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A nonlinear mechanism for the generation of sea waves

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Recent observations of the growth of sea waves under the action of wind have established that the rate of growth is several times greater than has yet been accounted for. In this paper a new mechanism of wave generation is proposed, based on the idea of a maser-like action of the short waves on the longer waves.

It is shown that when surface waves decay they impart their momentum to the surrounding fluid. Short waves are readily regenerated by shear instability. But a longer wave passing through shorter waves causes the short waves to steepen on the long-wave crests. Hence the short waves impart more of their momentum to the crests of the long waves, where the orbital motion of the long waves is in the direction of wave propagation. If the short waves are decaying only weakly (under the action of viscosity), the effect on the long waves is slight. But when the short waves are forced to decay strongly by breaking on the forward slopes of the long waves the gain of energy by the latter is greatly increased.

Calculations suggest that the mechanism is capable of imparting energy to sea waves at the rate observed.

1. INTRODUCTION

After a decade of intense study, which has seen the development of wave generation theories by Phillips (1957 to 1966), Miles (1957 to 1962), Hasselmann (1967) and others, it is now evident that the mechanism mainly responsible for the most rapid stage of growth of sea waves under the action of the wind still remains to be found.

The 'resonance' mechanism suggested by Phillips predicts a small but constant rate of growth of the energy in the initial stages of development, and gives correctly the initial angular distribution of the waves. The 'shear instability' mechanism proposed by Miles predicts an exponential rate of growth, which should eventually overtake the linear rate. Yet recent observations of the growth rates by Snyder & Cox (1966) and by Barnett & Wilkerson (1967) have shown, first, that the observed initial growth rate, though similar in form to that predicted by Phillips, is some 50 times greater than would be expected on the basis of the measured intensities of turbulence in air flow over rough surfaces; and secondly that the rate of growth in the main stage of development is roughly an order of magnitude larger than predicted by Miles's mechanism. Since the turbulent parameters of an airstream over a moving water surface are not yet well known the discrepancy in magnitude between the initial rate of growth and that predicted by Phillips's theory may still be soluble. The discrepancy between the later stages of growth and that predicted by Miles seems at present to be more serious.

In addition, neither of the above theories accounts for two well-marked features of wave generation: the existence of some wave energy in a frequency range corresponding to waves which travel *faster* than the wind-speed; and secondly the rapid damping of a swell by an adverse wind.

There has been some revival of interest in a previous theory proposed by Jeffreys (1924, 1925) that there is a separation of the airflow at the crests of the waves, leading to a sheltering of the lee slopes of the waves, and hence a net rate of working on the waves by normal pressure fluctuations. But it is difficult to see how long, low waves could be associated with this effect. Nor does it explain the generation of waves travelling faster than the wind.

Hasselmann (1967) has recently proposed that the waves react in a nonlinear way with turbulent components in the airstream. But so little being known about the atmospheric turbulence, and the difficulty of observation being so great, it seems unlikely that this mechanism can ever be satisfactorily tested.†

The purpose of the present paper is to point out another nonlinear mechanism, which is demonstrably operating in a normal sea state and which appears to be capable of supplying enough energy to the waves to account for the observed rates of growth.

The essence of the mechanism may be stated quite briefly. With any train of surface waves there is associated both an energy density E, say, and a horizontal momentum density M, related to E by the simple equation

$$E = Mc \tag{1.1}$$

where c denotes the phase velocity. If the wave decays under the action of viscosity, or even more drastically by breaking, it gives up a proportion of its energy. Consequently, it must impart an identical proportion of its momentum to the surrounding fluid.

Consider now a train of short waves riding on the crests of longer waves. It can be shown that the short waves tend to be both shorter and steeper at the crests of the longer waves than they are in the long-wave troughs, being compressed by the horizontal orbital motion of the long waves. Hence the short waves have a pronounced tendency to break on the crests of the longer waves, rather than in the troughs. In breaking they give up a significant proportion γ of their momentum to the longer waves. But since the orbital velocity u_2 of the longer waves is positive at the wave crests, the energy so imparted to the longer waves is also positive, and at most equal to $\gamma M u_2$.

So we have the following picture: the wind continually supplies energy to the shorter waves, imparting to them a momentum at a rate comparable to the wind stress τ. The short waves cover the whole surface of the longer waves. The longer waves, however, travel with a greater velocity and so move through the short

† Stewart (1967) has pointed out a more serious objection to this mechanism, namely that the total energy in the atmospheric turbulence appears insufficient to generate ocean waves of the observed magnitude. In the same paper Stewart (1967) suggests that appreciable energy may be imparted by variations in the tangential stress of the wind on the sea surface. A correction to his calculation is given in another paper (Longuet-Higgins 1969*b*).

waves, causing the latter to break on the forward face of the long wave crests. In this way the long waves gain energy at a rate comparable to τu_2.

This sweeping up of short-wave momentum by long waves, in a way favourable to growth of the long waves, is similar to the action of a maser and is conveniently called the 'maser mechanism' of wave generation. It is shown in § 8 that the maser mechanism may indeed be of an order of magnitude sufficient to account for the main stage of growth of the sea waves, and accounts quite well for the observations of Barnett & Wilkerson.

First, however, in §§ 2 to 4, we give an account of the emergence of momentum from a slowly decaying wave train, and show how it may contribute to the momentum of its surroundings by a 'virtual wave stress' exerted by the boundary layer at the free surface. The presence of this virtual stress corresponds to a small but significant part of the total stress τ. Even if the short waves were not forced to break, they would still do some work on the lower waves since the virtual stress is greater at the long wave crests than it is in the troughs. Then in §§ 6 and 7 we discuss the much more drastic 'maser mechanism' which results from breaking of the short waves. Lastly in § 8 the consequences for generation of energy of the long waves are discussed.

It will be seen that the maser mechanism can account for both the generation of waves travelling faster than the wind, and the observed damping of waves by an adverse wind.

2. The mass-transport velocity

We first recall some known results from the theory of surface waves on deep water.

The surface elevation ζ in a progressive wave of small amplitude a may be described by the expression

$$\zeta = a\cos(kx-\sigma t) \tag{2.1}$$

where x is a horizontal coordinate, t is the time and k and σ denote the wavenumber and the radian frequency. The latter are connected by the relation

$$\sigma^2 = gk + (T/\rho)\,k^3 \tag{2.2}$$

in which g, ρ and T denote gravity, density and surface tension (Lamb 1932). Equation (2.1) is correct to order (ak), the maximum surface slope. To the same order, the components u, w of the particle velocity in the interior are given by

$$\left.\begin{aligned} u &= a\sigma\cos(kx-\sigma t)\,\mathrm{e}^{kz} \\ w &= a\sigma\sin(kx-\sigma t)\,\mathrm{e}^{kz} \end{aligned}\right\} \tag{2.3}$$

the vertical coordinate z being measured upwards from the free surface. Close to the surface, however, there is a thin boundary layer, with thickness of order

$$\delta = (\nu/\sigma)^{\frac{1}{2}} \tag{2.4}$$

where ν denotes the kinematic viscosity. This will be discussed in detail in § 3.

In the interior of the fluid the particle trajections are circles, to a first approximation in powers of (ak). But in the second approximation, as was pointed out by Stokes (1847), a marked particle possesses a small second-order mean velocity U given by

$$U = \overline{u} + \overline{\int u\,\mathrm{d}t\,\frac{\partial u}{\partial x}} + \overline{\int w\,\mathrm{d}t\,\frac{\partial u}{\partial z}} \tag{2.5}$$

where $\overline{u}$ denotes the mean value of u at a fixed point (the Eulerian mean) and the remaining terms arise from the orbital displacement of the particle combined with the gradients of the velocity field. The second term on the right of (2.5), when evaluated by (2.3), gives a positive contribution $\frac{1}{2}a^2\sigma k\,\mathrm{e}^{2kz}$. This arises because when the orbital displacement of a particle is positive, as it is on the rear slope of the wave, the horizontal gradient of the velocity field is also positive. Similarly, the third term on the right of (2.5) also gives a positive contribution $\frac{1}{2}a^2\sigma k\,\mathrm{e}^{2kz}$. This is because when a particle is at the top of its orbit its forwards velocity is greater than the velocity at the centre of the orbit and when the particle is at the bottom of its orbit the backwards velocity is less. Together the second and third terms on the right of (2.5) may be called the Stokes velocity; thus

$$U = \overline{u} + U_{\text{Stokes}} \tag{2.6}$$

where

$$U_{\text{Stokes}} = \overline{\int u\,\mathrm{d}t\,\frac{\partial u}{\partial x}} + \overline{\int w\,\mathrm{d}t\,\frac{\partial u}{\partial z}} \tag{2.7}$$

and for the interior of a progressive wave

$$U_{\text{Stokes}} = a^2\sigma k\,\mathrm{e}^{2kz} \tag{2.8}$$

If the motion is started from rest it is initially irrotational, and by a well-known theorem must remain irrotational in the interior until vorticity is diffused or convected inwards from the boundary. Under these conditions, if one chooses axes at rest relative to the deep water, we find that in the interior

$$\overline{u} = 0 \tag{2.9}$$

everywhere except near the upper boundary. Hence

$$U_{\text{irrot}} = U_{\text{Stokes}} \tag{2.10}$$

This is greatest near the surface and diminishes rapidly with depth (see figure 1). The gradient of U near the surface is given by

$$\left(\frac{\partial U}{\partial z}\right)_{\text{irrot}} = 2(ak)^2\sigma \tag{2.11}$$

The total forwards momentum within the wave is given by

$$M = \int_{-\infty}^{0} \rho U\,\mathrm{d}z = \tfrac{1}{2}\rho a^2\sigma \tag{2.12}$$

This forwards momentum is somewhat paradoxical. If one takes any volume within the fluid, wholly below the level of the wave troughs then since $\overline{u} = 0$ everywhere within this fixed space it appears that the total momentum contained within this volume is zero. Thus from the Eulerian viewpoint (Phillips 1966, §3.2) the whole momentum appears to be above the wave troughs: under the crests, where

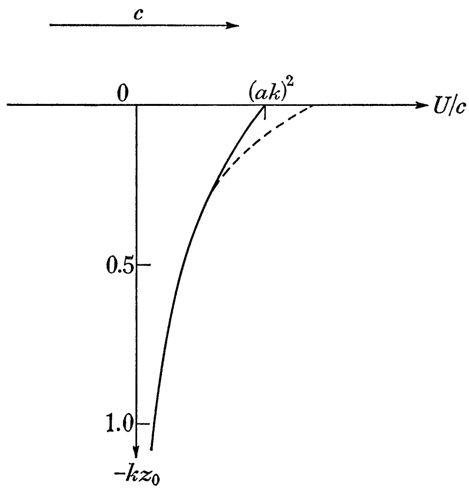

FIGURE 1. Profile of the mass-transport velocity in a progressive wave. ———, irrotational motion; – – – –, profile modified by viscosity.

there is an excess of fluid, the motion is forwards, and under the troughs, where there is a deficiency, the motion is backwards. Analytically this viewpoint is represented by the formula

$$M = \overline{\int_{-h}^{\zeta} \rho u \,\mathrm{d}z} = \rho\overline{u\zeta} \tag{2.13}$$

which on substitution from (2.1) and (2.2) gives the same result as equation (2.12).

The two viewpoints may be reconciled by noting that at any mean level z_0 within the fluid a surface $z = \zeta(x, z_0, t)$ may be drawn consisting of the same particles, and that by the same argument the total momentum contained below this surface is given by

$$M(z_0) = \overline{u\zeta(x, z_0, t)} \tag{2.14}$$

Since $\zeta \doteqdot \int w \,\mathrm{d}t$ it follows that

$$M(z_0) = \rho\overline{u\int w \,\mathrm{d}t} = \tfrac{1}{2}\rho a^2 \sigma \,\mathrm{e}^{2kz_0} \tag{2.15}$$

Hence the *distribution* of momentum within the fluid is given by

$$\rho U = \frac{\mathrm{d}M}{\mathrm{d}z_0} = \rho a^2 \sigma k \,\mathrm{e}^{2kz_0} \tag{2.16}$$

in agreement with (2.8).

There are good reasons, in the present context, for adopting the Lagrangian rather than the Eulerian viewpoint, that is to say for regarding the momentum as being attached to marked particles rather than to particular regions of space. This is because of the important role played by the viscous boundary layer at the surface (which will be described in the next section), combined with the fact that the thickness of the boundary layer is generally small compared to the vertical displacement of the surface itself. Hence we require coordinates and dynamical quantities related to the moving particles; in other words a Lagrangian description of the motion.

3. The generation of vorticity in the boundary layer

Let us take coordinates n and s normal and tangential to the free surface, as in figure 2. The boundary conditions at a free surface are that both the normal and the tangential stress shall vanish:

$$p_{nn} = p_{ns} = 0 \tag{3.1}$$

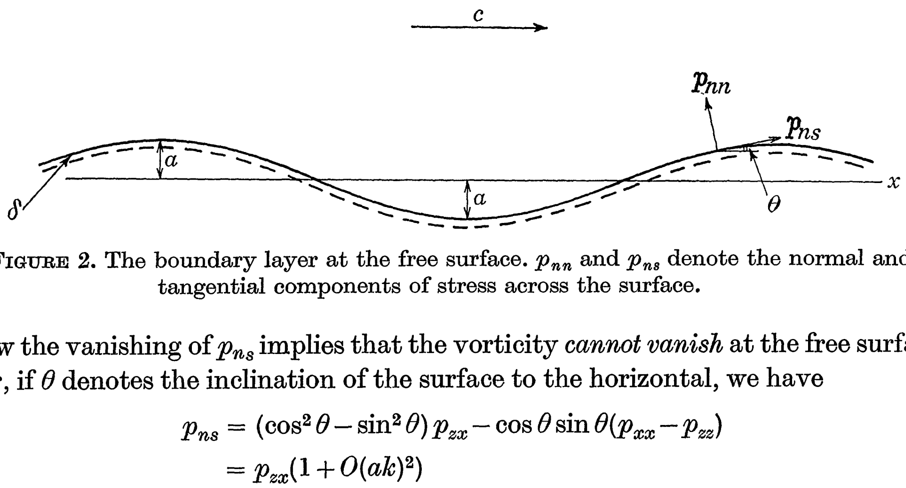

Figure 2. The boundary layer at the free surface. p_{nn} and p_{ns} denote the normal and tangential components of stress across the surface.

Now the vanishing of p_{ns} implies that the vorticity *cannot vanish* at the free surface. For, if θ denotes the inclination of the surface to the horizontal, we have

$$\begin{aligned} p_{ns} &= (\cos^2\theta - \sin^2\theta)\, p_{zx} - \cos\theta \sin\theta (p_{xx} - p_{zz}) \\ &= p_{zx}(1 + O(ak)^2) \\ &\doteqdot \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \end{aligned} \tag{3.2}$$

Therefore the vanishing of p_{ns} implies that

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} \tag{3.3}$$

and so

$$\omega \equiv \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = 2\frac{\partial w}{\partial z} \not\equiv 0 \tag{3.4}$$

in general. Since in the interior of the fluid the vorticity vanishes identically (to start with, at least) it follows that ω has a sharp gradient near the surface. A closer inspection (Longuet-Higgins 1953) shows that to the first order in ak we have

$$\omega = \omega_0 e^{\alpha n} \tag{3.5}$$

where

$$\omega_0 \doteqdot 2(\partial w/\partial z)_{z=0} \quad \alpha = (-i\sigma/\nu)^{\frac{1}{2}} \tag{3.6}$$

and n is the outwards normal. This represents an oscillating distribution of vorticity which does not penetrate beyond a distance of order δ, $= (\nu/\sigma)^{\frac{1}{2}}$, from the surface. However, to the second order in ak there is found, just beyond the boundary layer, a mean (second-order) vorticity

$$\overline{\omega} = 4\left(\overline{\frac{\partial w}{\partial x}\int \frac{\partial w}{\partial z}\,\mathrm{d}t}\right)_{z=0} = -2(ak)^2\sigma \tag{3.7}$$

which is independent of ν and of the boundary-layer thickness δ (see Longuet-Higgins 1953, 1960). This vorticity adds to the mass-transport gradient a term $2(ak)^2\sigma$ which is exactly equal to the irrotational gradient given by equation (2.11). Thus the total gradient of the mass-transport just outside the boundary layer is

$$(\partial U/\partial z)_{\text{viscous}} = 4(ak)^2\sigma \tag{3.8}$$

or just twice the Stokes gradient (see figure 1). The velocity gradient in gravity waves has been carefully checked by measurements in the laboratory (Longuet-Higgins 1960) and found to agree well with equation (3.8) and not with the irrotational formula (2.11).

We expect that the vorticity given by equation (2.7), being of constant sign, will gradually diffuse downwards from the boundary layer into the fluid At a time t, after initiating the wave motion, the depth of the fluid affected by the diffusion of vorticity will be of order $(\nu t)^{\frac{1}{2}}$.

The wave-induced vorticity (3.7) is in fact entirely equivalent to a virtual tangential stress.

$$\tau_{\text{wave}} = 2\rho\nu(ak)^2\sigma \tag{3.9}$$

applied to the surface of the fluid. We shall now interpret this stress in terms of the loss of momentum in a decaying wave.

4. The weak decay of a uniform wave train

If left to itself, a uniform train of free surface waves will decay under the action of viscosity. Thus we have, in the linear theory

$$a = a_0\,\mathrm{e}^{-t/t_0} \tag{4.1}$$

where the decay time t_0 is given by

$$t_0 = (2\nu k^2)^{-1} \tag{4.2}$$

(see Lamb 1932, §348). Now the original momentum M of the wave cannot be destroyed. How then is it redistributed?

One might expect it to be distributed with depth as in the original motion, that is to say proportionally to e^{2kz}. But the existence of the virtual tangential stress τ_{wave} shows that the decaying waves are in fact transferring all their momentum to the boundary layer at the free surface. For, if we calculate the total momentum M' transferred by the virtual stress (3.9) during the decay of the wave we find it to be given by

$$M' = \int_0^\infty \tau_{\text{wave}}\, \mathrm{d}t = \int_0^\infty 2\rho\nu(a_0 \mathrm{e}^{-t/t_0} k)^2 \sigma \, \mathrm{d}t \tag{4.3}$$

that is

$$M' = \rho\nu a_0^2 k^2 \sigma t_0 \tag{4.4}$$

On substituting for t_0 from equation (4.2) we find that

$$M' = \tfrac{1}{2}\rho a_0^2 \sigma = M \tag{4.5}$$

Hence all the momentum is transferred to the mean flow by the surface wave stress.

Hence the final distribution of momentum will be very different from the initial distribution ρU. It will be the result of downwards diffusion from the free surface. We shall have

$$\rho U' = \int_0^t f(t-t_1)\, \tau_{\text{wave}}(t_1)\, \mathrm{d}t_1 \tag{4.6}$$

where

$$f(t) = \frac{\exp(-z^2/4\nu t)}{\surd(\pi \nu t)}$$

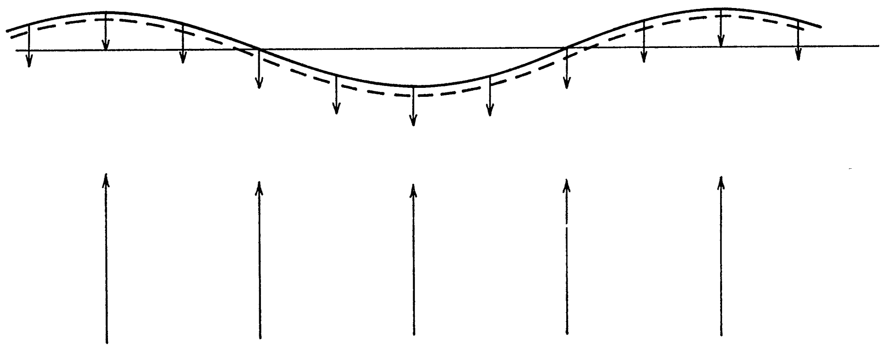

FIGURE 3. The flux of (Lagrangian) momentum in a damped water wave. The momentum is first driven upwards the surface and then diffused downwards from the boundary layer by viscosity.

After a time t of order t_0, the depth of the layer so affected will be of order k^{-1} which is of the same order as the depth to which the motion originally extended. But for much larger values of t the depth affected will increase like $(t/t_0)^{\frac{1}{2}}$.

This interpretation is illustrated in figure 3. The horizontal momentum ρU of the waves per unit depth (which initially may have been imported solely by normal stresses at the surface) is, during the process of decay, expelled upwards towards the free surface and then diffused downwards again by viscosity.

Thus the waves act somewhat as a reservoir of horizontal momentum for the sea surface. The momentum is drawn upon more or less gradually during the process of decay.†

† The upwards-pointing arrows in figure 3 represent the Lagrangian momentum flux. The Eulerian momentum flux vanishes.

In this process the boundary layer acts as an essential link. However, for long and steep waves it may, under the influence of intense shear, become unstable and break up spasmodically, shedding vorticity into the interior far more rapidly than by viscous diffusion.

5. Magnitude of the virtual wave stress

It is interesting to estimate the magnitude of the virtual stress

$$\tau_{\text{wave}} = 2\rho\nu(ak)^2\sigma \tag{5.1}$$

in a typical sea state.

The formula (5.1) holds for a discrete wave of amplitude a and maximum slope (ak). With a continuous slope spectrum $S(\sigma)$ defined by

$$\mathrm{d}(\tfrac{1}{2}a^2k^2) = S(\sigma)\,\mathrm{d}\sigma \tag{5.2}$$

equation (5.1) is replaced by

$$\tau_{\text{wave}} = 4\rho\nu\int_0^\infty \sigma S(\sigma)\,\mathrm{d}\sigma \tag{5.3}$$

Consider now the contribution to this integral from different parts of the frequency spectrum.

For the equilibrium range of gravity waves, in which the spectrum of the elevation ζ is given by

$$F(\sigma) = \alpha g^2\sigma^{-5} \quad (\sigma_1 < \sigma < \sigma_2) \tag{5.4}$$

we have simply

$$\sigma S(\sigma) = \alpha \tag{5.5}$$

where α is a constant determined experimentally (Phillips 1966) and theoretically (Longuet-Higgins 1969*a*) to be about 1.2×10^{-2}.

Table 1. Values of σ_3 and σ_4 as derived from the observations of Cox (1958), and a comparison of the virtual wave stress τ_{wave} with the total wind stress τ_{wind}.

U (cm/s)	σ_3 (rad/s)	σ_4 (rad/s)	τ_{wave} (dyn/cm²)	τ_{wind} (dyn/cm²)	$\tau_{\text{wave}}/\tau_{\text{wind}}$
318	35	300	0.15	1.2	0.13
608	25	900	0.45	4.3	0.11
920	15	1000	0.5	9.6	0.05
1202	12	1000	0.5	17	0.03

In the capillary range, the slope spectrum as measured by Cox (1958) in a wind-tunnel, is closely approximated by

$$\sigma S(\sigma) = \beta \quad (\sigma_3 < \sigma < \sigma_4) \tag{5.6}$$

where σ_3 and σ_4 depend to some extent on wind-speed and fetch and β is about 1.0×10^{-2}. Some typical values of σ_3 and σ_4 are given in table 1. It appears that at higher wind-speeds the two ranges (5.4) and (5.6) merge, and that over the combined range $\sigma_1 < \sigma < \sigma_4$ we have

$$\sigma S(\sigma) \doteqdot 10^{-2} \tag{5.7}$$

as pointed out by Phillips (1966). From (5.3) it then follows that

$$\tau_{\text{wave}} \doteqdot 0.04\rho\nu(\sigma_4 - \sigma_1) \tag{5.8}$$

Since $\sigma_4 \gg \sigma_1$ the lower frequency σ_1 can be omitted. Indeed, by far the largest part of the stress comes from the high-frequency end of the range. We can therefore take

$$\tau_{\text{wave}} = 0.04\sigma\rho\nu\sigma_4 \tag{5.9}$$

where σ_4 is the high-frequency cut-off.

The values of τ_{wave}, as determined by equation (5.9) and the observed values of σ_4 are shown in the fourth column of table 1; in the fifth column are shown the corresponding values of the total horizontal stress as determined by the empirical formula

$$\tau_{\text{wind}} = C\rho_{\text{air}} U^2 \tag{5.10}$$

where U is the wind-speed at a height of 4 cm above the free surface (as measured by Cox 1958) and C is the corresponding drag coefficient. We take $C = 6 \times 10^{-3}$. It can be seen that at the lower wind-speeds the capillary wave stress appears to account for a small but significant part of the total stress exerted by the wind. At higher wind-speeds the proportion appears to diminish.

However, it may be pointed out that if the laminar motion breaks down, as it probably will, the damping of the short waves may be greatly increased, leading to a corresponding increase in the virtual wave stress.

The part of the total wind stress in table 1 which is not accounted for by direct viscous decay of the wave field may be attributed to wave breaking and to the supply of momentum to increasingly long waves. We shall see in §7 that these two effects are closely related.

6. The weak decay of short waves riding on long waves

It is commonly observed that short gravity waves riding on the backs of longer waves are steeper on the crests of the longer waves than they are in the troughs (see figure 4). A quantitative analysis was carried out by Longuet-Higgins & Stewart (1960). The steepening is due to a combination of effects : the primary effect is a shortening of the wavelength due to the horizontal contraction of the surface near the crests of the long waves; next, the same horizontal contraction does work on the short waves, causing their amplitude to increase; thirdly, owing to the vertical acceleration in the long waves the ratio of the potential to the kinetic energy of the short waves is increased.

Let a_1, k_1 and σ_1 denote the amplitude, wave number and frequency of the short waves and a_2, k_2 and σ_2 the corresponding quantities for the longer waves, so that $k_1 \gg k_2$, $\sigma_1 \gg \sigma_2$. Then in the paper just quoted it was shown that if viscous dissipation is altogether neglected

$$\left.\begin{aligned} a_1 &= \bar{a}_1(1 + a_2 k_2 \cos(k_2 x - \sigma_2 t)) \\ k_1 &= \bar{k}_1(1 + a_2 k_2 \cos(k_2 x - \sigma_2 t)) \\ \sigma_1 &= \bar{\sigma}_1 \end{aligned}\right\} \tag{6.1}$$

to first order in $(a_1 k_1)$ and $(a_2 k_2)$. The ratio of $(a_1 k_1)^2 \sigma_1$ at the crests to the corresponding value in the troughs, is thus

$$r = \left(\frac{1+a_2 k_2}{1-a_2 k_2}\right)^4 \tag{6.2}$$

For small values of $(a_2 k_2)$ (to which the theory strictly applies) this ratio is equal to

$$r = 1 + 8a_2 k_2 \tag{6.3}$$

If for example $a_2 k_2 = 0.1$, then r $= 1.8$. Thus the virtual stress of the short waves will be considerably greater at the crests of the long waves, where the long wave orbital velocity is forwards, than in the troughs, where it is backwards.

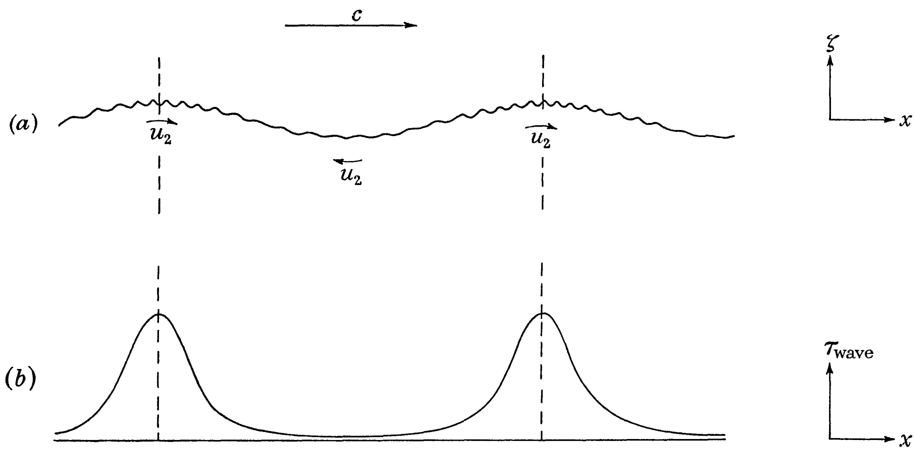

FIGURE 4. (*a*) A long wave of amplitude a_2 passing through a train of short waves of amplitude a_1, when the short waves do not break. (*b*) The virtual stress $2\rho\nu(a_1 k_1)^2 \sigma_1$ of the short waves.

Suppose now that the short waves are subject to viscous damping, but that the rate of working by the wind is such as to keep the wave amplitude a_1 steady and given by equation (6.1). The net work done by the long wave against the radiation stresses in the short waves is then zero.

From equations (5.1) and (6.1), the virtual stress τ_1 of the short waves is given by

$$\tau_1 = 2\rho\nu(\bar{a}_1 \bar{k}_1)^2 \bar{\sigma}_1 (1 + 4a_2 k_2 \cos(k_2 x - \sigma_2 t)) \tag{6.4}$$

to order $(a_2 k_2)$. Now the work W done by a small tangential stress τ_1 on the energy of a wave motion in which the orbital velocity is a_2 is given by

$$W = \overline{\tau_1 u_2} \tag{6.5}$$

the bar denoting the mean value with respect to time†. But near the surface,

$$u_2 \doteqdot a_2 \sigma_2 \cos(k_2 x - \sigma_2 t).$$

† This can be justified by a simple boundary-layer argument (see Longuet-Higgins 1969*b*).

So on substituting from (6.4) and taking mean values we find

$$W = 8\rho\nu(\bar{a}_1\bar{k}_1)^2\bar{\sigma}_1 a_2^2 k_2 \sigma_2 \tag{6.6}$$

and denoting the energy density $\frac{1}{2}\rho g a_2^2$ of the long waves by E_2 we have

$$\frac{\dot{E}_2}{E_2} \sim \frac{8\nu}{g^2}(\bar{a}_1\bar{k}_1)^2\sigma_1\sigma_2^3 \tag{6.7}$$

If we take, say $(\bar{a}_1\bar{k}_1)^2\sigma_1 \sim 10^{-2}\sigma_4$ where σ_4 is the cut-off frequency for the short waves, *ca.* 10^3 rad/s, then we have, in c.g.s. units,

$$\dot{E}_2/E \sim 10^{-6}\sigma_2^3 \tag{6.8}$$

This rate of growth depends rather critically on the frequency of the longer waves. For waves of period 6 s ($\sigma_2 \doteqdot 1$) it is negligible, but for waves of period 0.5 s we have

$$\dot{E}_2/E \sim 2\times 10^{-3} \tag{6.9}$$

which corresponds to a time constant of 250 s.

7. The breaking of short waves on long waves

We have so far assumed a small steepness for the longer waves ($a_2 k_2 \ll 1$). If $a_2 k_2$ is no longer small, it can be seen qualitatively from equation (6.2) that the steepening of the shorter waves becomes much more drastic. For example, on putting $a_2 k_2 = 0.5$ in (6.2) we find $r = 81$.

Hence the short waves must frequently be forced to break on the forward slopes of the longer waves, and to give up a large part of their momentum to the latter. In fact, when the long waves are on the point of breaking they are incapable of supporting any further gravity waves near the crest. The short waves then lose presumably all their energy in breaking on the forward face of the long waves.

This is confirmed by the visual observation that long steep waves are often very smooth on their rear faces, while their forwards faces are quite rough.

One might say that a long, steep wave passing through a field of short waves tend to 'clean up' the short waves by causing them to break in the forwards face of the long waves (see figure 5).

Now when the short waves give up their momentum to the longer waves they contribute to the energy of the latter at a rate proportional to the orbital velocity in the long waves. Since this is forwards on the upper part of the wave, the short waves supply a positive amount of energy to the long waves.

Meanwhile between the crests of the long waves the momentum of the short waves is replenished, mainly by the wind, and to a less extent by the radiation stresses.

Let us attempt a quantitative estimate of this effect. Suppose that on passing through each crest of the long waves the short waves lose on the average a proportion γ of their energy (or momentum) where γ is of order 1. We suppose that the proportion of momentum lost over the remainder of the wave is small compared to γ. It follows that *nearly all* of the momentum supplied by the wind to the short waves is ultimately imparted to the long waves on the forwards faces of the long-wave crests.

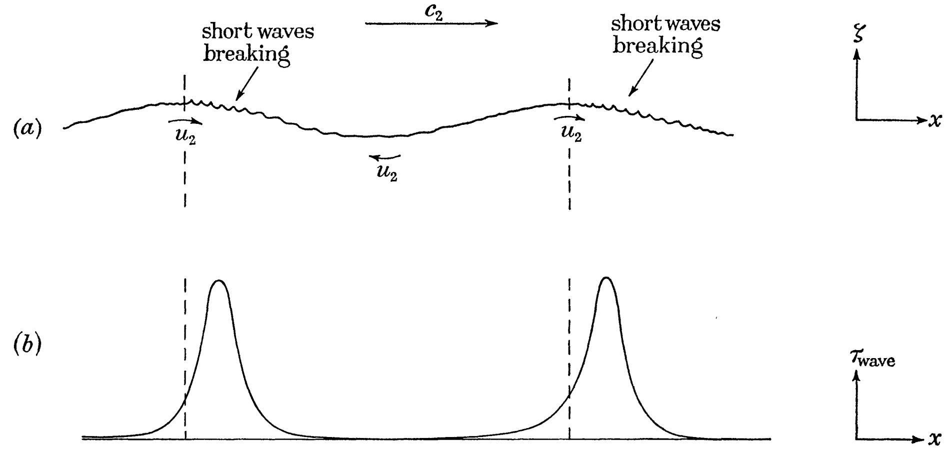

FIGURE 5. (*a*) The breaking of short waves on the forward face of a longer wave. (*b*) The distribution of the virtual stress.

If the wind-stress τ is assumed to supply momentum solely to the shorter waves it follows that the rate of energy supply to the longer waves is given by

$$W \sim \tau|u_2| \tag{7.1}$$

where $|u_2| = a_2\sigma_2$ denotes the orbital velocity of the longer waves.

It is important to show that the energy supplied to the longer waves in this way is not appreciably reduced by the work done by the long waves against the radiation stress.† Now if E_1 denotes the energy density of the short waves per unit distance, the momentum density per unit distance is E_1/c_1. Hence the momentum lost to the short waves, per unit time in one wavelength of the long waves, is

$$\gamma(E_1/c_1)\,c_2 \tag{7.2}$$

The energy supplied to the long waves per unit time, per wavelength, is thus

$$\gamma(E_1/c_1)\,c_2|u_2| \tag{7.3}$$

On the other hand the rate of working by the long waves on the short waves through the radiation stress, per wavelength, is of order

$$S\frac{\partial u_2}{\partial x}\Big/k_2 \sim S|u_2| \tag{7.4}$$

where S, $= \frac{1}{2}E$, denotes the radiation stress in the shorter waves. Comparing this with (7.3) we see that the latter is negligible provided that

$$c_1/c_2 \ll \gamma \tag{7.5}$$

which is true by hypothesis, since $\gamma = O(1)$.

† Phillips (1963) has taken into account only the work done by the radiation stresses and so concludes that the long waves are damped.

It may be noted that equation (7.1) is the same relation that would have been obtained had we assumed that all the wind stress were applied tangentially at the crests of the longer waves. But we emphasize that this is not the present assumption. Rather, the longer waves sweep up the momentum that was imparted to the short waves (possibly by normal stresses) over the whole extent of the longer waves.

In practice the amplitude of the long waves is variable (having a Rayleigh distribution; see Longuet-Higgins 1952) and in equation (7.1) $\bar{u}_2$ must be replaced by some mean value

$$W \sim \tau|\bar{u}_2| \sim \tau\overline{a_2 k_2}c_2 \tag{7.6}$$

to first order. However, the greater the value of $a_2 k_2$, the higher the proportion of energy swept up by the long wave, so that (7.6) may be an underestimate.

The mean value of the wave steepness $(a_2 k_2)$ may be determined either from observation or from theoretical considerations (see below) to be of order 10^{-1}, if the highest waves are breaking. Hence we have

$$W \sim 0.1\tau c_2 \tag{7.7}$$

where c_2 denotes the velocity of the longer waves.

This last estimate of the energy input may be compared with the estimate

$$W \sim \tau u_* \tag{7.8}$$

where u_* denotes the friction velocity, defined by $u_*^2 = \tau/\rho_{\text{air}}$. The two estimates (7.7) and (7.8) are equal if

$$u_* \sim 0.1c_2 \tag{7.9}$$

If we denote by C the drag coefficient, defined as u_*^2/U^2. where U denotes the wind velocity at some standard height, then the condition (7.9) is equivalent to

$$C \sim 10^{-2}(c_2/U)^2 \tag{7.10}$$

which is consistent with observation (see Phillips 1966, p. 144).

To show theoretically that $(a_2 k_2)$ is of order 10^{-1} we may note that in the equilibrium spectrum

$$F(\sigma) = \alpha g^2 \sigma^{-5} \tag{7.11}$$

the ratio of the breaking wave amplitude a_0 to the r.m.s. amplitude $\bar{a}$ is given by

$$\frac{a_0^2}{\bar{a}^2} \sim \frac{1}{8\alpha} \sim 10 \tag{7.12}$$

(see Longuet-Higgins 1969*a*). Assuming that the value of $(a_2 k_2)$ appropriate to a breaking wave is 0.5, this gives

$$(a_2 k_2)_{\text{r.m.s.}} \sim 0.5 \times 10^{-\frac{1}{2}} = 0.16 \tag{7.13}$$

Then assuming that the slope $(a_2 k_2)$ has a Rayleigh distribution it follows that

$$\overline{a_2 k_2} = \tfrac{1}{2}\pi^{\frac{1}{2}}(a_2 k_2)_{\text{r.m.s.}} \sim 0.13 \tag{7.14}$$

which is of order 10^{-1} as stated.

8. Discussion

Let us explore some of the consequences of equation (7.1) for wave generation. We shall deal only with orders of magnitude.

Assuming that the long waves are steep enough for equation (7.1) to apply, but not so steep as to be limited by breaking, then their rate of growth, in a spacially homogeneous ocean unlimited by the fetch, will be given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\rho g a^2\right) \sim \tau a \sigma \tag{8.1}$$

that is to say

$$\frac{\mathrm{d}a}{\mathrm{d}t} \sim \frac{\tau}{\rho g}\sigma \tag{8.2}$$

This represents a linear rate of growth† for the wave amplitude:

$$a \sim \frac{\tau}{\rho g}\sigma t \tag{8.3}$$

and for the wave steepness

$$ak \sim \left(\frac{\tau k}{\rho g}\right)\sigma t = \frac{\tau}{\rho c^2}\sigma t \tag{8.4}$$

at least before dissipation of the long waves by breaking becomes important.

Let us consider the order of magnitude of this growth rate. Since $\tau = C\rho_{\mathrm{air}}U^2$, equation (8.4) can be written

$$ak \sim 2\pi C\left(\frac{\rho_{\mathrm{air}}}{\rho_{\mathrm{water}}}\right)\left(\frac{U}{c}\right)^2\left(\frac{t}{T}\right) \tag{8.5}$$

where $T = 2\pi/\sigma$ denotes the wave period. Thus (t/T) denotes the number N of wave cycles. On substituting the numerical values

$$C = 1.5\times 10^{-3} \quad \text{and} \quad (\rho_{\mathrm{air}}/\rho_{\mathrm{water}}) = 1.3\times 10^{-3}$$

we obtain

$$ak \sim 1.2\times 10^{-5}(U/c)^2 N \tag{8.6}$$

Now the maximum value of $\overline{ak}$ corresponding to breaking waves is, as we saw earlier, of order 10^{-1}. Hence if we consider the growth of those waves whose phase speed c is equal to the wind-speed U the number N of wave cycles required for them to achieve their maximum steepness would be, according to equation (8.6), of order 10^4. This is in agreement with wave observations at sea (see, for example, Sverdrup & Munk 1947).

In a situation where the wave field is limited by the fetch x rather than by the duration t we may substitute for t in equation (8.3) using the relation x/t = group velocity $= \frac{1}{2}c$, that is to say $t = 2x/c$. This gives

$$a \sim \frac{2\tau}{\rho c^2}x \sim 2C\frac{\rho_{\mathrm{air}}}{\rho_{\mathrm{water}}}\left(\frac{U}{c}\right)^2 x \tag{8.7}$$

† If σ is assumed constant. If, on the other hand, σ is allowed to decrease gradually with the time t then (8·3) represents a lower bound for the wave amplitude.

or with the same numerical values as before,

$$a \sim 0.4 \times 10^{-5} (U/c)^2 x \tag{8.8}$$

This formula may be compared with the recent observations of Barnett & Wilkerson (1967) who contoured spectral density (in m²/Hz) against fetch x and frequency f (Hz) for a wind-speed U of about 15 m/s (see figure 6). Consider, for example, the situation when $x = 200\,\text{km} = 2 \times 10^5\,\text{m}$. Formula (8.8) then gives

$$a \sim 0.8\,(U/c)^2\,\text{metres} \tag{8.9}$$

The peak frequency for this distance in figure 6 is about 0.105 Hz, corresponding to a wave period of 9.5 s and hence a phase velocity $c = 15$ m/s. Hence $(U/c) \sim 1$, and (8.9) gives $a \sim 0.8$ m. On the other hand, the total mean-squared surface elevation,

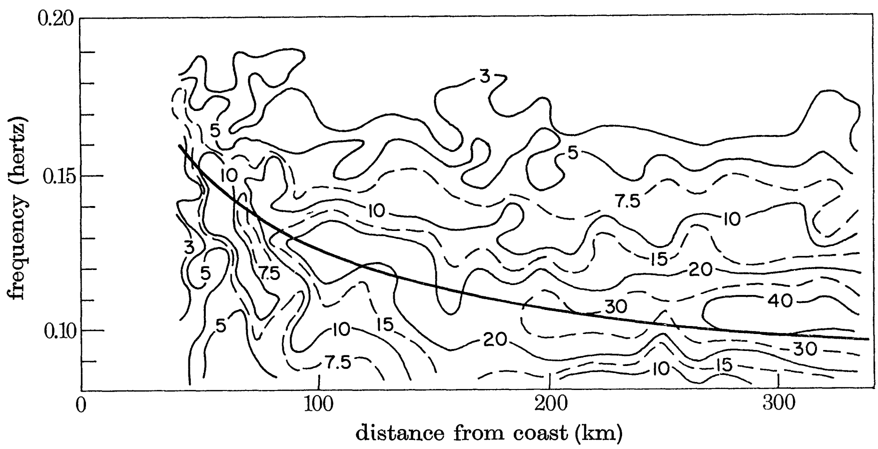

FIGURE 6. (From Barnett & Wilkerson, 1967.) Contours of spectral density as a function of fetch (distance from shore) and frequency.

from the section of the contour map at $x = 200$ km, is about 0.5 m². Equating this to $\frac{1}{2}a^2$ we should have $a = 1.0$ m. Hence the order of magnitude of the total energy transfer predicted by (8.8) appears to be correct.

Since the wave-number k is equal to g/c^2, equation (8.8) also predicts that before the waves are limited by breaking

$$ak \propto (U/c)^2\,(gx/c^2) \tag{8.10}$$

Hence the distance x at which the wave steepness ak achieves a given value is proportional to c^4. Since $c = g/\sigma$, we might expect the frequency $f = 2\pi\sigma$ corresponding to the peak spectral density in figure 6 to be proportional to $x^{-\frac{1}{4}}$. Such a curve has been drawn in figure 6. The constant of proportionality has been adjusted so as to give the best fit to the spectral peaks. One sees that the fit is fairly good, though the curve is evidently rather too high for the shorter fetches and too low for the longer fetches.

On the high-frequency side of the spectral peak, the spectral density is presumably limited by wave breaking. But on the low-frequency side, before the waves are limited by breaking, it may be justifiable to assume that the spectral density is given by a formula analogous to equation (8.8), namely

$$F(\sigma) = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\tfrac{1}{2}a^2\right) = K\left(\frac{U}{c}\right)^4 \frac{x^2}{\sigma} \tag{8.11}$$

where K is a constant. For fixed values of U and x this give$_a$

$$F(\sigma) \propto \sigma^3 \tag{8.12}$$

Although such a conclusion is not inconsistent with the spectral densities at the shorter fetches in figure 6, nevertheless the very steep lower face of the spectrum in the lower right-hand corner of figure 6 suggests that the rapid rate of growth on the low-frequency side of the peak is probably due to the operation of another mechanism.

We suggest that this mechanism may be as follows. The phase velocity of a wave of finite amplitude is somewhat greater than that of a small-amplitude wave of the same length, by an amount of order $(ak)^2 c$. It is thus plausible that such a wave should interact with a lower wave of slightly greater length (and lower frequency) but travelling with the same phase velocity—particularly if the wave groups are of finite length. In other words here may be a transfer of energy to a lower frequency.

A manifestation of this same mechanism is the instability of surface waves discovered by Benjamin & Feir (1967) in which the main wave gives up energy to each of two side-bands. For steep waves, the side-band of higher frequency would be limited by breaking, more than the side-band of lower frequency. Hence the energy would appear to be shifted continually towards slightly lower frequencies. This effect will be further investigated in a subsequent paper.

An interesting consequence of the maser mechanism described earlier is that the phase velocity of the longer waves is not limited to be less than the wind velocity, as it would be if only the resonance or shear instability mechanisms were operating. For, in order that energy be imparted to the longer waves by maser action it is necessary only that the wind generate short waves at some point on the long-wave profile, and this it can theoretically do no matter how great the phase speed of the long waves.

We see also that some damping of long waves by an adverse wind is also to be expected by the maser action of the short waves. For, the momentum of the short waves will be in the same direction as the wind and therefore opposite to the orbital velocity u_2 at the crests of the longer waves. So in breaking, the short waves will take energy away from the longer waves.

The damping action of an adverse wind may in fact be more pronounced than the generating action of a following wind. For by considering the waves in a frame of reference moving with the velocity of the longer waves, the longer waves are reduced to a steady stream in the same direction as the wind. The shorter waves are propagated on the stream in the same direction as the stream. However, because of the orbital velocity of the long waves the speed of the stream varies with distance (according to Bernoulli's law). Now if the adverse velocity in the stream exceeds the group velocity $\frac{1}{2}c_1$ of the shorter waves the latter cannot be propagated against the stream, and must be reflected or break (see, for example, Longuet-Higgins & Stewart 1961). In either case they give up their momentum to the stream, so that the maser action is clearly very effective.

It will be seen that some explanation is still required for the generation of the short gravity waves. These can be attributed to the maser action of even shorter gravity waves, and so on down to capillary wavelengths. The latter may be due to shear instability, essentially as described by Miles (1962).

It would be interesting to record instrumentally the form of the surface elevation ζ in ocean waves under the action of wind. If it can be established that the short waves on the forward faces of long wave crests are significantly steeper than those on the rear faces, so that the proportion γ of the energy difference is of order unity, one of the critical assumptions of the present theory would be verified.

Some indications of this effect, though on a small scale are already given by the observations of Cox (1958) in a laboratory flume. These need to extend to oceanic scales. Cox indeed observed steeper capillary wave action on the forward faces of longer, gravity waves than on the rear slopes. Some capillary waves were found even in the absence of the wind—a phenomenon attributable to the action of surface tension at the sharp gravity wave crests (Longuet-Higgins 1963). However, in the presence of the wind the shorter waves were of far greater amplitude.

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