Introduction

By G. E. R. Deacon, F.R.S.

A hundred years ago the Society often listened to papers about the ocean, but the rapid growth of science, especially the development of those aspects that can be carried through with the help of laboratory experiments, has led to some neglect of large-scale natural processes, in favour of those that can be more readily formulated and imitated in models. Although aiming at better understanding and use of the world we live in, we have concentrated on the most approachable and profitable aspects; in consequence, studies of the earth, oceans and atmosphere, have become 'fringe subjects'. But they are staging a come-back. The application of modern theoretical and practical methods to these problems brings results as exciting as any that can be obtained in a laboratory, and just as suitable for Ph.D. courses, which is one of the things that seems to matter, if they were better known. We must be grateful to the Society for giving us this opportunity to talk about the oceans. I am sure the papers will show the wide range of interest and excitement as well as the difficulties.

The directional spectrum of ocean waves, and processes of wave generation

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[Plate 21]

This paper describes some recent observations of the directional spectrum of sea waves and of air pressure fluctuations at the sea surface, and discusses their implications for theories of wave generation.

The angular spread of the wave energy in the generating area is found to be comparable with the 'resonance angle' $\sec^{-1}(\sigma U/g)$ (σ = wave frequency, U = wind speed) but lies slightly below it in the middle range of frequencies. The best fit to the directional spectrum $F(\sigma, \phi)$ is shown to be a cosine-power law: $F(\sigma, \phi) \propto \cos^{2s}(\frac{1}{2}\phi)$, where s decreases as σ increases. At the higher frequencies the total spectrum satisfies the equilibrium law: $F(\sigma) \propto \sigma^{-5}$.

The initial stages of wave generation are attributed to turbulence in the air stream, and the main stage of growth to the shear instability mechanism described by Miles. At the highest frequencies the form of the spectrum suggests that wave breaking plays a predominant part, as proposed by Phillips. The broadening of the angular distribution at the highest frequencies may also be due partly to third-order 'resonant' interactions among components of the wave spectrum.

The air-pressure fluctuations are nearly in phase with the vertical displacement of the sea surface (over most of the frequency range) and are consistent with the shear-flow model proposed by Miles. The turbulent component of the air pressure is much smaller than was previously supposed.

Introduction

The common observation that a wind blowing over a water surface may generate travelling waves is so familiar to us that we are apt to forget how little we understand of the actual process of wave generation. Not only is our lack of understanding philosophically unsatisfactory, but it is a practical hindrance when we try to forecast sea waves and swell for shipping, coast protection and many other purposes. The various empirical formulae which have had to be relied upon (for example, Darbyshire 1955, 1959*a*; Pierson, Neumann & James 1955) appear to be incomplete and often inapplicable outside the particular areas for which they were designed.

Only a few years ago attention was drawn to the unsatisfactory state of our theoretical knowledge by Ursell (1956). Since that time a number of promising ideas on the generation of sea waves have been published, and we have the benefit of better and more complete wave observations made instrumentally in areas of wave generation. The purpose of this paper is to review some of the recent ideas, and to discuss some measurements of the directional spectrum made by a new technique (Longuet-Higgins, Cartwright & Smith 1962). Some evidence on the distribution of air pressure over the sea surface will also be discussed.

The statistical representation of the sea surface is introduced in § 1; this is fundamental to all the more recent theoretical work, and some of the experimental results supporting it is described. In § 2 we discuss the theories of wave generation due to Miles and Phillips, and the transfer of energy between different parts of the spectrum due to non-linear interactions. The experiments relating to wave generation are described in §§ 3 and 4. A final section summarizes the conclusions and makes suggestions for further research.

1. Representation of the sea surface

In the classical theories of Airy (1842) and Stokes (1847) a gravity wave of small steepness in water of uniform depth h is represented approximately by the equation for the surface elevation

$$\zeta = a\cos(\mathbf{k}.\mathbf{x}-\sigma t), \tag{1·1}$$

where $\mathbf{x}$ and t denote the horizontal co-ordinate and the time. The wave number $\mathbf{k}$ and frequency σ are related by

$$\sigma^2 = gk\tanh kh \qquad (k = |\mathbf{k}|). \tag{1·2}$$

For water so deep that e^{-kh} is negligible this relation becomes

$$\sigma^2 = gk. \tag{1·3}$$

In the derivation of these equations it is assumed that the waves are of sufficiently small steepness ($ak \ll 1$ and $ak \ll (kh)^3$) to justify neglect of all non-linear terms in the equations of motion and in the boundary conditions at the free surface.

So far from being regular and periodic, however, the sea surface generally presents a very irregular appearance, with short wave crests and undulations of widely different scales. A more realistic representation of the surface was therefore introduced by Longuet-Higgins & Barber (1946) in which the single sine-wave of (1·1) was replaced by a sum

$$\zeta = \sum_n a_n \cos(\mathbf{k}_n . \mathbf{x} - \sigma_n t + \epsilon_n), \tag{1·4}$$

where the wave numbers $\mathbf{k}_n$ and frequencies σ_n are related by

$$\sigma_n^2 = gk_n \tanh k_n h \qquad (k_n = |\mathbf{k}_n|). \tag{1·5}$$

The phases ϵ_n are chosen at random. Thus the sea surface is considered as one of a statistical *ensemble* of surfaces. Attention is confined to certain average properties of the surface, as in other branches of statistical mechanics, the averages being calculated over the distribution of the phases.†

At a late stage in the analysis the number of component sine waves is made to tend to infinity and the amplitudes a_n to tend to zero in such a way that the sum of the squares of the amplitudes in a small (but fixed) element of wave number $\mathrm{d}\mathbf{k}$ is given by

$$\sum_{\mathrm{d}\mathbf{k}} \tfrac{1}{2} a_n^2 = E(\mathbf{k})\,\mathrm{d}\mathbf{k}, \tag{1·6}$$

where $E(\mathbf{k})$ is a continuous function. This function summarizes the useful information that can be obtained about the linear aspects of the surface. It is called the spectral density, or energy spectrum. From (1·4) and (1·6) we have

$$\overline{\zeta^2} = \sum_n \tfrac{1}{2} a_n^2 = \iint E(\mathbf{k})\,\mathrm{d}\mathbf{k}, \tag{1·7}$$

so that $E(\mathbf{k})\,\mathrm{d}\mathbf{k}$ is the contribution to the mean square wave height from wave components in the infinitesimal range of wave numbers $\mathrm{d}\mathbf{k}$.

The above type of formalism was used, notably by Rice (1944, 1945) for the representation of random noise in electrical circuits, and has since been widely employed in other branches of physics (for a recent list of references, see Tukey 1959). A thorough theoretical treatment is given, for example, by Doob (1953). Thus ζ might be considered as a stationary stochastic process, and (1·4) as its spectral representation.

Actually we know that the phys'cal process is not quite stationary statistically, for the horizontal extent of any storm is finite, and storms also grow and decay in time. This limits the accuracy with which the spectral density $E(\mathbf{k})$ may be measured. Thus in equation (1·6) $\mathrm{d}\mathbf{k}$ cannot be taken to be of smaller area A^{-1}, where A is the area of the sea surface which is statistically homogeneous. Similar limits are imposed by the development of the wave spectrum as a function of time.

Equations (1·4) to (1·6) may be called the linear model of the sea surface. To satisfy the boundary conditions at the free surface more exactly one may add to the right-hand side of (1·4) further terms proportional to the squares, products and higher powers of the amplitudes a_n. The nature of these higher-order terms will be discussed in § 2·3. In particular the third-order terms produce a small but ultimately appreciable transfer of energy between different wave numbers, comparable with the rate of growth of the waves under the wind. Fortunately it

† Phase averages such as (1·7) are assumed equal (with probability 1) to averages calculated with respect to $\mathbf{x}$ or t for a given member of the ensemble of surfaces.

appears that for many purposes, though always for a limited duration of time, the linear model is a good first approximation, and that the non-linear corrections to the model are small.

By contrast, in the theory of homogeneous turbulence the non-linear interactions between different parts of the spectrum cannot be neglected (see Batchelor 1953).†

For sea waves, the linear model has been checked experimentally in a variety of ways:

(1) The relation (1·5) between frequency and wave number has been verified within about 5 % even within the generating area (see Longuet-Higgins *et al.* 1962) at least for the range of frequencies containing most energy.

(2) The speed of propagation of wave energy outside the generating area has been shown to be equal approximately to the appropriate group velocity in deep water

$$\mathrm{d}\sigma/\mathrm{d}k = g/2\sigma$$

(Barber & Ursell 1948). In some remarkable observations in California swell has been received from distances of nearly 12000 miles, and the corresponding accuracy in determining the speed of propagation is very great (Munk *et al.*, in preparation).

(3) A consequence of the linear model (1·4) is that, with only mild restrictions on the order of magnitude of the a_n (see Rice 1944, §2·10) the probability density of ζ is normal with mean value 0. Under corresponding conditions the derivatives of ζ, up to any order, are also distributed normally. An example of the Gaussian distributions for ζ, $\partial\zeta/\partial x$ and $\partial\xi/\partial y$ being roughly satisfied is shown in figures 6(*a*), (*b*), (*c*) of the present paper. Earlier (Longuet-Higgins 1952; Cartwright & Longuet-Higgins 1956) the distribution of the heights of the *maxima* in a wave record was shown to agree well with that corresponding to a Gaussian surface. However, it should be borne in mind that these data do not take account of the highest frequencies present in the sea surface, which have been filtered out by the method of observation. Thus, if the distribution of surface slopes under a wind is measured optically (Cox & Munk 1954, 1956) some skewness in the down-wind direction is discernible. A slight skewness in the distribution of ζ is reported by Kinsman (1960). The surface curvatures, measured optically, can be very non-Gaussian (Schooley 1955).

In shallow water waves, where kh may be small, some evidence of departures from the Gaussian distribution have been noted by Birkhoff & Kotik (1952) and by Darbyshire (1959*b*). Such departures are less surprising in shallow water since one of the conditions for linearity in the classical theory of water waves was seen to be that $ak \ll (kh)^3$.

In passing, we may note some of the other fundamental properties which make the linear model, when applicable, most convenient in practical use.

(1) On passage through a linear filter the spectral density is multiplied by a function of $\mathbf{k}$ and σ only, independently of E. For example, a pressure recorder

† A spectral representation such as (1·4) can of course be defined for any absolutely continuous, statistically stationary process. However, the phases ϵ_n are then not independent, and generally the Gaussian distribution, discussed below, does not apply. Nor does the relation (1·5) between frequency and wave number.

placed on the sea bed records a signal whose spectrum, in theory, would be that of the surface elevation multiplied by the factor $(\rho g/\sinh kh)^2$. (However, at depths of more than one-quarter of a wavelength this factor is so small that second-order pressure fluctuations may predominate; see Cooper & Longuet-Higgins 1951.)

The motions of ships may be calculated in a similar way, the ship being treated as a linear filter (Pierson & St Denis 1953; Cartwright & Rydill 1957). Each harmonic component of the ship's heave, for example, can be related to certain harmonic components of the sea surface spectrum by factors depending on the wave number and the speed and dimensions of the ship.

(2) The spectral density $E(\mathbf{k})$ is transformed in a very simple way when energy is freely propagated from one part of the ocean to another (see Barber 1958). A simple transformation also applies when the waves are refracted near the shore by water of varying depth (Longuet-Higgins 1957). Thus, though the wave number $\mathbf{k}$ of a refracted wave varies along the ray path, the spectral density $E(\mathbf{k})$ remains a constant. As mentioned by Dorrestein (1960) this property constitutes part of a deep analogy between the statistical mechanics of particles and of waves.

(3) Once the spectrum of any derived motion or property has been determined, the corresponding statistical properties can be derived, as for example the distribution of the maxima, the distribution of the intervals between successive zeros or maxima, and so on. For a review of such statistical properties, see Longuet-Higgins (1961). Many interesting statistical properties involve the corresponding spectrum only through certain moments of the form

$$m_{pqr} = \iint E(\mathbf{k})\, k_x^p\, k_y^q \sigma^r \,\mathrm{d}^k, \quad (k_x k_y) = \mathbf{k}.$$

Thus whereas the statistical distributions themselves do not transform directly by any simple law, they are conveniently related to the corresponding spectra, which do.

The linear model defined in this section will be mainly relied upon as a basis for the subsequent discussion. However, instead of the spectral density $E(\mathbf{k})$ it is slightly more convenient to define the *directional spectrum* $F(\sigma, \phi)$, such that

$$\sum_{\mathrm{d}\sigma\,\mathrm{d}\phi} \tfrac{1}{2}a_n^2 = F(\sigma, \phi)\,\mathrm{d}\sigma\,\mathrm{d}\phi,$$

where σ denotes the frequency and ϕ the direction of propagation of each harmonic component. In other words $F(\sigma, \phi)\,\mathrm{d}\sigma\,\mathrm{d}\phi$ denotes the contribution to the mean-square value of ζ from frequencies in the range $(\sigma, \phi + \mathrm{d}\sigma)$ and directions in the range $(\phi, \phi + \mathrm{d}\phi)$. We have

$$\mathbf{k} = (k_x, k_y) = (k\cos\phi,\, k\sin\phi); \quad \frac{\partial(k_x, k_y)}{\partial(k, \phi)} = k,$$

and so

$$F = \frac{\partial(k_x, k_y)}{\partial(\sigma, \phi)} E = k\frac{\mathrm{d}k}{\mathrm{d}\sigma} E = \frac{2k^2}{\sigma} E$$

in deep water. Hence if E is known, so also is F, and vice versa.

We shall use $F(\sigma)$ to denote the spectral density of ζ with regard to frequency only. Thus $F(\sigma)\,\mathrm{d}\sigma$ denotes the contribution to $\overline{\zeta^2}$ from the range of frequencies $(\sigma, \sigma + \mathrm{d}\sigma)$, and

$$F(\sigma) = \iint_0^{2\pi} F(\sigma, \phi)\,\mathrm{d}\phi.$$

2. Processes of wave generation

We now review briefly some of the more important theories of wave generation, that have been suggested since Ursell's review (1956).

2·1. *Generation of waves by atmospheric turbulence*

Suppose a turbulent stream of air begins to flow over a water surface which initially is at rest. Associated with the turbulent eddies in the air stream are fluctuations in the air pressure, which must act on the water surface so as to produce waves. The turbulent pressure fluctuations are assumed essentially independent of any waves present at the time.

Following an earlier attempt by Eckart (1953), this theory was extensively developed by Phillips (1957). In Phillips's formulation U denotes the convection velocity of the turbulence, that is to say the (horizontal) velocity of the frame of reference in which the characteristic development time of the eddies appears greatest. U is plausibly identified with the mean wind speed, at some height comparable with the scale of the eddies. Phillips showed that in the main stage of development of the waves (that is to say at times t large compared with the development time of the eddies) the directional spectrum of the waves was given by

$$F(\sigma,\phi) \sim \frac{k^2\sigma t}{2(g\rho_w)^2}\int_0^\infty \Pi(\mathbf{k},\tau)\cos\left[\left(\frac{U\cos\phi}{c}-1\right)\sigma\tau\right]d\tau, \qquad (2\cdot1\cdot1)$$

where ρ_w denotes the density of water; $\Pi(\mathbf{k},\tau)$ denotes the spectrum of the pressure fluctuations at the water surface at time τ in the moving frame of reference; $\sigma^2 = gk$ and $c = g/\sigma$. In (2·1·1) the chief approximation made, apart from the linearization and neglect of viscosity, is omission of a term of much smaller magnitude than the right-hand side, corresponding to waves propagated in the opposite sense.

If the pressure spectrum $\Pi(\mathbf{k}, t)$ is fairly isotropic, with no preferential directions for k, it can be shown that the integral on the right of (2·2·1), considered as a function of the azimuth ϕ for fixed σ, is greatest when the coefficient of $\sigma\tau$ vanishes, that is to say when

$$U\cos\phi = c = g/\sigma. \qquad (2\cdot1\cdot2)$$

The interpretation of this is very simple. Let us confine attention to the pressure fluctuations which have a certain scale $k = |\mathbf{k}|$. These fluctuations tend to produce waves of the same scale k, travelling at various angles ϕ to the wind. Generally the wind speed will be different from that of the corresponding waves. However, if the component of the wind speed resolved in the direction of the waves just equals the speed of a free gravity wave (g/σ), then there is a matching between the pressure input and the free mode of oscillation; a kind of resonance takes place, and the wave amplitude builds up more quickly. The corresponding angle

$$\phi = \alpha = \sec^{-1}(U/c) \qquad (2\cdot1\cdot3)$$

may be called the 'resonance' angle.

On this theory, then, we might expect, at least for $U \geqslant g/\sigma$, that the wave energy in each frequency band would be found travelling mostly in two directions $\phi = \pm\alpha$ (figure 1) to either side of the mean wind direction.

At the time that this theory was proposed (1957) very little was known about the turbulent pressure fluctuations in the atmosphere, especially over the sea, and in the later part of his paper Phillips made the estimate

$$\overline{p^2} = 0{\cdot}1\rho_a^2 U^4, \tag{2·1·4}$$

where ρ_a is the density of the air. With the aid of some rough approximations to the integral (2·1·1) this brought the theory into order-of-magnitude agreement with the observed heights of storm waves. But we now have direct evidence, to be described later in this paper, that the estimate (2·1·4) is too high by a factor of the order of 10^2.

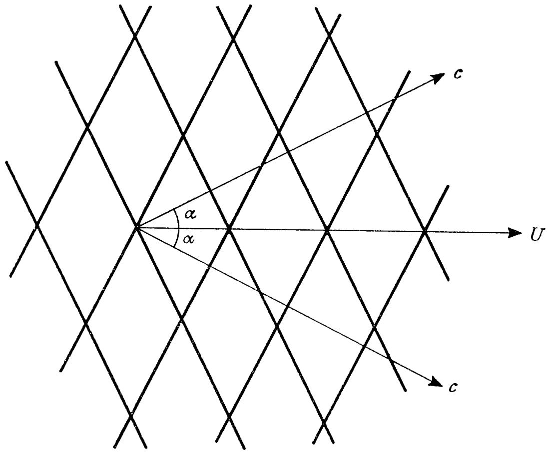


FIGURE 1. The pattern of waves generated by an isotropic travelling pressure disturbance of fixed scale.

This does not necessarily disprove Phillips's (1957) theory; the approximations involved in the later part of his paper, involving the equality of a differential and integral time-scale, may not be correct.† However, a more serious objection has been brought forward by Miles, in the course of a theory which is described next.

2·2. *Generation of waves by shear-flow instability*

The physical model suggested by Miles (1957) was quite different. He considered the flow of air over a fluid boundary in which there was already present a small sinusoidal displacement. He then estimated the pressure on the water surface resulting from the perturbation of the original flow. The component of the pressure in phase with the surface elevation, imparts no energy to the waves; but the component in quadrature generally results in work being done on the water which, if it exceeds the loss due to molecular or eddy viscosity, causes the wave amplitude to grow (figure 2).

In some respects the theory bears a resemblance to Jeffreys's theory of wave generation by sheltering (1925, 1926). However, there are some essential differences: in Jeffreys's theory the sheltering coefficient s is not known *a priori* and

† In Eckart's rather different formulation of the problem (1953) the assumed pressure pulses on the surface were also found insufficient to account for all the wave energy.

must be estimated from observational data; in different circumstances widely different values of s are found (Ursell 1956). On the other hand, theory in Miles's the sheltering coefficient is calculated from the physical model.

For Miles's theory it is necessary that the wind profile be non-uniform; the energy input into a wave of phase velocity c is related to the *curvature* of the velocity profile at the height where $U = c$. By contrast, the mechanism for wave generation suggested by Kelvin and Helmholtz, which depends on the in-phase component of the pressure, is not effective till much higher wind speeds; see Miles (1959*b*).

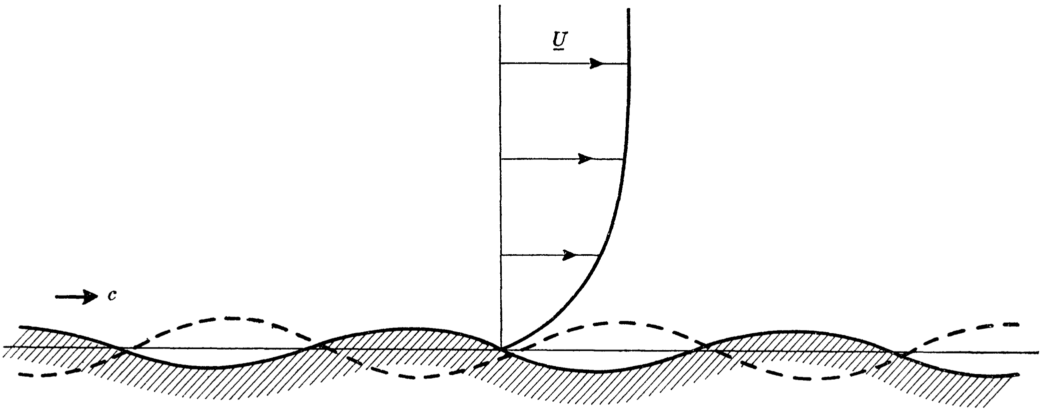


FIGURE 2. The pressure distribution due to shearing flow over a wave.

The estimated phase shift (and hence the sheltering coefficient) depends critically on the velocity profile of the air stream. Some presently available observations (Roll 1948; Hay 1955) support a logarithmic form for the velocity profile

$$U = U_1 \log(z/z_0), \tag{2·2·1}$$

where U_1 is a reference velocity† and z_0 is a roughness parameter. Empirical formulae quoted by Sheppard (1958) for conditions of neutral thermal stability give

$$U_1 = (U_a/K)(0{\cdot}08 + 0{\cdot}00114 U_a)^{\frac{1}{2}} \times 10^{-3}, \tag{2·2·2}$$

where U_a is the 'anemometer wind speed' in cm/s. In typical cases U_1 is of order $\frac{1}{9}U_a$. The roughness parameter z_0 is conveniently expressed as

$$z_0 = \Omega U_1^2/g, \tag{2·2·3}$$

where Ω is a non-dimensional parameter. The data of Hay (1955) indicate a value of Ω consistently around $1{\cdot}3 \times 10^{-2}$; but Roll's data (1948) suggest that it can differ from this by a factor of 2 or 3.

Assuming a logarithmic wind profile, Miles (1957, 1959*a*) calculated the quadrature component of the pressure‡ and hence the energy input into the waves. He showed that the rate of growth by instability could account for the observed energies of storm waves.

† $U_1 = U^*/K$, where U^* is the 'friction velocity' and K is von Kármán's constant.
‡ At least the more appreciable non-viscous part; see also Brooke Benjamin (1959).

More recently (1960) Miles combined the two theories of wave generation by turbulent pressure fluctuations and by shear-flow instability. He showed that the effect of the instability on the main stage of growth of the waves is to multiply Phillips's expression (2·1·1) by the factor

$$\frac{e^{MT}-1}{MT} = f, \tag{2·2·4}$$

say, where

$$\left.\begin{aligned} M &= \frac{\rho_a}{\rho_w}\left(\frac{U_1 \cos\phi}{c}\right)^2 \frac{U_1}{c}\beta, \\ T &= gt/U, \end{aligned}\right\} \tag{2·2·5}$$

and β is given by the calculation. Now initially, that is to say as $t \to 0$ the factor f tends to unity; thus Phillips's expression (2·1·2) still holds. But as t grows large ($MT \gg 1$) so f increases exponentially, indicating that the instability mechanism is dominant. Miles calculated f for a few typical values of the parameters appropriate to ocean waves, and found that it could lie between 1 and 10^6, depending on the ratio of wind speed to wave speed, the angle ϕ and the total time t. The implication is that although the waves may be *started* by turbulent pressure fluctuations, the most rapid stages of growth are due to shearing instability and not to turbulence, except possibly when $U/c \doteqdot 1$.

Since also f varies with the angle ϕ, and is greatest at small angles (that is, for waves running nearest the direction of the wind) one may conclude that the effect of the instability is to increase most strongly those wave components which travel in directions nearest to the wind. Thus the instability tends to narrow the directional spread of the energy in any particular band of frequencies.

2·3. *Third-order wave interactions*

Consider two harmonic components having wave numbers $\mathbf{k}_1$, $\mathbf{k}_2$ and frequencies σ_1, σ_2 (not necessarily in the same direction). Each individually will satisfy, to the first order, the condition of constant pressure at the free surface provided that

$$\sigma_1^2 = g\,|\mathbf{k}_1|, \quad \sigma_2^2 = g\,|\mathbf{k}_2|. \tag{2·3·1}$$

In the first approximation the two waves do not interact; their sum also satisfies the linearized boundary condition.

In the second approximation (taking into account squares and products of the wave slopes) the boundary conditions are equivalent to a perturbation on the first-order solutions; the forcing function in the perturbation can be split up into terms with frequencies $2\sigma_1$, $2\sigma_2$, $(\sigma_1+\sigma_2)$, $(\sigma_1-\sigma_2)$ and corresponding wave numbers $2\mathbf{k}_1$, $2\mathbf{k}_2$, $(\mathbf{k}_1+\mathbf{k}_2)$, $(\mathbf{k}_1-\mathbf{k}_2)$. However, the relations (2·3·1) imply

$$\left.\begin{aligned} (2\sigma_1)^2 &\neq g\,|2\mathbf{k}_1|, \quad (\sigma_1+\sigma_2)^2 \neq g\,|\mathbf{k}_1+\mathbf{k}_2|, \\ (2\sigma_2)^2 &\neq g\,|2\mathbf{k}_2|, \quad (\sigma_1-\sigma_2)^2 \neq g|\mathbf{k}_1-\mathbf{k}_2|, \end{aligned}\right\} \tag{2·3·2}$$

except in the trivial case when σ_1 or σ_2 vanishes. Hence the perturbations do not produce a resonant response, and the second-order interactions are bounded.

However, Phillips (1960) has pointed out that in the third approximation the situation is very different. For example, there are perturbation terms with frequency $(2\sigma_1 - \sigma_2)$ and wave number $(2\mathbf{k}_1 - \mathbf{k}_2)$, and for a suitable choice of wave numbers it *is* possible to have

$$(2\sigma_1 - \sigma_2)^2 = g\,|2\mathbf{k}_1 - \mathbf{k}_2|, \tag{2·3·3}$$

consistently with (2·3·1). In that case the interaction between the two primary waves produces a resonant response which grows linearly with time; there is a small but (after a time) appreciable transfer of energy from the two primary wave numbers $\mathbf{k}_1$, $\mathbf{k}_2$ to the tertiary wave number $(2k_1 - \mathbf{k}_2)$. The characteristic time for the transfer of energy is of order $(s_1 s_2)^{-1}$ wave cycles, where s_1 and s_2 are the steepnesses of the two primary waves.

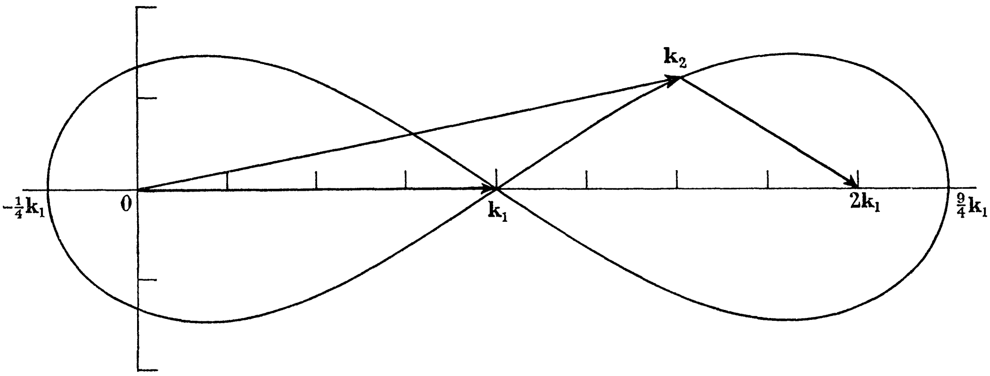


FIGURE 3. Relation of the wave numbers $\mathbf{k}_1$, $\mathbf{k}_2$, involved in the resonant interaction of two trains of waves.

Let $\mathbf{k}_1$ be fixed. Then in order to satisfy (2·3·1) and (2·3·3) Phillips (1960) shows that $\mathbf{k}_2$ must lie on a figure-of-eight with centre $\mathbf{k}_1$ and extremities $-\frac{1}{4}\mathbf{k}_1$ and $\frac{9}{4}\mathbf{k}_1$ (see figure 3). The interaction wave number is found by joining $\mathbf{k}_2$ to the point $2\mathbf{k}_1$, the image of O in the centre $\mathbf{k}_1$. The figure is symmetrical, and if $\mathbf{k}_2$ lies on the curve then so also does $(2\mathbf{k}_1 - \mathbf{k}_2)$, and vice versa.

In the special case when $\mathbf{k}_2$ coincides with $\mathbf{k}_1$ then the interaction wave number is also $\mathbf{k}_1$, and no new wave number is produced. In this case, and this case only, can the interaction be considered as equivalent to a small perturbation of the original wave frequency. Generally it appears that a transfer of energy must take place.

Such transfers of energy from one part of the frequency spectrum to another are recognized, for example, in the theory of turbulence. By analogy with turbulent flow we may conjecture that any spectrum that is non-isotropic will tend slowly towards isotropy. Moreover, the only absolutely stable spectrum (in the absence of external generating or dissipative forces) may be an isotropic spectrum.

2·4. *Transfer of energy by breaking*

Regular waves of a given length are capable of attaining only a certain amplitude before they break. One possible condition for breaking appears to be that the particles at the free surface shall attain a downwards acceleration equal to g. When this stage is reached, some at least of the wave energy must be lost or transferred to other parts of the spectrum.

Phillips (1958) has suggested that over a certain range of frequencies the spectral density may, for sufficiently high wind speeds, approach a saturated state determined by the process of breaking alone. This being independent of the wind speed and air density, and more or less independent of viscous dissipation, dimensional arguments show that

$$F(\sigma) = Cg^2\sigma^{-5}, \qquad (2\cdot4\cdot1)$$

where C is an absolute constant. Phillips showed that some data of Burling (1955) for rather short waves on a reservoir fitted this law well, provided

$$C = 1\cdot48 \times 10^{-2}. \qquad (2\cdot4\cdot2)$$

We shall see below that the same law applies to sea waves of much lower frequencies, and so much longer wavelengths, than Burling's original observations.

2·5. *Additional effects*

Among other processes affecting the waves may be mentioned the transfer of energy by 'tangential' wind stress; this is shown by Miles (1957) to be only a small part of the energy input from the shearing instability; the dissipation of wave energy by molecular viscosity, which is negligible for wave periods greater than about 1·5 s, except in shallow water; and the scattering of wave energy by turbulence (Phillips 1959) which also appears to be small.

3. Observations of the directional spectrum

The question of how to measure the directional spectrum $F(\sigma, \phi)$ is interesting in itself. A theoretical discussion of the use of linear arrays of wave records has been given by Barber (1957). The optimum spacing of such arrays depends somewhat on the wavelengths measured, and to the author's knowledge the only published example of such a measurement is for a single band of frequencies (Barber 1954).† A quite different but very neat suggestion is to use an aerial photograph of the sea surface as a diffraction grating for monochromatic light (Barber 1949); but the estimate of the spectrum thus obtained is only qualitative. The most complete evaluation of $F(\sigma, \phi)$ so far has been by the use of aerial stereophotography (Cote *et al.* 1960). However, this method requires considerable organization—involving two aircraft—and clear weather. In principle, the quantity finally obtained is not $F(\sigma, \phi)$ itself but $[F(\sigma, \phi) + F(\sigma, \phi + \pi)]$, for the method does not distinguish between waves travelling in directly opposite senses.

The observations to be described depend upon yet another method, first proposed by Barber (1946) and with developments by the present author (1946, 1955). This method makes use of the recorded motions of a freely floating buoy. A full account of both the theory and the observations is given in the paper by Longuet-Higgins *et al.* (1962); here we summarize only what is needed for an understanding of the results.

3·1. *Theory of the method*

From the motions of the buoy it is possible to obtain (within the framework of the linear theory) a record of the vertical displacement ζ of the free surface as a

† However, there is an ambiguity in the measurements which is resolved by assuming that the energy comes from one side of the array only.

function of the time t, in the neighbourhood of a fixed point with horizontal coordinates (x, y); and simultaneously a record of the two components of surface gradient, $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$. Thence, by calculating the auto-spectra and cross-spectra of ζ, $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ one derives the first five Fourier coefficients of $F(\sigma, \phi)$ with respect to ϕ, namely a_0, a_1, b_1, a_2, b_2, where

$$\left.\begin{aligned} a_n(\sigma) &= \frac{1}{\pi}\int_0^{2\pi} \cos n\phi F(\sigma, \phi)\,\mathrm{d}\phi, \\ b_n(\sigma) &= \frac{1}{\pi}\int_0^{2\pi} \sin n\phi F(\sigma, \phi)\,\mathrm{d}\phi. \end{aligned}\right\} \tag{3·1·1}$$

The higher coefficients (a_n, b_n; $n > 2$) can be found only if the higher derivatives, $\partial^2\zeta/\partial x^2$, $\partial^2\zeta/\partial x\,\partial y$, etc., are measured s multaneously.

One can make use of the information so obtained in various ways. In the first place, one can form the finite sum

$$F_1(\sigma, \phi) = \tfrac{1}{2}a_0 + (a_1\cos\phi + b_1\sin\phi) + (a_2\cos 2\phi + b_2\sin 2\phi), \tag{3·1·2}$$

which, by (3·1·1), can be expressed as

$$F_1(\sigma, \phi) = \frac{1}{2\pi}\int_0^{2\pi} F(\sigma, \phi')\; W_1(\phi' - \phi)\,\mathrm{d}\phi', \tag{3·1·3}$$

where
$$W_1 = 1 + 2\cos(\phi' - \phi) + 2\cos 2(\phi' - \phi) = \frac{\sin\frac{5}{2}(\phi' - \phi)}{\sin\frac{1}{2}(\phi' - \phi)}. \tag{3·1·4}$$

Thus F_1 is the smoothed average of F by the weighting function W_1. Since W_1 may be negative it is possible that F_1 takes some negative values, whereas F itself is essentially positive. Accordingly one may prefer to take some alternative approximation to F, for example

$$F_3(\sigma, \phi) = \tfrac{1}{2}a_0 + \tfrac{2}{3}(a_1\cos\phi + b_1\sin\phi) + \tfrac{1}{6}(a_2\cos 2\phi + b_2\sin 2\phi), \tag{3·1·5}$$

which corresponds to the weighted average of $F(\sigma, \phi)$ by the weighting function

$$W_3(\phi' - \phi) = 1 + \tfrac{4}{3}\cos(\phi' - \phi) + \tfrac{1}{3}\cos 2(\phi' - \phi) = \tfrac{8}{3}\cos^4\tfrac{1}{2}(\phi' - \phi). \tag{3·1·6}$$

W_3 is not only non-negative, but it is also a decreasing function of $|\phi' - \phi|$.

Apart from these weighted averages $F_i(\sigma, \phi)$, one can use the coefficients a_0, a_1, b_1, a_2, b_2 to provide useful parameters of the actual distribution $F(\sigma, \phi)$.

The simplest of these is a_0, which is proportional to the total energy per unit of frequency summed over all directions; in other words $a_0(\sigma) \propto F(\sigma)$.

Secondly, as a measure of the directional properties at each frequency we may define the two directions, ϕ_1, ϕ_2 which 'best' fit the distribution in the following sense. Consider the integral

$$I = \frac{1}{2\pi}\int_0^{2\pi} 16\sin^2\frac{\phi - \phi_1}{2}\sin^2\frac{\phi - \phi_2}{2}F(\sigma, \phi)\,\mathrm{d}\phi, \tag{3·1·7}$$

where ϕ_1 and ϕ_2 are, in the first place, arbitrary angles. I may easily be expressed in terms of ϕ_1 and ϕ_2 the five known coefficients a_0, a_1, b_1, a_2, b_2. We choose ϕ_1, ϕ_2 so as to make I a minimum. When the spectrum is not too broad it can be shown that

$$\bar{\phi}_1 = \tfrac{1}{2}(\phi_1 + \phi_2) \tag{3·1·8}$$

is approximately the mean direction, while

$$\psi_1 = \tfrac{1}{2}|\phi_1 - \phi_2| \tag{3·1·9}$$

is approximately the r.m.s. angular width of the directional distribution.

Finally, if $I_{\text{min.}}$ denotes the minimum value of (3·1·7) (corresponding to the two 'best' angles ϕ_1, ϕ_2) then $I_{\text{min.}}/a_0$ is an indicator of the *shape* of the directional distribution. For example, very small values of $I_{\text{min.}}/a_0$ imply that practically all the energy in that frequency band is being propagated in the directions ϕ_1, ϕ_2. In general, $I_{\text{min.}}/a_0$ is related to the fourth moment of the distribution of energy about the mean direction.

3·2. *The observations*

The apparatus consisted of a flat, circular buoy, 6 ft. in diameter and about 2 ft. in depth (see figure 4 (*a*), plate 21). The total weight (about 11 cwt.) was adjusted so that, when floating, the buoy was submerged as far as the upper rim of the vertical sides. Contained in the buoy were instruments to record the two angles of inclination, the 'pitch' and 'roll', and the vertical acceleration—which was integrated twice electronically so as to give a measure of the vertical displacement.

The response of the buoy to waves of different wavelengths was calibrated in the 30 ft. wide wave channel at the Ship Hydrodynamics Laboratory, Feltham. It was found that the buoy responded linearly to the surface displacement, with response factors and phase shifts dependent on frequency.

The buoy was kept in line with the local wind by a simple arrangement of drogue and pellet.

Simultaneously with the buoy's motions, the air-pressure fluctuations on the upper surface of the buoy were also recorded (see § 4).

TABLE 1

	number of record	date	time G.M.T.	position		wind speed and direction (ship's anemometer)
cruise I	1	31. v. 55	0915 to 0935	41° 08′ N	14° 37′ W	19 kt from 340°
	2	31. v. 55	1435 to 1455	41° 08′ N	14° 37′ W	14 kt from 350°
	3	3. vi. 55	0910 to 0930	39° 16′ N	11° 53′ W	17 kt from 320°
cruise II	4	30. x. 56	1450 to 1510	50° 58′ N	12° 15′ W	8 kt from 080°
	5	1. xi. 56	1525 to 1545	50° 19′ N	11° 54′ W	23 kt from 065°

Sixteen records of the buoy's motions, together with the air pressure fluctuations, were obtained during 1955 and 1956; but only five were sufficiently complete to be suitable for a harmonic analysis. Data relevant to these five records are shown in table 1. The windspeeds ranged from 8 to 23 knots. The corresponding weather charts at about the times of recording are shown in figure 5. In most of these the weather situation is not simple, and there is a possibility of swell being recorded from other than local winds. Fortunately, however, in one case (figure 5 (*d*)) the wind system was very simple; an anticyclone centred off the west coast of Scotland remained practically stationary for 2 days before the time of recording, and resulted in a steady north-easterly wind at a point off the south coast of Ireland where the waves were recorded. The corresponding record is no. 5 (windspeed 23 knots).

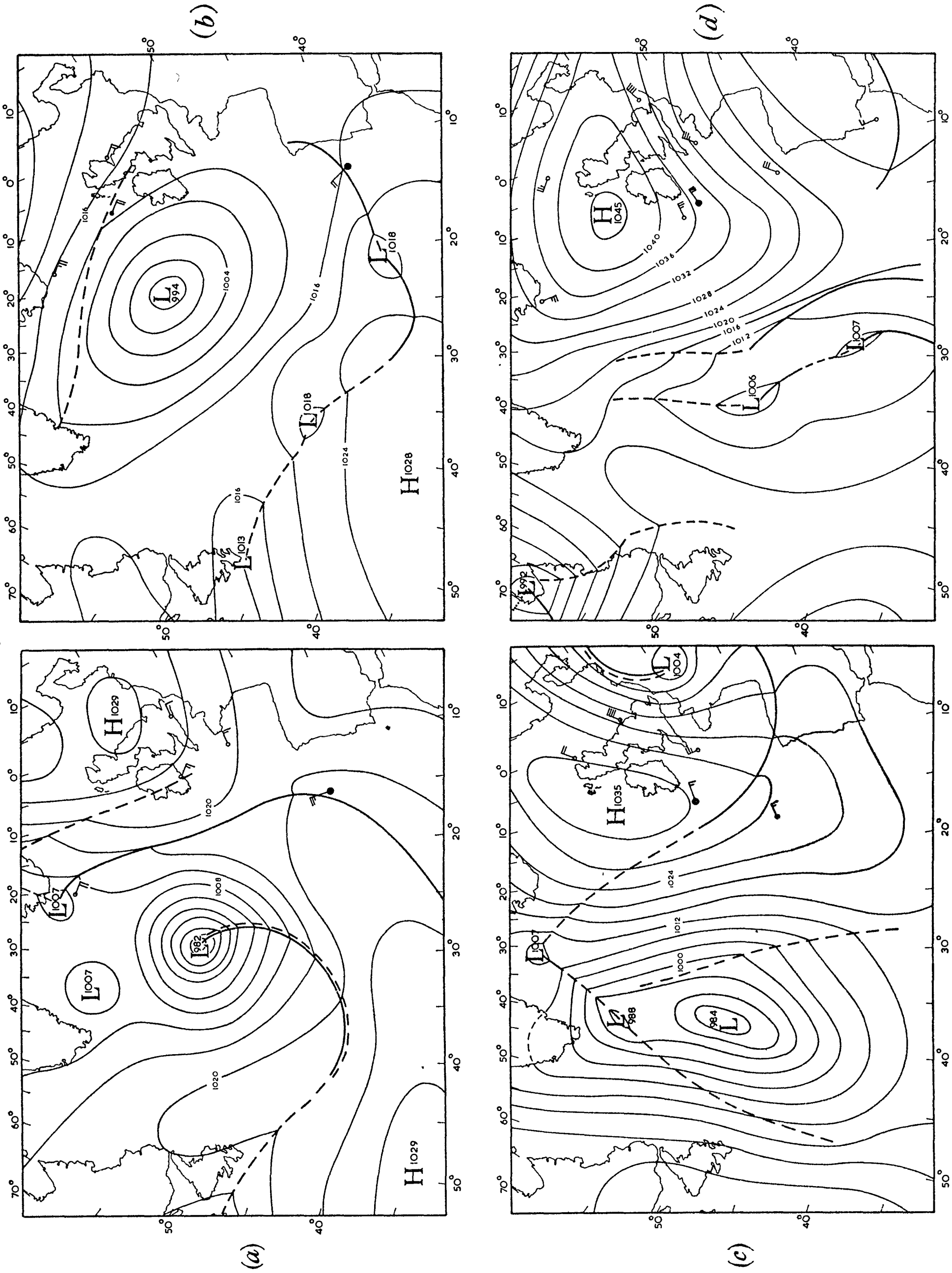


FIGURE 5. Charts showing the weather situations at about the times of wave recording.

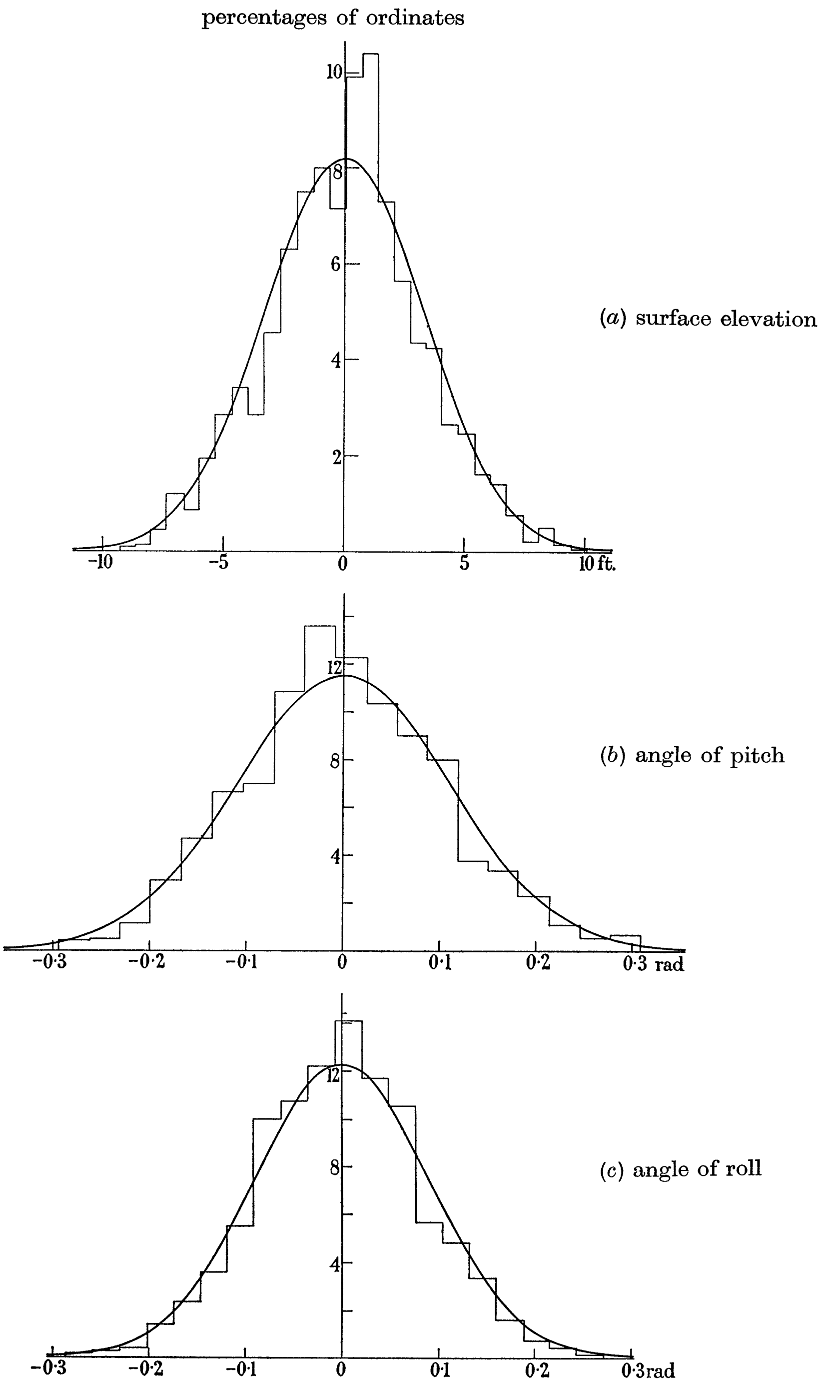
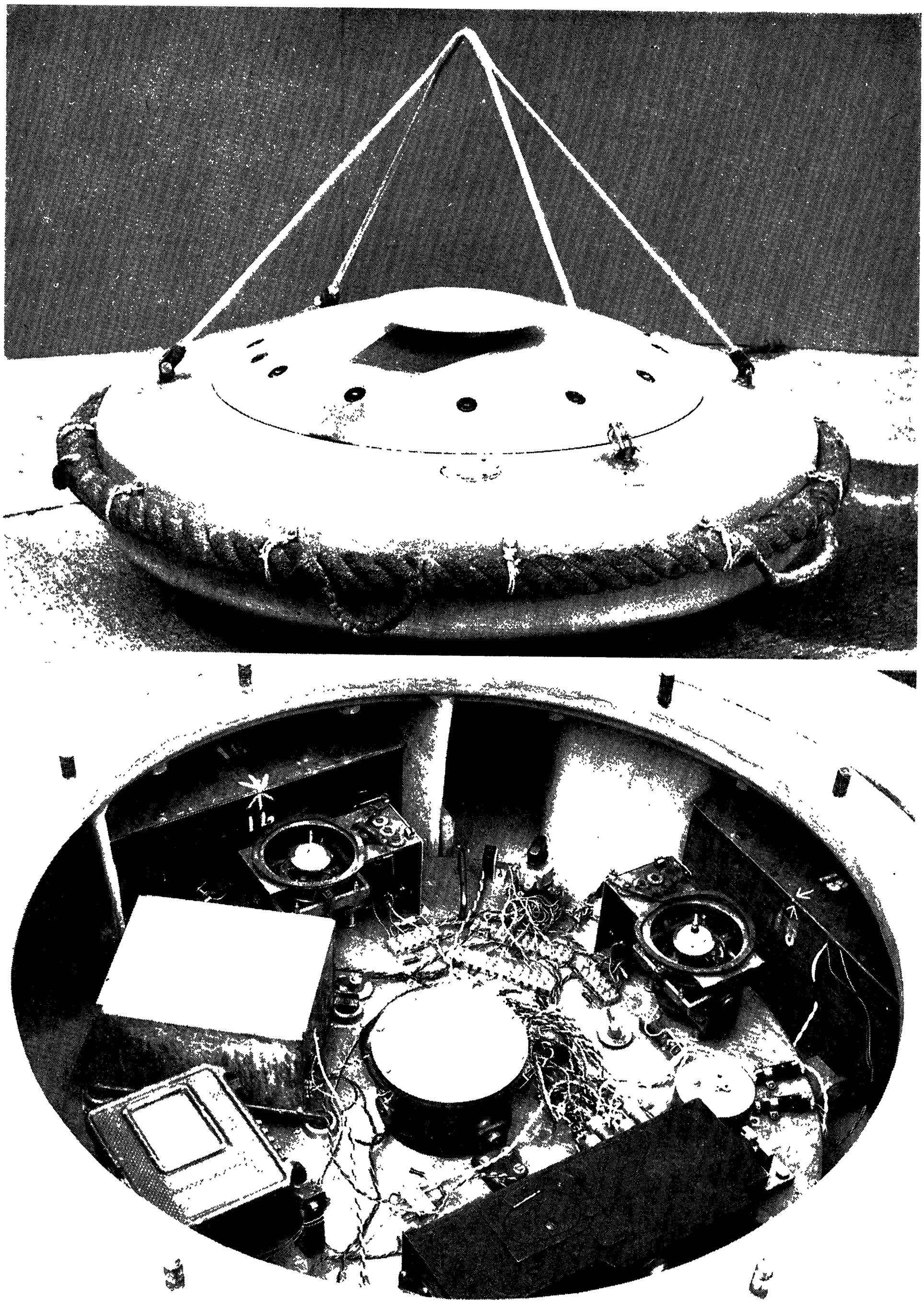


FIGURE 6. Histograms of (*a*) heave, (*b*) pitch, (*c*) roll of the buoy in record no. 5 (wind speed 23 knots).



(*a*)

(*b*)

FIGURE 4. (*a*) Exterior view of the wave recording buoy; (*b*) the instrument panel.

(*Facing p.* 300)

3·3. *The frequency spectra*

The total frequency spectra $F(\sigma)$, regardless of direction, are shown for each record in figure 7, (i) to (v). The scales are logarithmic, and the straight line which has been inserted in each diagram represents the limiting or equilibrium spectrum given by Phillips's law (2·4·1) with the constant C determined by Burling's data. It will be seen that the closest agreement is obtained at the highest wind speed (figure 7(v)). The results show that Phillips's law holds good for much lower frequencies than the data to which it was originally fitted.

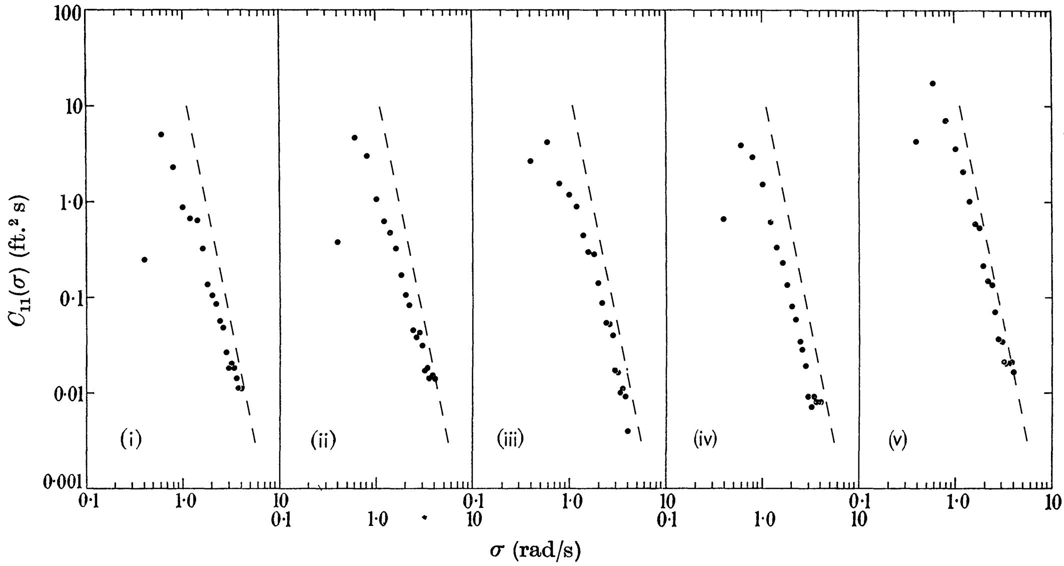


FIGURE 7. The total frequency spectra $F(\sigma)$.

3·4. *Angular width of the spectrum*

The parameter ψ which defines the angular width of the directional distribution $F(\sigma, \phi)$ was computed for each frequency band in records 3 and 5, that is to say for the two records in which the wind direction was most nearly constant. The results are shown plotted in figure 8. The abscissa is the non-dimensional ratio

$$U_1/c = \sigma U_1/g$$

which is proportional to the frequency, in each record. In the same diagram have been plotted curves corresponding to the resonance angle (2·1·3), assuming that the convection velocity U can be identified with the mean velocity at a height $z = 2\pi/k$, where $k = \sigma^2/g$. Writing $z = 2\pi/k$ in (2·2·1) gives

$$\sec\alpha = \frac{U}{c} = \frac{U_1}{c}\log\frac{2\pi}{\Omega(U_1/c)^2}, \tag{3·4·1}$$

where Ω is given by (2·2·3). The full curve in figure 8 corresponds to the assumed value $\Omega = 1{\cdot}3\times10^{-2}$; the upper and lower broken curves correspond to values of

Ω five times less than this and five times greater respectively—or equivalently to values of z five times greater or less.

It will be seen (1) that the angular width tends generally to increase with the ratio U_1/c (the points in brackets at the lowest frequencies may correspond to external swell); (2) that the resonance angle is not critically dependent on the assumed parameters in the wind profile; (3) that the observed width of the spectrum is quite comparable with the resonance angle, but tends to lie somewhat below it in the middle part of the frequency range.

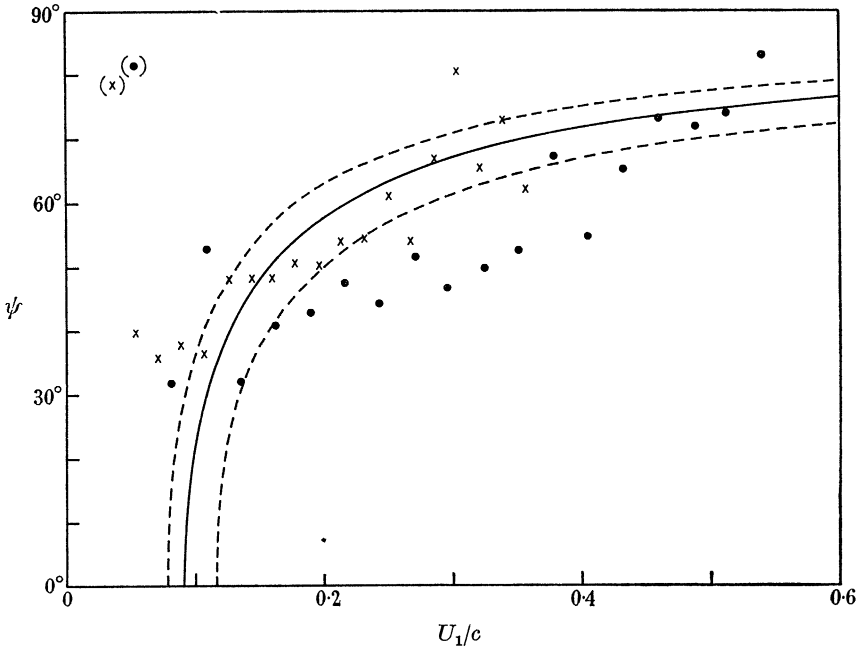


FIGURE 8. The angular width of the spectrum, compared with the theoretical resonance angle.

Recalling the effect of shear instability mentioned in §2·2, namely, that it may tend to produce a narrowing of the directional distribution of the wave energy, we have calculated the values of the parameter M as a function of ϕ (see equation (2·2·5)) assuming values of U_1 appropriate to record 5. Thus we have chosen

$$U_a = 23 \text{ knots} = 1180 \text{ cm/s}, \quad U_1 \text{ (from equation (2·2·2))} = 134 \text{ cm/s},$$

$$U_1/c = \sigma U_1/g = 0{\cdot}136\sigma \ (\sigma \text{ in rad/s}).$$

Using Miles's values of β, and taking $\Omega = 1{\cdot}3 \times 10^{-2}$ as before we obtain the values of M shown in figure 9.

To calculate the distortion factor f of equation (2·2·4) we must also estimate the duration t. According to the weather charts, the time since the wind began to blow was about 45 h. However, the fetch L being only 300 miles, this was the limiting factor at the lower frequencies. We have taken $t \leqslant L \div$ group velocity $= 2\sigma L/g$ and so

$$T = \frac{gt}{U_1} \times \min \begin{cases} 0{\cdot}72 \times 10^5 \sigma, \\ 1{\cdot}28 \times 10^5. \end{cases}$$

With these values, the factor f is as shown in figure 10.

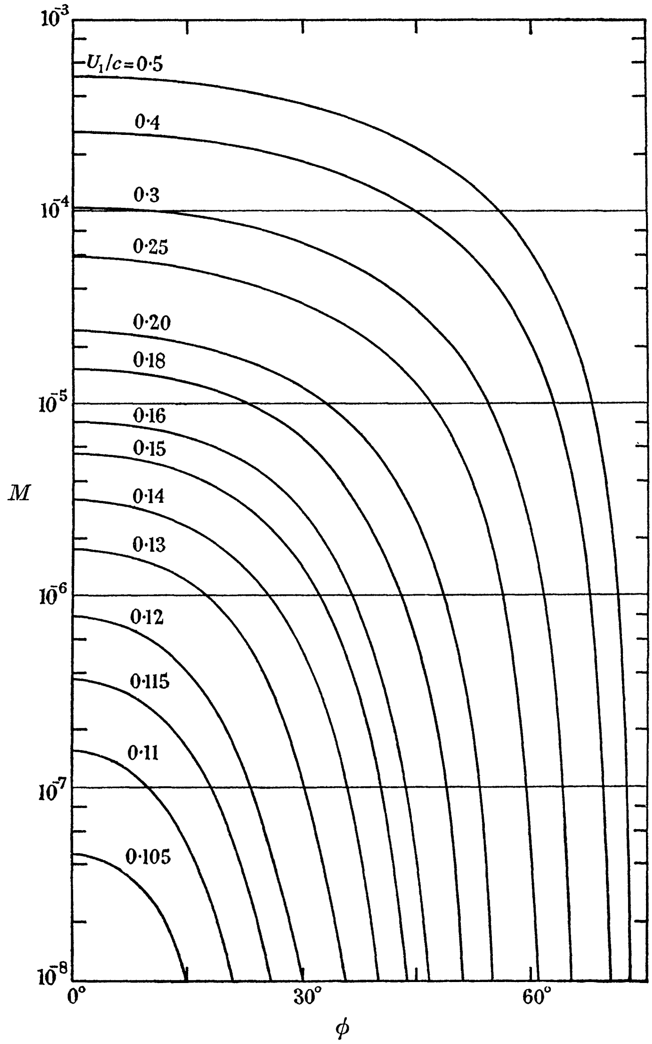


FIGURE 9. M as a function of the angle ϕ (taking $\Omega = 1{\cdot}3 \times 10^{-2}$).

At the lowest frequencies, and so the lowest values of U_1/c, the distortion factor is practically unity at all angles. As the frequency increases, so also does the distortion factor, for any particular angle but the factor is greatest at small angles and falls off at the larger angles. At $\sigma = 2{\cdot}0\ \mathrm{s}^{-1}$ the waves in the direction of the wind are amplified relative to those at an angle 60° by a factor 10^3. No doubt it is this effect which has caused the reduction in angular spread below the resonance angle already seen in figure 8.

However, at the highest frequencies the distortion factors become so large that it would not be surprising if the energy at each wave number were limited by breaking. That this is indeed so is suggested by the fact that $F(\sigma) \propto \sigma^{-5}$, as seen in figure 7 (v).

If the amplification factor f in figure 10 is bounded roughly at the value 10^5, say, it can be seen that with increasing frequency the corresponding curve for the

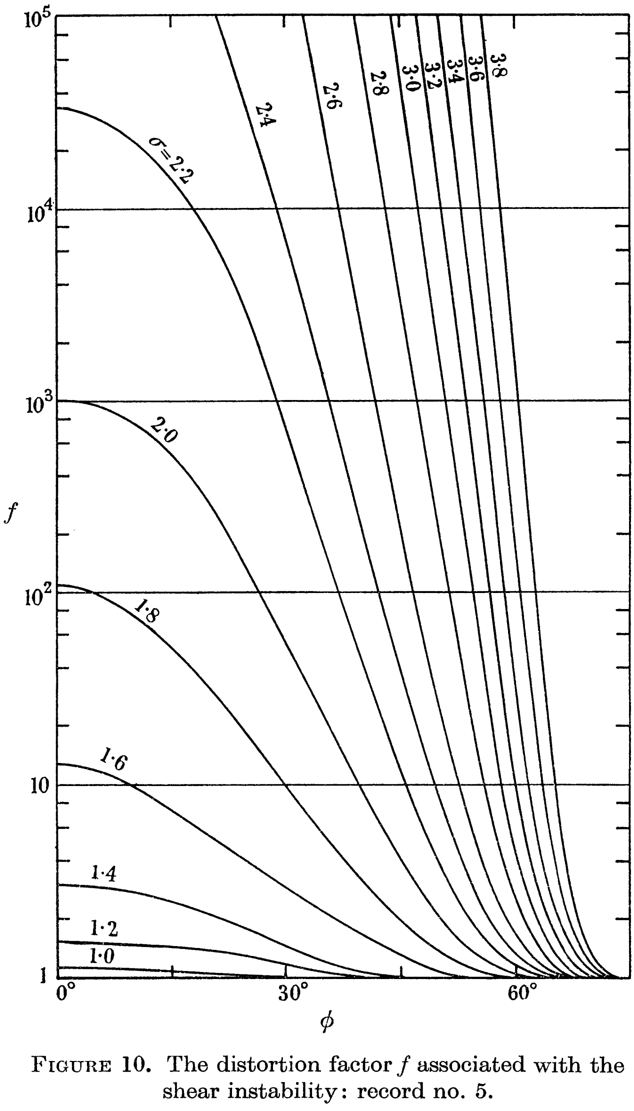


FIGURE 10. The distortion factor f associated with the shear instability: record no. 5.

distortion factor becomes somewhat broader with increasing frequency. However, the width never exceeds about 65°. At the highest frequencies in figure 8 even this width is exceeded.

Now we saw in § 2·3 that non-linear interactions between different parts of the spectrum may tend to broaden the directional distribution of energy, and that this effect will be most pronounced at the highest frequencies, when the characteristic time for the non-linear transfer of energy is the least. Thus it seems not unreasonable to attribute some of the boundary of the spectrum at the highest frequencies to non-linear (i.e. tertiary) interactions among the different parts of the spectrum.

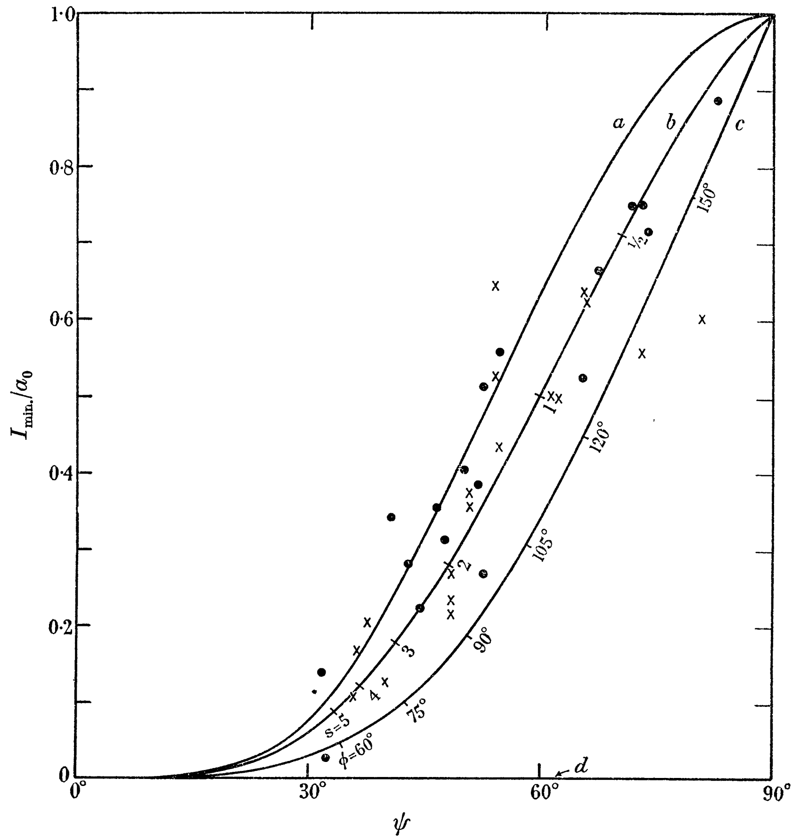


FIGURE 11. A joint plot of two parameters $I_{min.}/a_0$ and ψ, indicating the shape of the directional spectrum. ×, record 3; ●, record 5. Curve *a*, quasi-normal; *b*, cosine power; *c*, square-topped; *d*, two delta-functions.

3·5. *The shape of the angular distribution*

It was seen earlier that an indicator of the shape of the spectrum is the ratio $I_{min.}/a_0$. In figure 11 this ratio has been plotted against the corresponding angular half-width ψ for the two records 3 and 5 for which the wind system was simplest (greater weight should perhaps be given to record 5, the data for which are indicated by circles).

For comparison the same figure shows the locus of values of $I_{min.}/a_0$ and ψ corresponding to some very simple distributions:

(1) The line drawn along the ψ axis corresponds to an ideal distribution consisting of at most two narrow bands of long-crested waves

$$F(\sigma, \phi) \propto \delta(\phi - \phi_1) + \delta(\phi - \phi_2). \tag{3·5·1}$$

For such a distribution we have seen that $I_{min.}/a_0$ vanishes.

(2) The lowest of the three curves corresponds to a 'square-topped' angular distribution, of width $2\phi_0$

$$F(\sigma, \phi) \propto \begin{cases} 1 & (|\phi| < \phi_0), \\ 0 & (|\phi| > \phi_0). \end{cases} \tag{3·5·2}$$

(The corresponding values of ϕ_0 are indicated along the curve.)

(3) The middle continuous curve corresponds to the cosine-power distribution

$$F(\sigma, \phi) \propto (1 + \cos\phi)^s \propto \cos^{2s}(\tfrac{1}{2}\phi). \tag{3·5·3}$$

(The value of the index s is indicated along the curve.) When $s = 0$ the distribution is independent of ϕ, and as s increases the distributions become more and more

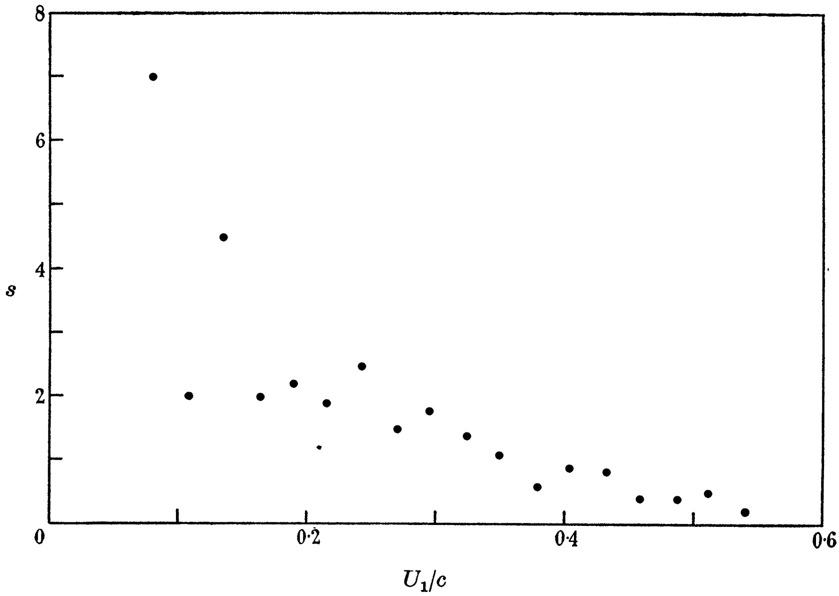


FIGURE 12. The closest values of the parameter s corresponding to the plotted points of figure 11 (data from record 5).

concentrated about the mean direction $\phi = 0$. When s is large, the distribution is approximately normal, with angular variance $2/s$.

(4) The uppermost curve corresponds to the 'quasi-normal' distribution

$$F(\sigma, \phi) \propto e^{-2\Delta^{-2}\sin^2\frac{1}{2}\phi}, \tag{3·5·4}$$

which when Δ is small also approximates a normal distribution. (In general, the Fourier coefficients a_n of (3·5·4) involve simple Bessel functions of Δ^{-2}.)

Of the four laws considered, it appears that the cosine power (3·5·3) gives on the whole the best fit to the observations (though not necessarily for any individual frequency). Corresponding to each observation a corresponding value of s may be allotted by going to the point on the 'cosine-power' curve which is nearest to the plotted point. The values of s so obtained for record 5 have been plotted in figure 12 against the value of U_1/c for each observation. It is seen that generally s decreases with U_1/c, as would be expected from the fact that the angular width s generally increases with U_1/c.

Other types of distribution cannot necessarily be ruled out, but they must differ from the cosine-power distribution only in the third and higher harmonics of ϕ.

If wave generation by turbulent pressure fluctuations (§2·1 above) were the only mechanism involved, one would expect a 'bimodal' distribution with maxima at $\phi = \pm\alpha$—but only at the higher values of U_1/c; for there would in any case be some spread of energy about the critical directions, and if these were close together the combined distribution $F(\sigma, \phi)$ could still have a single maximum at $\phi = 0$.

Thus the fact that the distribution approximates to a unimodal curve is not conclusive evidence against the generating mechanism being turbulence alone, at the lower frequencies; but it is strong evidence at the higher frequencies.

4. The atmospheric pressure fluctuations

4·1. *General remarks*

In both of the mechanisms of wave generation discussed in §§2·1 and 2·2, the transfer of energy to the waves is associated with fluctuations in the air pressure on the surface of the water. If the pressure fluctuations are due mainly to atmospheric turbulence they will be almost uncorrelated with the surface elevation ζ; if they are coupled to the existing waves, as in Miles's theory, there should be a high correlation with ζ. The variances of the coupled and uncoupled components of the pressure fluctuations are additive.

Table 2. Variances of the atmospheric pressure fluctuations

record no.	U_a (ft./s)	$0{\cdot}09\ U_a^4/g^2$ (ft.^{2})	$\overline{(p/g\rho_a)^2}$ (ft.^{2})
1	32	91	7·3
2	24	29	8·2
3	29	62	5·2
4	14	3	9·8
5	39	203	13·1

In the wave observations discussed in the preceding section, a simultaneous record was made of the air-pressure fluctuations on the upper surface of the buoy. Here we shall discuss only the results; for a description of the microbarograph and other experimental details, see the paper by Longuet-Higgins *et al.* (1962).

The pressure fluctuations are conveniently expressed in terms of the equivalent vertical displacement in still air, at a standard atmospheric density. The observed variances of the pressure fluctuations are shown in the last column of table 2. In the preceding column is shown the turbulent fluctuations according to Phillips's estimate (2·1·4). It will be seen that the observed pressure fluctuations are generally much the smaller.

In the following section it will be shown that a substantial part of the observed pressure fluctuations can be attributed to the flow of air over the undulating surface of the sea.

4·2. *Calculation of the non-turbulent pressure spectrum*

In Miles's (1957, 1959*a*) model the aerodynamical pressure exerted on a sinusoidal boundary

$$\zeta = \mathscr{R} a \mathrm{e}^{\mathrm{i}(\mathbf{k}.\mathbf{x}-\sigma t)} \tag{4·2·1}$$

by an air stream in the direction of wave propagation has the form

$$p = \mathscr{R}(\alpha+\mathrm{i}\beta)\rho_a U_1^2 ka \mathrm{e}^{\mathrm{i}(\mathbf{k}.\mathbf{x}-\sigma t)}, \tag{4·2·2}$$

where α and β are real, non-dimensional quantities depending on the wind profile. To (4·2·2) we must add the static pressure term $-g\rho_a\zeta$. Thus the total pressure measured by an apparatus floating in the surface is

$$p = \mathscr{R} - g\rho_a[1-(\alpha+\mathrm{i}\beta)(U_1/c)^2]\, a \mathrm{e}^{\mathrm{i}(\mathbf{k}.\mathbf{x}-\sigma t)} \tag{4·2·3}$$

The phase lag χ of the pressure relative to the surface depression $(-\zeta)$ is given by

$$\chi = \tan^{-1}\frac{\beta(U_1/c)^2}{1+\alpha(U_1/c)^2}. \tag{4·2·4}$$

From the numerical values given by Miles (1959) for the logarithmic profile we have computed χ (see figure 13).† It appears that over the range $0 < U_1/c < 0·5$ the phase angle does not exceed 0·35, or $\cos^{-1} 0·94$. Hence the *amplitude* of the pressure fluctuation is due almost entirely to the in-phase component of the pressure

$$\left|\frac{p}{g\rho_a\zeta}\right|^2 \doteqdot 1-\alpha(U_1/c)^2 \tag{4·2·5}$$

with an error of at most 6 %.

We have computed the right-hand side of (4·2·5) (denoted by ϖ) from the numerical values given by Miles (1959*a*) and the results are shown in figure 14, for representative values of Ω. It will be seen that ϖ has a minimum at around $U_1/c = 0·11$, that is, at around $U_a/c = 1$.

The behaviour of ϖ can be understood if we consider the Kelvin–Helmholtz model in which the wind velocity U is constant. In that case (see Lamb 1932, p. 370) it is easily found that

$$\varpi = 1+(U/c-1)^2 \tag{4·2·6}$$

which is clearly a minimum when $U = c$, that is to say when the wind speed just equals the phase velocity of the waves.

More generally, Brooke Benjamin (1959) has shown that an approximation to the in-phase component of the pressure is given by

$$\varpi_1 = 1+\int_0^\infty (U/c-1)^2 \mathrm{e}^{-k\eta}\, \mathrm{d}(k\eta) \tag{4·2·7}$$

† Miles's figure 6 gives the angle $\tan^{-1}\beta/(-\alpha)$.

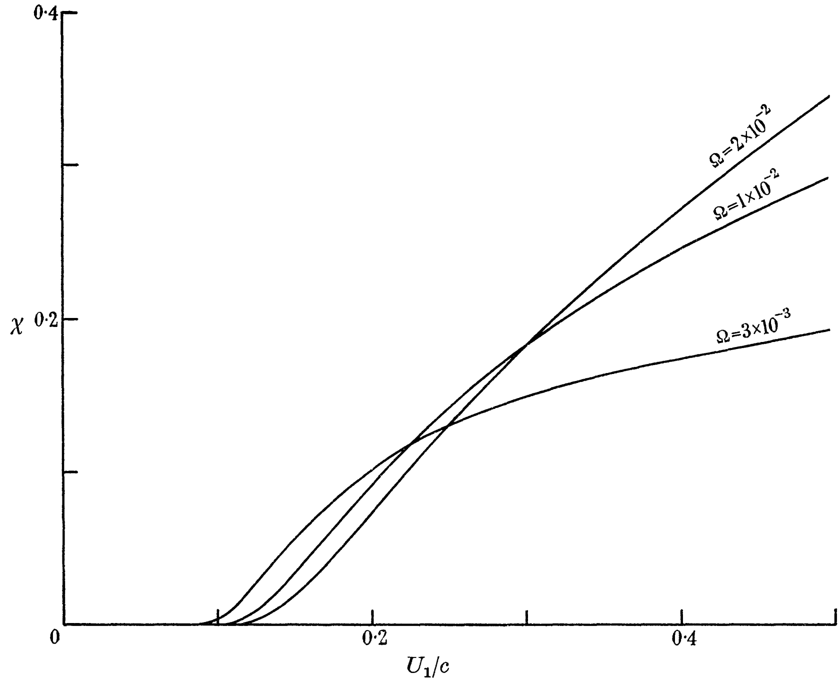


FIGURE 13. The theoretical phase angle between the air pressure and the depression of the surface, on Miles's shear-flow model.

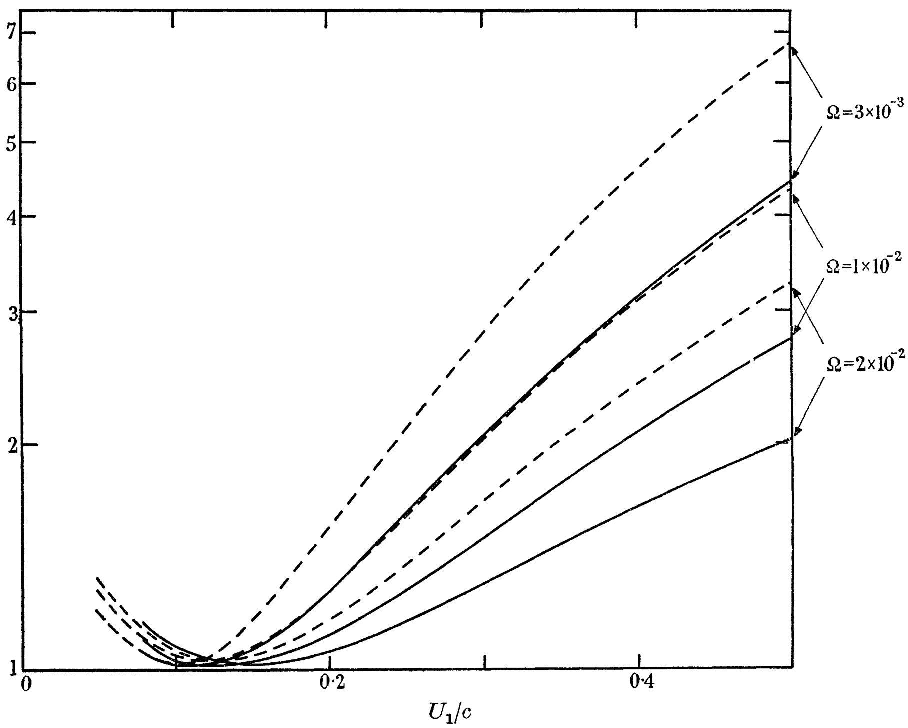


FIGURE 14. The theoretical in-phase component of the pressure in flow over a single sine wave: ——, Miles (1959); -------, Brooke Benjamin (1959).

(we have added the statical term), where η is a co-ordinate orthogonal to the free surface. On substituting for U from equation (2·2·1) and replacing z by η we find, on evaluating the integral,

$$\varpi_1 = 1+\left[1+\frac{U_1}{c}\ln(\gamma\Omega U_1^2/c^2)\right]^2+\tfrac{1}{6}\pi^2(U_1/c)^2, \tag{4·2·8}$$

where $\gamma = \exp(0·5772\ldots)$. The curves for ϖ_1 have been plotted in figure 14 for comparison with Miles's numerical results. It will be seen that ϖ_1 somewhat exceeds ϖ, but that the general behaviours of ϖ and ϖ_1 are very similar.

Consider now the more general case of a sine wave travelling at an arbitrary angle ϕ relative to the wind. By Squire's theorem the component of the wind parallel to the crests has no effect on the pressure perturbations, which may thus be calculated as though the mean wind field were equal to $U\cos\phi$ in the direction of wave propagation. Returning to (2·2·1) we see that the effective wind profile $U\cos\phi$ remains logarithmic; but to maintain the *form* of the results the parameter Ω must be multiplied by $\sec^2\phi$. Since the dependence of ϖ upon Ω cannot be readily expressed analytically,† we use as an approximation equation (4·2·8), generalized to arbitrary directions of propagation; that is to say

$$\varpi_1 = 1+\left[1+\frac{U_1}{c}\cos\phi\ln(\gamma\Omega U_1^2/c^2)\right]^2+\tfrac{1}{6}\pi^2\left(\frac{U_1\cos\phi}{c}\right)^2. \tag{4·2·9}$$

Let the right-hand side of (4·2·9) be denoted by $\varpi_1(\sigma,\phi)$. We see that the spectrum of the pressure is then

$$C_{44}(\sigma) \doteq \int_0^{2\pi} F(\sigma,\phi)\varpi_1(\sigma,\phi)\,\mathrm{d}\phi, \tag{4·2·10}$$

and since $\varpi_1(\sigma,\phi)$ involves only the fourth power of $\cos\phi$, $C_{44}(\sigma)$ may be expressed in terms of the coefficients a_n, b_n up to $n = 4$. On division by

$$C_{11}(\sigma) = \int_0^{2\pi} F(\sigma,\phi)\,\mathrm{d}\phi = \pi a_0 \tag{4·2·11}$$

we have the ratio C_{44}/C_{11} in terms of a_n/a_0 and b_n/a_0 up to $n = 4$. In particular for a symmetrical spectrum ($b_n = 0$) we find

$$\frac{C_{44}}{C_{11}} = 4+8P\frac{a_1}{a_0}+(4P^2+2Q^2)\left(1+\frac{a_2}{a_0}\right)$$
$$+P(P^2+Q^2)\left(3\frac{a_1}{a_0}+\frac{a_3}{a_0}\right)+\tfrac{1}{8}(P^2+Q^2)\left(3+4\frac{a_2}{a_0}+\frac{a_4}{a_0}\right), \tag{4·2·12}$$

where
$$P = \frac{U_1}{c}\ln(\gamma\Omega U_1^2/c^2), \quad Q = \frac{\pi^2}{6}\frac{U_1^2}{c^2}. \tag{4·2·13}$$

† Also the numerical values of α and β are probably sensitive to actual departures from the logarithmic wind profile.

The curves drawn in figure 15 illustrate the ratio $(C_{44}/C_{11})^{\frac{1}{2}}$, computed for the cosine-law spectrum (3·4·3). The (constant) values of U_1 and Ω are those appropriate to the data of record 5. It appears that the behaviour of the observation (a minimum at around $U_1/c = 0{\cdot}11$) corresponds quite well to the behaviour of the theoretical curves. It should be borne in mind that at larger values of U_1/c the theoretical curves may be somewhat high, since ϖ_1 generally exceeds ϖ. Nevertheless, there is qualitative agreement even in this part of the range of U_1/c. There is a tendency for the equivalent value of s to diminish with U_1/c, as shown independently in figure 12.

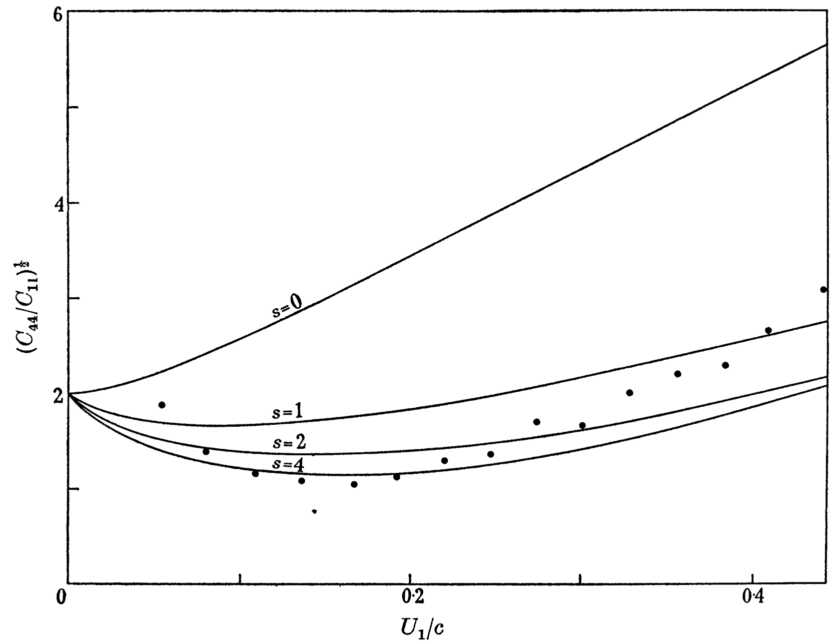


FIGURE 15. Values of $(C_{44}/C_{11})^{\frac{1}{2}}$ for record 5, giving the ratios of the spectral densities of pressure (in feet of air) and of surface elevation.

From this comparison it appears that for the most part the pressure fluctuations at the surface are simply the aerodynamical pressure changes due to the flow of air over the undulating surface, together with the statical pressure changes arising from the buoy's vertical displacement.

4·3. *The phase angles between p and ζ*

Confirmation is provided by considering the phase-differences between the pressure and the surface elevation. If the pressure fluctuations were due only to uncoupled turbulence, there would be no definite phase relation between p and ζ If, however, the pressure fluctuations were due mainly to the local shear flow, and not to the turbulence, then from figure 13 we expect the phase differences between p and $-\zeta$ to be small.

Owing to the combined uncertainties in the phase calibrat'ons of the microbarograph, the accelerometer and its integrating circuit, and of the heaving motion of the buoy, the phases could be determined only to within about 10° for $\sigma \leqslant 3{\cdot}2$ and within wider limits at higher frequencies. The estimated phase differences are shown in table 3.

TABLE 3. PHASE LAG OF PRESSURE BEHIND WAVE HEIGHT (RECORD 5)

σ(rad/s)	phase lag on film	instrumental corrections	corrected phase lag
0·4	214°	−50°	164°
0·6	220	−36	184
0·8	210	−30	180
1·0	209	−26	183
1·2	207	−24	183
1·4	207	−23	184
1·6	210	−22	188
1·8	202	−22	180
2·0	205	−22	183
2·2	206	−22	184
2·4	213	−22	191
2·6	211	−22	189
2·8	197	−23	174
3·0	230	−23	207
3·2	224	−22	202
3·4	221	−13	208
3·6	222	+14	236
3·8	221	38	259
4·0	225	86	311

It will be seen that over the most energetic part of the spectrum, p and ζ are about 180° out of phase.

From this it may be concluded that over the most energetic part of the spectrum not more than 10 % of the observed pressure fluctuations can be attributed to turbulence in the air stream; so that the variance of the turbulent pressure fluctuations cannot be more than about 1 % of the observed variance.

4·4. *Discussion*

The above measurements of the air-pressure fluctuations, though they set an upper limit to the turbulent component of the pressure, do not rule out the possibility that turbulence contributes the greater part of the wave energy at the lower wave frequencies in record 5. At the higher frequencies such a possibility is ruled out by the directional distribution.

The mean rate of work done by the normal pressures depends on the value of $\overline{p\,\partial\zeta/\partial t}$ and hence on the component of air pressure that is in quadrature with ζ. Owing to the small phase angle between p and $-\zeta$ the quadrature component could not be determined with accuracy sufficient to assign a significant† value to $\overline{p\,\partial\zeta/\partial t}$. However, even supposing that the phase angles could be roughly determined, still further measurements might be necessary to determine the relative importance of turbulence and shear instability. For the phase angles, even on Miles's simplified model, depend critically on the profile of the wind velocity.

† A claim to have measured $\overline{p\,\delta\xi/\partial t}$ was made by Kolesnikov (1960) at the Helsinki meeting of the U.G.G.I. The details of this work are not yet available.

An accurate measure of $\overline{p\,\partial\zeta/\partial t}$ could, however, serve to confirm that normal pressures, of one kind or another, do not play a negligible part in generation of sea waves.

5. Conclusions

The observations available at present of the angular distribution of wave energy, and of the pressure fluctuations in the atmosphere, are consistent with the following description of how sea waves are generated.

The initial disturbance of the water surface may be due to turbulent pressure fluctuations in the air flow, as described by Phillips. Although the pressure fluctuations are much weaker than was first estimated, it is still conceivable that such a process may account for most of the energy at the lower frequencies in the wave spectrum, for which U/c differs little from unity. The main stage of wave growth appears to be due to the shearing flow in the air, as described by Miles; this would tend to produce the observed reduction in the angular spread of the energy. At the highest frequencies, the spectrum is probably controlled by the breaking of the waves; this is strongly indicated by the dependence of the spectrum on the fifth power of the frequency. Concurrently with the last two processes there is probably a slow modification of the spectrum due to the 'resonant' third-order wave interactions.

Fundamental to the processes of wave generation are the profile of the wind velocity and the amplitude and scale of the turbulent pressure fluctuations. Of these our knowledge is very incomplete, and further efforts to observe them might be well repaid. Moreover, significant observations of the wave spectrum itself are very few, and all those in existence contain important ambiguities. Further observations of the directional spectrum by one or other of the techniques described in §3 would be valuable; it may prove possible, for example, to extend the measurements with the floating buoy so as to obtain the higher angular harmonics of $F(\sigma, \phi)$. Observations with a linear array of buoys would be useful over a certain range of frequencies.

On a model scale, it is clear that experiments in wind tunnels will be of limited value unless the wind profile is accurately reproduced and the turbulence correctly scaled. However, experiments with free waves should be worth while. For example, it would be of great interest to measure the tertiary wave interactions described in §3. Similarly, a model study might provide information on the transfer of energy by wave breaking.

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Auto-suspension of transported sediment; turbidity currents

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It has been apparent for some time past that fine sediment material carried in suspension by a turbulent water stream flowing by gravity is apt to behave inconsistently with conventional theory. This demands that the concentration of suspended solids, which being heavier than the fluid tend to fall through it, must always increase downwards towards the bed. In fact, however, the concentrations