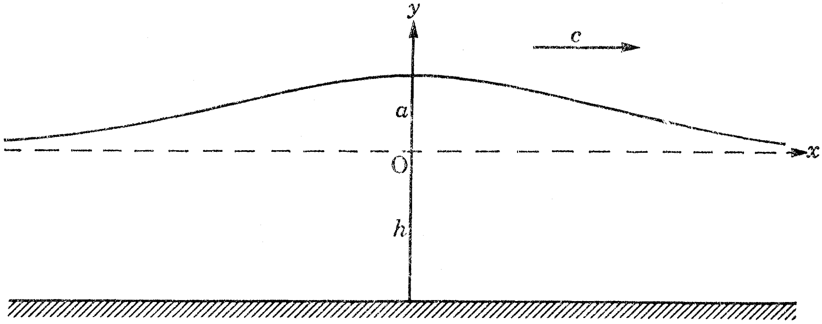

FIGURE 1. Notation and coordinates for the solitary wave.

We may define then the excess mass of the wave

$$M = \int_{-\infty}^{\infty} \eta \,\mathrm{d}x,$$

the total momentum (or impulse)

$$I = \int_{-\infty}^{\infty}\int_{-h}^{\eta} u \,\mathrm{d}y \,\mathrm{d}x,$$

the kinetic energy

$$T = \int_{-\infty}^{\infty}\int_{-h}^{\eta} \tfrac{1}{2}(u^2+v^2) \,\mathrm{d}y \,\mathrm{d}x,$$

and the potential energy

$$V = \int_{-\infty}^{\infty} \tfrac{1}{2}g\eta^2 \,\mathrm{d}x.$$

We also define the total *circulation*

$$C = \int_{-\infty}^{\infty} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{s} = [\phi]_{-\infty}^{\infty},$$

where ϕ is the velocity potential (the motion being assumed irrotational) and $\boldsymbol{u} = (u, v)$, a function of $(x-ct)$. The integral is taken along any streamline. Although $\boldsymbol{u} = \nabla\phi$ vanishes at infinity, nevertheless since the particle motion in a solitary wave is always in the direction of wave propagation, the circulation C is generally positive. In a periodic wave of finite length one can of course choose a frame of reference in which C vanishes, but for a wave of infinite length this is no longer possible.

Between the above quantities there are the following known relations

$$I = cM, \tag{A}$$

$$2T = c(I-hC), \tag{B}$$

and

$$3V = (c^2-gh)\,M. \tag{C}$$

Equation (A) has been pointed out by Starr (1947). (B) was proved by McCowan (1891), and (C) was first prove by Starr (1947). For convenience we give short but direct proofs of these results in the Appendix.

To these we shall add in the present paper the new relation

$$\mathrm{d}\left(\frac{T}{c^2}\right) = \frac{1}{c^2}\,\mathrm{d}V, \tag{D}$$

the increment being with respect to the parameter F or the proportional wave amplitude a/h. From (D) a number of other relations can be deduced (see § 3), in particular an expression for the Lagrangian $(T-V)$.

In § 2 we derive some other relations between ϕ and η, and in particular an identity involving η alone, namely

$$\int_{-\infty}^{\infty} [(h+\eta)\,(1-2\eta/F^2h)^{\frac{1}{2}}\,(1+\eta'^2)^{\frac{1}{2}} - h]\,\mathrm{d}x = 0 \tag{E}$$

which was pointed out to the author by Dr Byatt-Smith. We shall here give a shorter proof.

In § 5 we shall discuss some previous calculations of the solitary wave, and in § 6 we shall show how the above relations can be used to deduce a very simple but close approximation to the profile of the solitary wave of maximum amplitude.

2. Expressions for the kinetic energy

We note first that equation (B) enables us to write a simple expression for the kinetic energy in terms of the velocity potential ϕ. For we have

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} \phi\,\mathrm{d}y = \int_{-h}^{\eta} \frac{\partial \phi}{\partial x}\,\mathrm{d}y + (\phi)_s\frac{\partial \eta}{\partial x},$$

where $(\phi)_s$ denotes the value of ϕ at the free surface. So on integrating with respect to x,

$$hC = I + \int (\phi)_s\,\mathrm{d}\eta.$$

From equation (B) we have then

$$2T = -c\int (\phi)_s\,\mathrm{d}\eta, \tag{2.1}$$

or on integration by parts

$$2T = c\int \eta\,\mathrm{d}\phi. \tag{2.2}$$

To express the kinetic energy T entirely in terms of the surface elevation η, let

$$\Phi = \phi - cx$$

denote the velocity potential of the fluid motion as seen by an observer travelling with the phase velocity c, and let $q\ (>0)$ denote the speed at the free surface. We have then

$$\mathrm{d}\phi = \mathrm{d}\Phi + c\,\mathrm{d}x = -q\,\mathrm{d}s + c\,\mathrm{d}x.$$

Also
$$\mathrm{d}s = (1+\eta'^2)^{\frac{1}{2}}\,\mathrm{d}x$$

and by Bernoulli's equation for the steady motion

$$q^2 = c^2 - 2g\eta = F^2gh(1-2\eta/hF^2).$$

Choosing for convenience units in which $g = h = 1$ we have

$$\mathrm{d}\phi = F[1-(1-2\eta/F^2)^{\frac{1}{2}}(1+\eta'^2)^{\frac{1}{2}}]\,\mathrm{d}x.$$

So from the definition of C in § 1 we have

$$C = F\int_{-\infty}^{\infty}[1-(1-2\eta/F^2)^{\frac{1}{2}}(1+\eta'^2)^{\frac{1}{2}}]\,\mathrm{d}x \tag{2.3}$$

and from (B)

$$T = \tfrac{1}{2}F^2\int_{-\infty}^{\infty}[\eta-1+(1-2\eta/F^2)^{\frac{1}{2}}(1+\eta'^2)^{\frac{1}{2}}]\,\mathrm{d}x. \tag{2.4}$$

Alternatively we have from equation (2.2)

$$T = \tfrac{1}{2}F^2\int_{-\infty}^{\infty}\eta[1-(1-2\eta/F^2)^{\frac{1}{2}}(1+\eta'^2)^{\frac{1}{2}}]\,\mathrm{d}x, \tag{2.5}$$

which, besides serving as a convenient check, has some advantages over equation (2.4). For, the integrand is immediately seen to be $O(\eta^2)$ for small η and so tends to zero more rapidly as $|x| \to \infty$. We note also that for small values of η equation (2.5) gives

$$T \approx \tfrac{1}{2}F^2\int_{-\infty}^{\infty}\eta[1-(1-\eta/F^2)]\,\mathrm{d}x = V,$$

showing that for solitary waves of small amplitude the kinetic and potential energies are nearly equal (cf. McCowan 1891).

Comparing equations (2.4) and (2.5) we have the identity

$$\int_{-\infty}^{\infty}[(1+\eta)(1-2\eta/F^2)^{\frac{1}{2}}(1+\eta'^2)^{\frac{1}{2}}-1]\,\mathrm{d}x = 0. \tag{E}$$

This relation, as pointed out by J. G. B. Byatt-Smith (personal communication), can be derived from his integral equation for the solitary wave (Byatt-Smith 1970). We give here a more direct proof. Let Φ and Ψ denote velocity-potential and stream function in the steady motion. Then since $(x+\mathrm{i}y)$ is an analytic function of $(\Phi+\mathrm{i}\Psi)$, we have

$$\iint\frac{\partial y}{\partial\Psi}\mathrm{d}\Phi\,\mathrm{d}\Psi = \iint\frac{\partial x}{\partial\Phi}\mathrm{d}\Phi\,\mathrm{d}\Psi,$$

so on integrating between the distant limits $x = \pm X$ we obtain

$$\int(h+\eta)\,\mathrm{d}\Phi = 2X\int\mathrm{d}\Psi = -2Xch = -ch\int\mathrm{d}x$$

from which (E) follows at once.

3. A parametric relation between T and V

To establish the relation (D) we shall use a variational argument similar to that of Benjamin (1973), though with some essential differences. Let partial differentiation be denoted by suffixes. Then the motion being progressive, the kinematical and dynamical conditions at the free surface can be written

$$\begin{cases} \phi_x \eta_x - \phi_y = c\eta_x, & (3.1) \\ \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + \eta = c\phi_x. & (3.2) \end{cases}$$

The total energy $E = T + V$ may be written

$$E = \int_{-\infty}^{\infty} \int_{-h}^{\eta} \tfrac{1}{2}(\phi_x^2 + \phi_y^2)\,\mathrm{d}y\,\mathrm{d}x + \int_{-\infty}^{\infty} \tfrac{1}{2}\eta^2\,\mathrm{d}x.$$

Now let $\delta\phi$ and $\delta\eta$ denote arbitrary small variations of the velocity potential and surface elevation, such that $\delta\phi_x$ and $\delta\eta$ (but not necessarily $\delta\phi$) vanish as $|x| \to \infty$. Then applying Green's theorem we have

$$\begin{aligned} \delta E &= \int_{-\infty}^{\infty} \int_{-h}^{\eta} (\phi_x\,\delta\phi_x + \phi_y\,\delta\phi_y)\,\mathrm{d}y\,\mathrm{d}x + \int_{-\infty}^{\infty} [\tfrac{1}{2}(\phi_x^2 + \phi_y^2) + \eta]\,\delta\eta\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} (\delta\phi)_{\mathrm{s}} \frac{\partial\phi}{\partial n}\,\mathrm{d}s + \int_{-\infty}^{\infty} c(\phi_x)_{\mathrm{s}}\,\delta\eta\,\mathrm{d}x \end{aligned}$$

by equation (2.3). But from (3.1) we have at the free surface

$$\frac{\partial\phi}{\partial n} = (-\phi_x\eta_x + \phi_y)\frac{\mathrm{d}x}{\mathrm{d}s} = -c\eta_x \frac{\mathrm{d}x}{\mathrm{d}s}.$$

Therefore

$$\delta E = c\int_{-\infty}^{\infty} [(\phi_x)_{\mathrm{s}}\,\delta\eta - (\delta\phi)_{\mathrm{s}}\,\eta_x]\,\mathrm{d}x. \tag{3.3}$$

On the other hand, since

$$I = \int_{-\infty}^{\infty} \int_{0}^{\eta} \phi_x\,\mathrm{d}y\,\mathrm{d}x$$

we have

$$\delta I = \int_{-\infty}^{\infty} \int_{0}^{\eta} \delta\phi_x\,\mathrm{d}y\,\mathrm{d}x + \int_{-\infty}^{\infty} (\phi_x)_{\mathrm{s}}\,\delta\eta\,\mathrm{d}x. \tag{3.4}$$

Also since

$$\frac{\partial}{\partial x}\int_{0}^{\eta} \delta\phi\,\mathrm{d}y = \int_{0}^{\eta} \delta\phi_x\,\mathrm{d}y + \eta_x(\delta\phi)_{\mathrm{s}},$$

we have, after integration with respect to x,

$$h\,\delta C = \int_{-\infty}^{\infty} \int_{0}^{\eta} \delta\phi_x\,\mathrm{d}y\,\mathrm{d}x + \int_{-\infty}^{\infty} \eta_x(\delta\phi)_{\mathrm{s}}\,\mathrm{d}x. \tag{3.5}$$

From (3.3), (3.4) and (3.5) there follows the general relation

$$\delta E = c(\delta I - h\,\delta C). \tag{3.6}$$

This is a generalization of the result due to Benjamin (1973), who considered only the case $\delta C = 0$.† Equation (3.6) is true for arbitrary (smooth) variations $\delta\phi$, $\delta\eta$ subject only to the conditions that $\delta\phi_x$ and $\delta\eta$ vanish at ∞.

Suppose now that the variations $\delta\phi$, $\delta\eta$ are consistent with the 'growth' of the solitary wave so as to slightly change its amplitude a (or Froude number F) while keeping the mean depth h constant. Then from (3.6) and equation (B) it follows that

$$\mathrm{d}E = c\,\mathrm{d}(2T/c). \tag{3.7}$$

This interesting relation may be put in a number of different forms. Writing $E = T + V$ we find

$$\frac{\mathrm{d}T}{\mathrm{d}c} - \frac{2T}{c} = \frac{\mathrm{d}V}{\mathrm{d}c}.$$

Hence
$$\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{T}{c^2}\right) = \frac{1}{c^2}\frac{\mathrm{d}V}{\mathrm{d}c}, \tag{D}$$

so
$$T = c^2\int\frac{\mathrm{d}V}{c^2} = V - c^2\int V\,\mathrm{d}\left(\frac{1}{c^2}\right). \tag{D$'$}$$

Alternatively, if we introduce the Lagrangian

$$L = T - V,$$

we may write
$$\frac{\mathrm{d}L}{\mathrm{d}c} = \frac{2T}{c}, \tag{D$''$}$$

and
$$\frac{\mathrm{d}(L/c)}{\mathrm{d}(1/c)} = -E, \tag{D$'''$}$$

or in integral form

$$L = -c^2\int V\,\mathrm{d}\left(\frac{1}{c^2}\right) = \int\frac{T}{c^2}\,\mathrm{d}c^2 = -c\int E\,\mathrm{d}\left(\frac{1}{c}\right). \tag{D$^{\mathrm{iv}}$}$$

It is worth noting that if in place of V one introduces the total potential energy

$$V^* = \int_{-X}^{X}\tfrac{1}{2}(h+\eta)^2\,\mathrm{d}x = V + ghM + gh^2X$$

and considers the variation of $(T + V^*)$, admitting variations $\mathrm{d}h$ in the depth h, then one arrives at the more general result that

$$\mathrm{d}(T - V) - 2T(\mathrm{d}c/c) + [(c^2 - gh)\,M - 2T]\,\mathrm{d}h/h = 0.$$

When $\mathrm{d}h = 0$ then this reduces to equation (D). But when F is kept constant and h is varied, then since $T \propto h^3$, $V \propto h^3$ and $c \propto h^{\frac{1}{2}}$ it follows that

$$3(T - V) - T + [(c^2 - gh)\,M - 2T] = 0,$$

which is equivalent to relation (C) of § 1.

† It appears from equation (2.6) of Benjamin (1973) that he also assumed it possible to take $C = 0$.

4. The small-amplitude approximation

Writing $g = h = 1$ and introducing the parameter

$$\alpha = F^2 - 1 \quad (F = c) \tag{4.1}$$

we have the well-known approximate expressions for waves of small amplitude:

$$\left.\begin{aligned} \eta &\approx \alpha\,\mathrm{sech}^2\,\beta(x-ct), \\ \phi + \mathrm{i}\psi &\approx (\alpha/\beta)\tanh\beta(x+\mathrm{i}y-ct), \end{aligned}\right\} \tag{4.2}$$

where

$$\beta \approx \tfrac{1}{2}\sqrt{3}\,\alpha^{\frac{1}{2}} \tag{4.3}$$

(Boussinesq 1871; Rayleigh 1876; Lamb 1932). From these it is easy to find that

$$\left.\begin{aligned} M \approx I \approx C &\approx \frac{4}{\sqrt{3}}\alpha^{\frac{1}{2}}, \\ V \approx T &\approx \frac{4}{3\sqrt{3}}\alpha^{\frac{3}{2}}, \end{aligned}\right\} \tag{4.4}$$

and so from equation (D)

$$L \approx \frac{8}{15\sqrt{3}}\alpha^{\frac{5}{2}} \tag{4.5}$$

Further terms in these expansions will be given in a subsequent paper by the present author and Dr Fenton (in preparation).

5. The form of the solitary wave of maximum amplitude

In table 1 are listed a number of calculations that have been made of the height and form of a solitary wave of maximum amplitude. The accuracy of these calculations has not been easy to assess. Yamada (1957) satisfied the free surface condition numerically, but at only 13 points on his profile. Byatt-Smith (1970) and Fenton (1972) both calculated profile of waves *less* than the maximum amplitude and extrapolated the results to the highest wave. But because the behaviour of the wave

Table 1. Estimates of the speed and height of a solitary wave of maximum amplitude

author	F^2	a/h
McCowan (1894)	1.56	0.78
Yamada (1957)	1.656	0.828
Lenau (1966)	1.654	0.827
Byatt-Smith (1970)	1.72	0.86
Strelkoff (1971)	1.70	0.85
Fenton (1972)	1.70	0.85

height as a function of wave velocity is unknown near the maximum amplitude, their extrapolation is necessarily in some doubt. Lenau (1966) and Strelkoff (1971) both solved integral equations for the wave profile of maximum amplitude. This method also involves some uncertainty, in that the appropriate expansion of the profile in the neighbourhood of the crest is still not known (see Grant 1973).

Strelkoff did not publish the details of his profile but only quoted his result: $a/h \approx 0.85$.

6. A simple approximation to the wave of maximum amplitude

For some purposes it is convenient to have a simple but accurate approximation to a function which otherwise requires numerical tabulation. We shall now show how the energy relations obtained earlier can be used to obtain such an approximation to the profile of the solitary wave.

Stokes (1880) showed that at a sharp wave crest the angle at the corner is 120°, so that the surface gradient is $1/\sqrt{3}$. Since also a particle at the surface must come to rest at the corner (in the frame of reference moving with the wave) we have the two necessary relations

$$\eta(0) = \tfrac{1}{2}F^2, \quad \eta'(+0) = -1/\sqrt{3}. \tag{6.1}$$

We know also (see Lamb 1932, § 252) that in the outer fringes of the wave the profile η behaves asymptotically like $e^{-\mu|x|}$, where

$$\tan\mu/\mu = F^2 \tag{6.2}$$

exactly. Suppose then that we approximated the profile by a simple exponential

$$\eta = A\,e^{-\mu|x|}, \tag{6.3}$$

we should obtain

$$\tan\mu = 2/\sqrt{3} = 1.155, \quad F^2 = 1.347, \tag{6.4}$$

a somewhat low value. This is because the single term (6.3) can satisfy, in effect, only two conditions, one at the crest and one at infinity.

Suppose, however, that we introduce a second term:

$$\eta = A\,e^{-\mu|x|} + B\,e^{-2\mu|x|}. \tag{6.5}$$

This is still consistent with (6.2) and in addition we can now satisfy the integral condition

$$\int_{-\infty}^{\infty} [\tfrac{3}{2}\eta^2 - (F^2-1)\,\eta]\,dx = 0. \tag{6.6}$$

Substituting in equations (6.1), (6.2) and (6.6) we have

$$\left.\begin{aligned} A + B &= \tfrac{1}{2}F^2 = \tan\mu/2\mu, \\ A + 2B &= 1/\sqrt{3}\,\mu, \\ 3(\tfrac{1}{2}A^2 + \tfrac{2}{3}AB + \tfrac{1}{4}B^2) &= (2A+B)\,(F^2-1). \end{aligned}\right\}$$

These equations are easily solved to give

$$A = 1.116, \quad B = -0.284, \quad \mu = 1.052, \quad F^2 = 1.665. \tag{6.7}$$

The value of F^2 is now close to that found by earlier authors (see table 1).

By introducing a further term:

$$\eta = A\,e^{-\mu|x|} + B\,e^{-2\mu|x|} + C\,e^{-3\mu|x|} \tag{6.8}$$

we can satisfy also the integral condition

$$\int_{-\infty}^{\infty} [(1+\eta)(1-2\eta/F^2)^{\frac{1}{2}}(1+\eta'^2)^{\frac{1}{2}}-1]\,\mathrm{d}x = 0. \tag{6.9}$$

In this way we obtain

$$\left.\begin{matrix} A = 1.3117, & B = -0.6522, & C = 0.1795, \\ \mu = 1.0577, & F^2 = 1.6780, & a/h = 0.8390. \end{matrix}\right\} \tag{6.10}$$

TABLE 2. THE PROFILE OF THE WAVE OF MAXIMUM AMPLITUDE, AS GIVEN BY EQUATIONS (6.8) AND (6.12), COMPARED WITH THAT GIVEN BY YAMADA (1957)

Yamada (1957)		equation (6.8)		equation (6.12)	
x/h	η/h	η/h	difference	η/h	difference
0.0000	0.8278	0.8390	0.0112	0.8296	0.0019
0.1631	0.7363	0.7490	0.0127	0.7380	0.0017
0.3462	0.6414	0.6573	0.0159	0.6426	0.0012
0.4968	0.5698	0.5847	0.0149	0.5707	0.0009
0.6370	0.5086	0.5230	0.0144	0.5093	0.0007
0.7751	0.4533	0.4666	0.0133	0.4540	0.0007
0.9172	0.4014	0.4133	0.0199	0.4023	0.0009
1.0690	0.3514	0.3615	0.0101	0.3527	0.0013
1.2381	0.3016	0.3101	0.0085	0.3037	0.0021
1.4368	0.2508	0.2576	0.0068	0.2540	0.0032
1.6880	0.1969	0.2025	0.0056	0.2017	0.0048
2.0505	0.1369	0.1412	0.0045	0.1436	0.0069
2.7971	0.0596	0.0663	0.0067	0.0699	0.0103
∞	0.0000	0.0000	0.0000	0.0000	0.0000

A comparison of the profile (6.8) with Yamada's profile is given in table 2, showing that the differences are only of order 0.01. The corresponding values of the mass and of the kinetic and potential energies are

$$M = 1.977h^2, \quad V = 0.447gh^3, \quad T = 0.547gh^3. \tag{6.11}$$

If, on the other hand, we adopt the simpler form

$$\eta = A\,\mathrm{e}^{-\mu|x|} + B\,\mathrm{e}^{-\nu|x|} \quad (\nu > \mu), \tag{6.12}$$

(a suggestion that I owe to Dr Packham) and then choose A, B, μ, ν so as to satisfy the same four conditions, we obtain

$$\left.\begin{matrix} A = 1.5389, & B = -0.7093, \\ \mu = 1.0495, & \nu = 1.4630, \\ F^2 = 1.6592, & a/h = 0.8296. \end{matrix}\right\} \quad 6.13)$$

From table 2 it will be seen that the agreement with Yamada's profile is now even closer, the differences over most of the range being only of order 0.002. Graphically (figure 2) the profile (6.12) is almost indistinguishable from that of Yamada or Lenau. Corresponding to (6.12) we find

$$M = 1.963h^2, \quad V = 0.431gh^3, \quad T = 0.527gh^3. \tag{6.14}$$

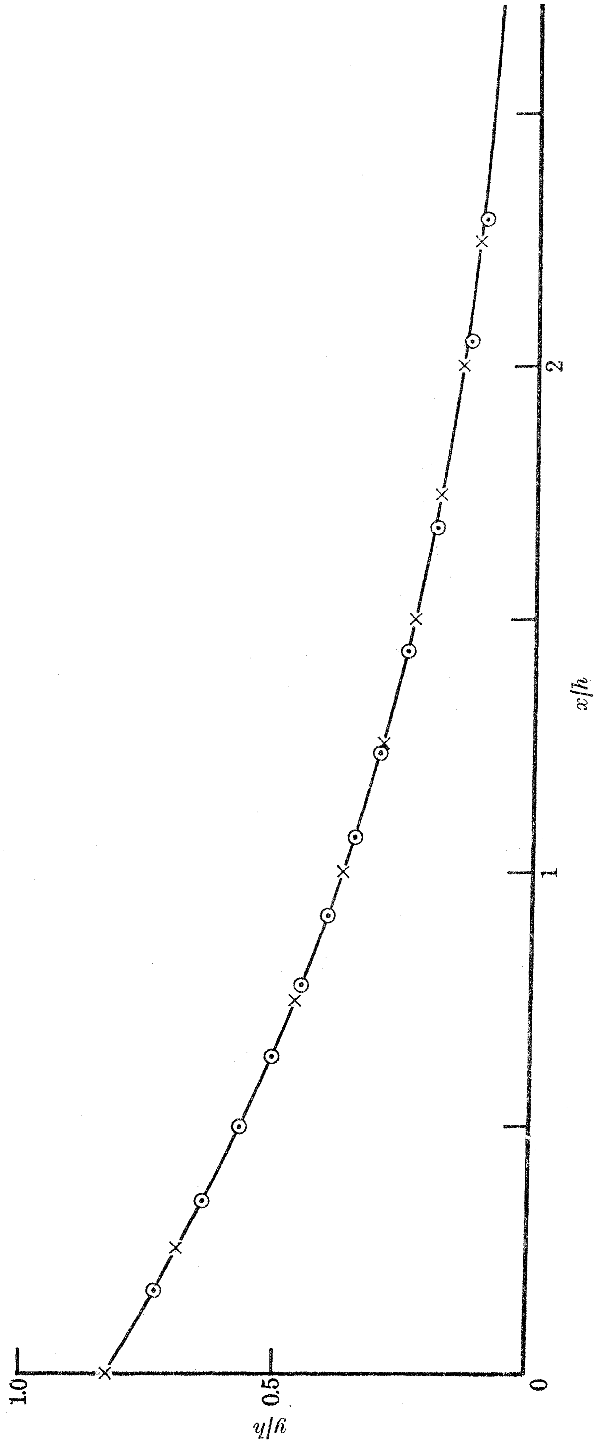

FIGURE 2. The profile of the wave of maximum amplitude as given by equation (6.12) (solid line), compared with the numerical calculations of Yamada (1957) (circles) and Lenau (1966) (crosses).

7. Conclusion

We have derived some exact integral relations for the solitary wave and used these to obtain simple approximations for the wave of maximum amplitude. These agree well with the more lengthy calculations of Yamada and Lenau. In fact both Yamada's and Lenau's profiles lie very slightly below our approximations, but there is some evidence from the work of Strelkoff (1971), Fenton (1972) and Byatt-Smith (1970) that the maximum surface elevation may indeed slightly exceed the values obtained by Yamada and Lenau. There is clearly still a need for a definitive calculation of the profile of the solitary wave of maximum amplitude.

In a further paper we shall pay attention to improving the calculation of waves of less than the maximum amplitude. Here the parametric relations derived in § 3 may be expected to be particularly useful.

Appendix. Proof of the relations (A), (B) and (C)

To prove (A), consider the motion relative to axes moving with the phase-velocity c. The particle velocity is then $(u-c, v)$ and by continuity of mass

$$\int_{-h}^{\eta} (u-c)\,\mathrm{d}y = \text{constant} = -ch$$

so

$$\int_{-h}^{\eta} u\,\mathrm{d}y = \int_{0}^{\eta} c\,\mathrm{d}y$$

which is equivalent to (A).

Following McCowan (1891), let Φ and Ψ denote the velocity potential and stream-function in the relative motion, so that

$$\Phi = \phi - cx, \quad \Psi = \psi - cy$$

and

$$\frac{\partial(\Phi, \Psi)}{\partial(x, y)} = U^2 + V^2$$

where (U, V) denotes $(u-c, v)$. Then on integrating over the region of flow contained between the two distant limits $x = \pm X$, say, we have

$$\begin{aligned}
2T &= \iint [(U+c)^2 + V^2]\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint (U^2+V^2)\,\mathrm{d}x\,\mathrm{d}y + 2c\iint (U+c)\,\mathrm{d}y\,\mathrm{d}x + c^2 \int\!\!\int \mathrm{d}y\,\mathrm{d}x \\
&= \iint \mathrm{d}\Phi\,\mathrm{d}\Psi + 2c\int_{-X}^{X} (-hc)\,\mathrm{d}x + c^2 \int_{-X}^{X} (h+\eta)\,\mathrm{d}x \\
&= -ch[\Phi]_{-X}^{X} - 4c^2hX + c^2(M + 2hX) \\
&= -ch[\phi]_{-X}^{X} + c^2 M.
\end{aligned}$$

Letting $X \to \infty$ and using equation (A) we see that (B) follows immediately.

To prove (C) let p denote the pressure and write Bernoulli's equation (multiplied by two) in the form

$$[p+(u-c)^2]+(gy-c^2)+v^2+(p+gy) = 0.$$

Now from the equation of vertical momentum we have

$$(y+h)\left[\frac{\mathrm{D}v}{\mathrm{D}t}+\frac{\partial}{\partial y}(p+gy)\right] = 0,$$

where $\mathrm{D}/\mathrm{D}t$ denotes differentiation following the motion. Adding, we get

$$[p+(u-c)^2+(gy-c^2)]+\frac{\mathrm{D}}{\mathrm{D}t}[(y+h)\,v]+\frac{\partial}{\partial y}[(y+h)\,(p+gy)] = 0. \qquad \text{(A 1)}$$

But by using the constancy of momentum flux in the moving frame of reference, we have

$$\int_{-h}^{\eta}[p+(u-c)^2+(gy-c^2)]\,\mathrm{d}y = \int_{-h}^{0}[-gy+c^2]\,\mathrm{d}y+\int_{-h}^{\eta}(gy-c^2)\,\mathrm{d}y$$
$$= \int_{0}^{\eta}gy-c^2)\,\mathrm{d}y.$$

So on integrating over the whole wave,

$$\int_{-\infty}^{\infty}\int_{-h}^{\eta}[p+(u-c)^2+(gy-c^2)]\,\mathrm{d}y\,\mathrm{d}x = V-c^2M.$$

Also we have, with the same limits of integration

$$\iint\frac{\mathrm{D}}{\mathrm{D}t}[(y+h)\,v]\,\mathrm{d}y\,\mathrm{d}x = \frac{\mathrm{D}}{\mathrm{D}t}\iint(y+h)\,v\,\mathrm{d}y\,\mathrm{d}x = 0$$

and
$$\iint\frac{\partial}{\partial y}[(y+h)\,(p+gy)]\,\mathrm{d}y\,\mathrm{d}x = \int(\eta+h)\,g\eta\,\mathrm{d}x = 2V+ghM.$$

So from equation (A 1) we have altogether

$$(V-c^2M)+(2V+ghM) = 0,$$

from which (C) follows immediately. It will be noted that this proof does not explicitly introduce the kinetic energies of the horizontal and vertical motion, as is done by Starr (1947) and by Keady & Pritchard (1973).

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