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On wave breaking and the equilibrium spectrum of wind-generated waves

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A theoretical calculation is made of the loss of energy by wave breaking in a random sea state in terms of the spectral density function. In the special case of the equilibrium spectrum $F(\sigma) = \alpha g^2\sigma^{-5}$ the proportion ϖ of energy lost per mean wave cycle is found to be given by

$$\varpi \doteqdot e^{-1/8\alpha}$$

irrespective of the low-frequency cut-off in the spectrum.

Assuming that in the equilibrium state the loss of energy by breaking is comparable to that supplied by the wind, one can estimate the constant α in terms of the drag coefficient of the wind on the sea surface. It is found that

$$\alpha \doteqdot -\tfrac{1}{8}/\ln[1600C^{3/2}(\rho_{\text{air}}/\rho_{\text{water}})].$$

Taking a representative value of C one finds $\alpha \doteqdot 1{\cdot}3\times 10^{-2}$, which falls within the range of observed values of α. The above equation for α is rather insensitive to the various assumptions made in the analysis.

There is some evidence, derived from observation, that α may not in fact be quite constant, but may decrease slightly as the wave age (gt/U) or the non-dimensional fetch (gx/U^2) is increased. It is suggested that the drag coefficient may behave similarly.

INTRODUCTION

The 'equilibrium law' for wind-generated surface waves, as first derived by Phillips (1958) is that over a certain range of wave frequencies σ the spectral density of the surface elevation ζ is given by

$$F(\sigma) = \alpha g^2\sigma^{-5} \qquad (\sigma_1 < \sigma < \sigma_2), \tag{1·1}$$

where g denotes the acceleration of gravity and α is an absolute constant. The frequency-dependence of the spectrum (1·1) has been approximately verified by observations over a wide range of frequencies (Phillips 1966). Some experimentally determined values of α, as quoted by Phillips, are shown in the last column of table 1.

Although there appear to be significant differences among the entries in the last column, there is at least an order-of-magnitude agreement among them. However, no independent theoretical estimate of α has yet been given.

Now the basic idea underlying Phillips's derivation of the equilibrium law is that over this range of frequencies the spectrum is limited by wave breaking. It seems natural then to inquire how much energy is indeed lost by wave breaking in a spectrum of the form (1·1), and to relate this to the energy input by the wind, which can be estimated independently. This is the purpose of the present note.

The estimate of energy loss must of necessity be somewhat rough, since we are dealing with a nonlinear phenomenon. Fortunately, however, it turns out that the appropriate value of α is fairly insensitive to the various assumptions that will be made.

TABLE 1. VALUES OF α AS DETERMINED BY MEASUREMENTS OF THE SPECTRUM OF SURFACE WAVES

	x (m)	U (m/s)	gx/U^2	α
Burling (1959)	70	3·9	180	$1{\cdot}48\times10^{-2}$
Hicks (1960)	55	1·2	380	$1{\cdot}21\times10^{-2}$
Kinsman (1960)	2500	5·6	800	$1{\cdot}04\times10^{-2}$
Pierson (ed.) (1960)	3×10^5	9·5	3300	$1{\cdot}33\times10^{-2}$
Longuet-Higgins, Cartwright & Smith (1963)	5×10^5	10·0	5×10^4	$0{\cdot}80\times10^{-2}$

The calculation is rendered remarkably straightforward by the demonstration, given in § 2, that the proportional loss of energy per mean wave cycle in the spectrum (1·1) is approximately given by

$$\varpi = \mathrm{e}^{-1/8\alpha} \tag{1·2}$$

independently of the frequency range $\sigma_1 < \sigma < \sigma_2$, provided $\sigma_1 \ll \sigma_2$. On the other hand, if we assume that the energy lost in breaking is comparable with that supplied by the wind, we are led to the conclusion (see § 2) that, to an order of magnitude

$$\varpi \sim 1600\, C^{\frac{3}{2}}(\rho_{\mathrm{air}}/\rho_{\mathrm{water}}), \tag{1·3}$$

where C is the 'drag coefficient' of the wind on the waves. On assuming the value $C = 1{\cdot}5\times10^{-3}$ and equating (1·2) and (1·3) we find

$$\alpha = 1{\cdot}3\times10^{-2}, \tag{1·4}$$

a value within the range of observation and moreover extremely insensitive to the rough assumptions made in the analysis.

The question is raised in § 4 whether α is an absolute constant or whether, on the other hand, it varies throughout the development of a wave spectrum. There is some evidence (derived from table 1) to suggest that α may in fact be a decreasing function of the 'wave age' as defined by Sverdrup & Munk (1947).

2. THE LOSS OF ENERGY BY WAVE BREAKING

A rough estimate of the energy lost to the breaking waves may be made in the following way. It is well known that in a standing wave, the maximum downward acceleration is equal to g (see Taylor 1953). It is less well known that in a progressive wave, whose limiting form at the crest is the Stokes 120° angle, the acceleration is equal to $\frac{1}{2}g$, directed away from the crest in all directions (Longuet-Higgins 1963).

This is illustrated in figure 1. In a typical sea state, where the energy is directed mainly within about 30° of the mean direction, it is reasonable to suppose that white-caps appear whenever the vertical acceleration at the crest approaches $-\frac{1}{2}g$.

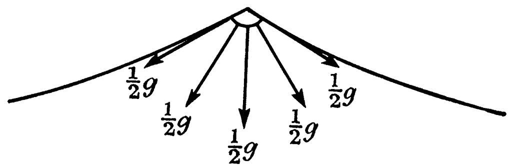

FIGURE 1. The limiting acceleration near the crest in a Stokes 120 ° angle.

If for a moment we assume that the wave is adequately described by the linear expression

$$\zeta = a\cos(kx - \sigma t) \qquad (\sigma^2 = gk), \tag{2·1}$$

it follows that the condition for breaking is simply

$$a\sigma^2 = \tfrac{1}{2}g. \tag{2·2}$$

It might be thought unacceptable to use the linearized theory of small-amplitude waves for an obviously nonlinear effect. But we note that on the same basis the maximum wave slope would be given by

$$(ak)_{\text{max.}} = (a\sigma^2/g)_{\text{min.}} = 0·5 \tag{2·3}$$

compared to the actual value

$$\arctan\tfrac{1}{6}\pi = 1/\surd 3 = 0·547\ldots, \tag{2·4}$$

and that the ratio of the wave height to the wavelength would be given by

$$\frac{2a}{L} = \frac{ak}{\pi} = \frac{a\sigma^2}{\pi g} = \frac{1}{2\pi} = 0·159\ldots, \tag{2·5}$$

compared to the computed value 0·142...(Michell 1893). Hence in adopting the criterion of equation (2·2) we may be in error by some 10 %.

Now in a sea state a narrow frequency spectrum it has been shown (Longuet-Higgins 1952) that the distribution of wave amplitudes a is theoretically a Rayleigh distribution:

$$p(a) = \frac{2a}{\bar{a}^2}\exp\{-a^2/\bar{a}^2\} \tag{2·6}$$

and, moreover, this distribution has been found to fit the observed distribution of wave heights remarkably well even when the spectrum is not narrow (Longuet-Higgins 1952; Watters 1953; Cartwright & Longuet-Higgins 1956). This distribution is shown in figure 2. It has a maximum density at $a/\bar{a} = 1/\surd 2$, and it decreases exponentially as $a/\bar{a} \to \infty$. The parameter $\bar{a}$ represents the root-mean-square wave amplitude, which for a narrow spectrum is related to the mean energy density E of the wave field per unit horizontal area by

$$E = \tfrac{1}{2}\rho g \bar{a}^2. \tag{2·7}$$

Now let us suppose that all those waves of amplitude greater than the critical amplitude a_0 will break, where

$$a_0 \bar{\sigma}^2 = \tfrac{1}{2} g \tag{2·8}$$

and $\bar{\sigma}$ is a mean frequency which may be determined from the spectral density $F(\sigma)$ by

$$\bar{\sigma}^2 \int_0^\infty F(\sigma)\,\mathrm{d}\sigma = \int_0^\infty \sigma^2 F(\sigma)\,\mathrm{d}\sigma. \tag{2·9}$$

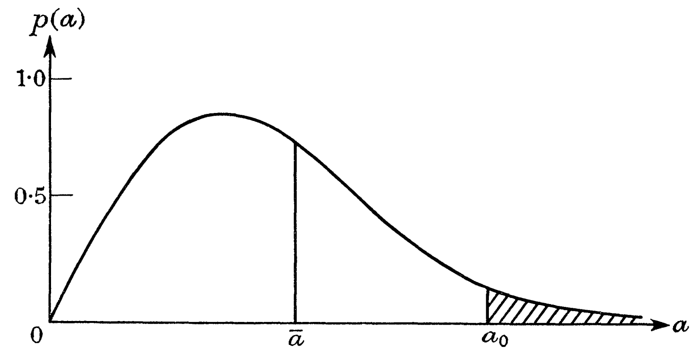

FIGURE 2. The probability density $p(a)$ of the wave amplitude a, given by

$$p(a) = 2(a/\bar{a}^2)\exp\{-a^2/\bar{a}^2\}$$

(the Rayleigh distribution). Here $\bar{a}$ denotes the r.m.s. wave amplitude and a_0 the critical amplitude for breaking waves.

If determined in this way, $\bar{\sigma}$ has the property that the curve $\zeta = \bar{a}\cos\bar{\sigma}t$ has the same mean number of zero-crossings per unit time as the record with spectrum $F(\sigma)$ (see Rice 1944). Let us further assume that if the wave amplitude a exceeds a_0, then wave breaking reduces the amplitude to a_0 exactly, so that the amount of energy lost is $\tfrac{1}{2}\rho g(a^2 - a_0^2)$. With these assumptions it follows that the mean loss of energy density in one wave cycle $T = 2\pi/\bar{\sigma}$ is given by

$$\begin{aligned} \int_{a_0}^\infty \tfrac{1}{2}\rho g(a^2 - a_0^2)\,p(a)\,\mathrm{d}a &= -\int_{a_0}^\infty \tfrac{1}{2}\rho g(a - a_0^2)\,\mathrm{d}(\exp\{-a^2/\bar{a}^2\}) \\ &= \int_{a_0}^\infty \exp\{-a^2/\bar{a}^2\}\,\mathrm{d}(\tfrac{1}{2}\rho g a^2) \\ &= \tfrac{1}{2}\rho g \bar{a}^2 \{\exp\{-a_0^2/\bar{a}^2\} \\ &= E\exp\{-E_0/E\}, \end{aligned} \tag{2·10}$$

where

$$E_0 = \tfrac{1}{2}\rho g a_0^2. \tag{2·11}$$

In other words, the *proportion* ϖ of wave energy lost per mean wave cycle is simply

$$\varpi = \exp\{-E_0/E\}. \tag{2·12}$$

But

$$E = \rho g \overline{\zeta^2} = \rho g \int_0^\infty F(\sigma)\,\mathrm{d}\sigma. \tag{2·13}$$

So from (2·8) and (2·11) this can also be written

$$\varpi = \exp\left\{\frac{-g^2}{8\overline{\sigma}^4}\bigg/\int F(\sigma)\,\mathrm{d}\sigma\right\}. \tag{2·14}$$

To make further progress we need to make a specific assumption as to the form of the spectral density $F(\sigma)$. Let us assume first that $F(\sigma)$ is the equilibrium spectrum with sharp low-frequency cut-off:

$$F(\sigma) = \alpha g^2 \sigma^{-5} \times \begin{Bmatrix} 0 & (\sigma < \sigma_1), \\ 1 & (\sigma > \sigma_1). \end{Bmatrix} \tag{2·15}$$

In this case we find

$$\int_0^\infty F(\sigma)\,\mathrm{d}\sigma = \frac{\alpha g^2}{4\sigma_1^4} \tag{2·16}$$

and

$$\overline{\sigma}^2 = \frac{\int_{\sigma_1}^\infty \alpha g^2 \sigma^{-3}\,\mathrm{d}\sigma}{\int_{\sigma_1}^\infty \alpha g^2 \sigma^{-5}\,\mathrm{d}\sigma} = 2\sigma_1^2. \tag{2·17}$$

Hence

$$\varpi = \exp\{-1/8\alpha\}, \tag{2·18}$$

a result independent of σ_1.

If, on the other hand, $F(\sigma)$ has a smooth cut-off as suggested by Neumann & Pierson (1966):

$$F(\sigma) = \alpha g^2 \sigma^{-5} \exp\{-\beta(\sigma/\sigma_1)^4\}, \tag{2·19}$$

where β is an absolute constant ($\doteqdot 0·74$) and $\sigma_1 = g/U$, U being a representative wind-speed, we find then

$$\int_0^\infty F(\sigma)\,\mathrm{d}\sigma = \frac{\alpha g^2}{4\beta\sigma_1^4}, \tag{2·20}$$

and

$$\overline{\sigma}^2 = \frac{\int_0^\infty \sigma^{-3}\exp\{-\beta(\sigma_1/\sigma)^4\}\,\mathrm{d}\sigma}{\int_0^\infty \sigma^{-5}\exp\{-\beta(\sigma_1/\sigma)^4\}\,\mathrm{d}\sigma} = (\pi\beta)^{\frac{1}{2}}\sigma_1^2. \tag{2·21}$$

Hence

$$\varpi = \exp\{-1/2\pi\alpha\}, \tag{2·22}$$

independently of both β and σ_1.

The two results (2·18) and (2·22) are quite similar. For the purpose of discussion we shall adopt equation (2·18).

3. An estimate of α

Some of the energy lost to a wave through breaking will doubtless be transferred to other parts of the gravity wave spectrum. A small but significant part may also be transferred to capillary waves (Longuet-Higgins 1963). However, the visual observations of breaking waves in a laboratory flume certainly suggests that the greater part of the wave energy lost through breaking normally goes directly into turbulence.

Because of these turbulent energy losses it follows that the energy involved in wave breaking must be comparable with, though in equilibrium less than, the total energy supplied to the sea surface by the wind. Thus if W denotes the mean rate of working by the wind,

$$\varpi E/T \sim W, \tag{3·1}$$

where E denotes the total wave energy density per unit area and T the mean period as defined in §2.

On the other hand since the equilibrium spectrum is limited by wave breaking, if we assume that the wind puts energy mainly into the longer wave components that are about to be limited by breaking, it is arguable that the loss of energy through breaking should be comparable with the rate of growth of the total wave energy:

$$\varpi E/T \sim \mathrm{d}E/\mathrm{d}t. \tag{3·2}$$

Of these two inferences, (3·1) appears the better founded. But if both inferences are correct, then it follows that

$$W \sim \mathrm{d}E/\mathrm{d}t \tag{3·3}$$

as suggested on other grounds by R. W. Stewart (1961). Of course it is always true that the left-hand side of (3·3) is greater than the right.

To estimate the rate of working of the wind we may adopt more than one method. Perhaps the simplest is to assume that

$$W \sim \tau u_*, \tag{3·4}$$

where τ denotes the wind-stress and u^* the friction-velocity, defined by

$$\tau = \rho_{\mathrm{air}} u_*^2. \tag{3·5}$$

Since it is known empirically that

$$\tau = C\rho_{\mathrm{air}} U^2, \tag{3·6}$$

where U denotes the mean wind-speed at a height 10 m above the sea surface, and $C \sim 1·5 \times 10^{-3}$, it follows that, on this basis, $u_*^2 = CU^2$ and so

$$W \sim \rho_{\mathrm{air}} u_*^3 = C^{\frac{3}{2}}\rho_{\mathrm{air}} U^3. \tag{3·7}$$

Alternatively we may adopt the suggestion of Stewart & Grant (1962) and assume that, since energy and momentum are transferred to a wave in the ratio of the phase velocity, then

$$W = \tau c_1 = C\rho_{\mathrm{air}} U^2 c_1 \tag{3·8}$$

where c_1 denotes the phase velocity of the waves that are receiving most of the energy. The two estimates are the same if it is supposed that

$$c_1 \sim C^{\frac{1}{2}} U \sim 0·04U. \tag{3·9}$$

Without making this assumption we shall simply adopt equation (3·7), which is almost certainly correct to within an order of magnitude.

Now ϖ, by equation (3·1), is given by

$$\varpi \sim TW/E. \tag{3·10}$$

We need then to estimate typical values of T and E. Referring to the observations of Sverdrup & Munk (1947) we take the phase speed c of the dominant waves to be equal to U nearly, so that

$$T = \frac{2\pi c}{g} \sim \frac{2\pi U}{g}, \tag{3·11}$$

and we take the 'significant wave height' H to be equal to $0{\cdot}25U^2/g$ approximately. Since for a Rayleigh distribution $H = 2{\cdot}83\bar{a}$ (Longuet-Higgins 1952) we have $\bar{a} \sim 0{\cdot}088U^2/g$ and so

$$E = \tfrac{1}{2}\rho g\bar{a}^2 \sim 3{\cdot}9\times 10^{-3}\rho U^4/g. \tag{3·12}$$

On substituting into equation (3·10) from equations (3·11), (3·12) and (3·7) we find that the terms in U and g cancel each other identically, giving

$$\varpi \sim 1600C^{\frac{3}{2}}\rho_{\text{air}}/\rho_{\text{water}} \tag{3·13}$$

or, with the values $C = 1{\cdot}5\times 10^{-3}$ and $\rho_{\text{air}}/\rho_{\text{water}}$ $1{\cdot}3\times 10^{-3}$,

$$\varpi \sim 1{\cdot}0\times 10^{-4}. \tag{3·14}$$

Having now estimated the numerical value of ϖ we are in a positionto find α from equation (2·18):

$$\alpha \doteqdot -\frac{1}{8\ln\varpi} \doteqdot 1{\cdot}35\times 10^{-2}. \tag{3·15}$$

We notice that this value of α falls within the range of the experimentally determined values of α as shown in table 1. However, we may remark that if any of the relations between W, E, T and U were to be altered by a factor of 2 this would change the value of $\ln\varpi$ by only one part in 12. Hence the corresponding value of α given by (3·13) would only vary by this amount and so would still be within the range of observation.

However, the replacement of equation (2·18) by equation (2·22) would put α just beyond the upper limit of the observed range.

4. Is α really constant?

For the appropriate values of E and T we selected, in the above analysis, the values given by Sverdrup & Munk (1947) for large values of the nondimensional fetch (gx/U^2), say $(gx/U^2) \geqslant 10^4$. In a sea state which is still growing under the action of the wind, smaller values of (gx/U^2) would be appropriate and hence smaller values of both E and T. The variation of T under different wind conditions is generally less marked than the variation of E. Hence ϖ, which is proportional to T/E, might be expected to *decrease* with $(gx/U)^2$, and so α, by (3·15), would also decrease with (gx/U^2).

This inference gains some support from the data in table 1. In the second and third columns are shown the logarithmic-mean values of the fetch x and the wind speed U for each of the observations quoted, and in the fourth column the logarithmic-mean value of (gx/U^2). The values of α in the last column do indeed decrease as the parameter (gx/U^2) increases, with the exception of the value attributed to Pierson (1960). This was derived from measurements with a vertical wave pole, the calibration of which was uncertain (see Chase *et al.* 1957). Apart from this observation all the values of α decrease monotonically with (gx/U^2).

That α should be a decreasing function of the nondimensional fetch or of the 'wave age' (gt/U) is in accord with the common observation that when a breeze *suddenly* starts to blow white caps appear initially to be more numerous, indicating a temporarily increased value of α.

The equation whether the wind-stress coefficient C is also a function of the time would be worth investigation. Some of the scatter in the experimental determinations of C (see Phillips 1966, figure 4·18) may perhaps be explained in this way.

Some further light is thrown on these conclusions by some recent observations of fetch-limited waves by Barnett & Wilkerson (1967). The observations were made in a practically uniform wind-field $U \doteqdot 16$ m/s at distances from the shoreline ranging from 50 to 300 km, so that

$$2000 < gx/U^2 < 12000.$$

The most striking observational conclusion was that as the fetch x increased, so the peak energy density in the spectrum (which occurs at the low-frequency end of the equilibrium range) first rose to a value 1·5 to 3 times greater than its final, equilibrium value. The authors suggest, in fact an equilibrium spectrum of the form

$$F(\sigma) = \alpha g^2 \sigma^{-5} f(\sigma/\sigma_0), \tag{4·1}$$

where σ_0 is the radian frequency of the maximum spectral density. This frequency diminishes as the fetch is increased. Clearly, if an attempt were made to make a best fit of the equilibrium law (1·1) to a typical spectrum, the constant α would have to be adjusted differently according to the fetch, the highest values necessarily occurring at the shortest fetches. This is apparently consistent with our conclusions.† Nevertheless an examination of the original spectra summarized in table 1 shows that many of them do not exhibit such a spectral peak as found by Barnett & Wilkerson, even at quite small values of (gx/U^2). We therefore prefer not yet to adopt any particular form of $f(\sigma/\sigma_0)$ apart from that implied by equation (1·1). There is no reason in principal why the calculation of ϖ in terms of $F(\sigma)$ given in § 2 should not be adapted to a function $f(\sigma/\sigma_0)$ of any reasonable form.

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† If the peak values of the spectral density are neglected, Barnett & Wilkerson find a rather low value of α, namely $0·6 \times 10^{-2}$.

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References

Barnett, T. P. & Wilkerson, J. C. 1967 On the generation of ocean wind waves as inferred from airborne measurements of fetch-limited spectra. *J. Mar. Res.* **25**, 292–321.

Burling, R. W. 1959 The spectrum of waves at short fetches. *Dtsch. Hydr. Z.* **12**, 45–64, 96–117.

Cartwright, D. E. & Longuet-Higgins, M. S. 1956 On the distribution of the maxima of a random function. *Proc. Roy. Soc.* A **237**, 212–232.

Chase, J. *et al.* 1957 The directional spectrum of a wind generated sea as determined from data obtained by the Stereo Wave Observation Project. *New York Univ. Coll. Eng. Rep.* July 1967.

Hicks, B. L. 1960 The energy spectrum of small wind waves. *Univ. Illinois, C.S.L. Rep.* M-91.

Kinsman, B. 1960 Surface waves at short fetches and low wind-speed—a field study. *Chesapeake Bay Inst.* Rep. no. 19.

Lamb, H. 1932 *Hydrodyanmics* (6th ed.). Cambridge University Press.

Longuet-Higgins, M. S. 1952 On the statistical distribution of the heights of sea waves. *J. Mar. Res.* **11**, 245–266.

Longuet-Higgins, M. S. 1963 The generation of capillary waves by steep gravity waves. *J. Fl. Mech.* **16**, 138–159.

Longuet-Higgins, M. S., Cartwright, D. E. & Smith, N. D. 1963 Observations of the directional spectrum of sea waves using the motions of a floating buoy. *Ocean wave spectra*, pp. 111–136, Englewood Cliffs, N.Y.: Prentice-Hall, Inc.

Michell, J. H. 1893 The highest waves in water. *Phil. Mag.* **36**, 430–437.

Neumann, F. & Pierson, W. J. 1966 *Principles of Physical Oceanography*. Englewood Cliffs, N. J., Prentice-Hall, 545 pp.

Phillips, O. M. 1958 The equilibrium range in the spectrum of wind-generated waves. *J. Fl. Mech.* **4**, 426–434.

Phillips, O. M. 1966 *The dynamics of the upper ocean*. Cambridge University Press.

Pierson, W. J. (ed.) 1960 The directional spectrum of a wind-generated sea as determined from data obtained by the Stereo Wave Observation Project. *NewYork Univ. Coll. Eng., Met. Pap.* **2**, no. 6.

Rice, S. O. 1944 The mathematical analysis of random noise. *Bell Syst. Tech. J.* **23**, 282–332.

Stewart, R. W. 1961. The wave drag of wind over water. *J. Fl. Mech.* **10**, 189–194.

Stewart, R. W. & Grant, H. L. 1962 Determination of the rate of dissipation of turbulent energy near the sea surface in the presence of waves. *J. Geophys. Res.* **67**, 3177–3180.

Stokes, G. G. 1847 On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.* **8**, 441–455.

Sverdrup, H. U. & Munk, W. H. 1947 Wind, sea and swell: theory of relations for forecasting. *U.S. Navy Hydrogr. Office Pub.* no. 601.

Taylor, G. I. 1953 An experimental study of standing waves. *Proc. Roy. Soc.* A **218**, 44–59.

Watters, J. K. 1953 Distribution of height in ocean waves. *N.Z. J. Sci. Tech.* B **34**, 409–422.