On the intervals between successive zeros of a random function

By M. S. Longuet-Higgins
National Institute of Oceanography, Wormley, Surrey

(*Communicated by G. E. R. Deacon, F.R.S.—Received* 13 *February* 1958)

A new approach is suggested to the problem of the statistical distribution of the intervals between successive zeros of a random, Gaussian function. Hence is derived a sequence of approximations $p_n(\tau)$ $(n=3, 4, 5, \ldots)$ to the desired probability density $p(\tau)$. The third approximation p_3 is already correct to order τ^4, and has the correct limiting form in the case of a narrow spectrum. The analysis also gives rise to an alternative approximation $p_n^*(\tau)$, less accurate for small values of τ, but possibly more accurate for larger values. Numerical computation of both p_3, p_4, p_5 and p_3^*, p_4^*, p_5^* is carried out for a low-pass spectrum, and the results are compared with observation.

Introduction

Let f denote a stationary random function of the time t, with mean value zero. What is the statistical distribution of the interval τ between two successive zeros of f, or between two successive maxima or minima?

The problem arises in connexion with the analysis of the sea surface, where $f(t)$ may represent, for example, the height of the surface above a fixed point. It has also been considered by Rice (1945) in connexion with the analysis of noise in electrical circuits.

As in a recent paper (1956), f will be assumed to be representable as the sum of an infinity of sine waves in random relative phase, and its energy spectrum will be assumed a continuous function of the frequency. Under general conditions (see Rice 1944) the statistical distribution of f itself is then normal.

The distribution of τ (which we denote by $p(\tau)$) has a mean value which is easily ound; it is reciprocal of the average number N_0 of zero-crossings per unit time, and Rice (1944, 1945) has shown this to be given by

$$N_0 = \frac{1}{\pi}\left(\frac{-\psi_0''}{\psi_0}\right)^{\frac{1}{2}}, \tag{0·1}$$

where ψ_τ denotes the correlation function of f, as defined in § 1 below.

To find the complete distribution of τ is somewhat more difficult. In certain limiting cases $p(\tau)$ is known: for example, when τ is small an approximate expression has been given by Rice (1945, p. 59) and when τ is very large it may be shown that $p(\tau)$ decreases exponentially (Kuznetsov, Stratonovich & Tikhonov 1954). Further, in the important case of a narrow spectrum, when f appears as a sine wave of slowly varying amplitude and phase, Rice gave an approximation to $p(\tau)$ valid for a limited range of τ around the mean value $\bar{\tau}$ (1945, p. 63). It is interesting to note that the same approximation may be derived by two alternative methods, either through the distribution of f''/f (Longuet-Higgins 1956) or through the distribution of the phase angle of f (Longuet-Higgins 1957, § 2·10).

The purpose of the present paper is to describe a fresh approach to the problem, by which successive approximations to $p(\tau)$ may be calculated. The method depends upon a simple relation (equation (2·1) below) between $p(\tau)$ and the function $U(\tau)$, defined as the probability that f is entirely positive over a fixed time interval of length τ. Since $U(\tau)$ may be approximated by the probability $U_n(\tau)$ that f be positive at n suitably chosen points in the interval (where n is sufficiently great), we thus obtain a set of successive approximations to $p(\tau)$, depending on n.

It is found that the third approximation $p_3(\tau)$ already has the correct gradient, curvature and derivatives up to the fourth order at the origin; moreover, it tends to the correct limiting form when the energy spectrum is narrow. The next two approximations, $p_4(\tau)$ and $p_5(\tau)$ can be evaluated without difficulty, though higher approximations require one or more additional integrations to be performed. Numerical computation of p_3, p_4 and p_5 is carried out for the case when f has a low-pass spectrum. The results are compared with experimental data, with encouraging agreement.

An alternative approximation $p_n^*(\tau)$ is also derived which is less accurate than p_n for small values of τ, but more accurate for larger values.

1. Definitions

We assume that $f(t)$ may be represented in the form

$$f(t) = \sum_n c_n \cos(\sigma_n t + \epsilon_n), \tag{1·1}$$

where the frequencies σ_n of the individual sine waves are distributed densely in the interval $(0, \infty)$; the phases ϵ_n are randomly and uniformly distributed in $(0, 2\pi)$ and the amplitudes c_n are such that over a small interval of frequency $(\sigma, \sigma + d\sigma)$

$$\sum_n \tfrac{1}{2}c_n^2 = E(\sigma)\, d\sigma, \tag{1·2}$$

where $E(\sigma)$ is a continuous function which will be called the energy spectrum of f. The moments of E about the origin, given by

$$m_r = \int_0^\infty E(\sigma)\, \sigma^r d\sigma \quad (r = 0, 1, 2, \ldots) \tag{1·3}$$

are assumed to exist up to all orders required.

The correlation function of f, defined by

$$\psi(\tau) = \overline{f(t)\, f(t+\tau)}, \tag{1·4}$$

where a bar denotes a mean value with respect to the phases or with respect to t, exists and is related to the spectrum by

$$\psi(\tau) = \int_0^\infty E(\sigma) \cos \sigma\tau\, d\sigma. \tag{1·5}$$

The derivatives of ψ at the origin are given by

$$\frac{d^r\psi}{d\tau^r} = \begin{cases} (-1)^{\frac{1}{2}r} m_r & r \text{ even,} \\ 0 & r \text{ odd.} \end{cases} \tag{1·6}$$

Thus if we write (in Rice's notation)

$$\psi(\tau) = \psi_\tau, \quad \mathrm{d}\psi_\tau/\mathrm{d}\tau = \psi'_\tau, \tag{1·7}$$

we have

$$\psi_0 = m_0, \quad \psi'_0 = 0, \quad \psi''_0 = -m_2, \quad \text{etc.} \tag{1·8}$$

The mean frequency in the spectrum may be defined by

$$\bar{\sigma} = m_1/m_0 \tag{1·9}$$

and the rth moment about the mean is then

$$\mu_r = \int_0^\infty E(\sigma)(\sigma-\bar{\sigma})^r \,\mathrm{d}r = m_r - \binom{r}{1} m_{r-1}\bar{\sigma} + \ldots (-1)^r m_0 \bar{\sigma}^r. \tag{1·10}$$

In particular,

$$\mu_0 = m_0, \quad \mu_1 = 0, \quad \mu_2 = m_2 - m_0\bar{\sigma}^2 = m_0\bar{\sigma}^2\delta^2, \tag{1·11}$$

where

$$\delta^2 = \frac{m_0 m_2 - m_1^2}{m_1^2}. \tag{1·12}$$

δ^2 is a non-dimensional parameter proportional to the variance of $E(\sigma)$; it may be expressed also in the form

$$\delta^2 = \frac{1}{2m_1^2}\int_0^\infty\int_0^\infty E(\sigma_1)\,E(\sigma_2)\,(\sigma_1-\sigma_2)^2\,\mathrm{d}\sigma_1\mathrm{d}\sigma_2. \tag{1·13}$$

When $\delta \ll 1$ the spectrum will be said to be *narrow*, and we see that in that case the energy is concentrated in a narrow range surrounding the mean frequency $\bar{\sigma}$. The correlation function may then be expanded asymptotically in the following way. In (1·5) let the term $\cos\sigma\tau$ be written

$$\begin{aligned}\cos\sigma\tau &= \cos(\sigma-\bar{\sigma})\tau\cos\bar{\sigma}\tau - \sin(\sigma-\bar{\sigma})\tau\sin\bar{\sigma}\tau \\ &= \left[1 - \frac{(\sigma-\bar{\sigma})^2\tau^2}{2!} + \ldots\right]\cos\bar{\sigma}\tau - \left[\frac{(\sigma-\bar{\sigma})\tau}{1!} - \ldots\right]\sin\bar{\sigma}\tau.\end{aligned} \tag{1·14}$$

On multiplying by $E(\sigma)$ and integrating term by term we have

$$\psi(\tau) = \left[\mu_0 - \mu_2\frac{\tau^2}{2!} + \ldots\right]\cos\bar{\sigma}\tau - \left[\frac{\mu_1\tau}{1!} - \ldots\right]\sin\bar{\sigma}\tau. \tag{1·15}$$

In the second bracket, μ_1 vanishes. Then assuming that μ_r is of order δ^r and neglecting $(\delta\bar{\sigma}\tau)^3$ we have

$$\psi(\tau) = A_\tau\cos\bar{\sigma}\tau, \tag{1·16}$$

where

$$A_\tau = \mu_0 - \tfrac{1}{2}\mu_2\tau^2 = \psi_0(1 - \tfrac{1}{2}\bar{\sigma}^2\tau^2\delta^2). \tag{1·17}$$

In other words, the correlation function then approximates to a sine wave of period $2\pi/\bar{\sigma}$ and of slowly varying amplitude A_τ.

It has been mentioned that the distribution of f is in general normal. Thus if $t_1, \ldots, t_n$ are n given values of t, and if $f(t_1)\ldots f(t_n)$ are denoted for short by $\xi_1, \ldots, \xi_n$, then the joint-probability density of $\xi_1 \ldots \xi_n$ is of the form

$$p(\xi_1, \ldots, \xi_n) = \frac{1}{(2\pi)^{\frac{1}{2}n}\Delta_n^{\frac{1}{2}}}\exp\left[-\tfrac{1}{2}\sum_{i,j} M_{ij}\xi_i\xi_j\right]. \tag{1·18}$$

In this expression the matrix (M_{ij}) is the inverse of the matrix of mean values (Ξ_{ij}), given by

$$\Xi_{ij} = \overline{\xi_i \xi_j} = \overline{f(t_i)\, f(t_j)} = \psi(t_i - t_j) = \psi_{ij}, \tag{1·19}$$

say; and

$$\Delta_n = |\,(\psi_{ij})\,| = |\,(M_{ij})\,|^{-1}. \tag{1·20}$$

It may be shown that (M_{ij}) is positive-definite (see, for example, Longuet-Higgins 1957).

2. A relation for $p(\tau)$

Our method depends upon the following lemma: let $U(\tau)$ denote the probability that f is positive over an arbitrary time-interval of length τ; then the distribution of the intervals between successive zeros is given by

$$p(\tau) = \frac{2}{N_0} \frac{\mathrm{d}^2 U}{\mathrm{d}\tau^2}, \tag{2·1}$$

where N_0 denotes the average number of zero crossings per unit time (equation (0·1)).

To prove this, let t', t'' be any two instants of time separated by an interval $\tau = t'' - t'$. Then we have

$$U(t''-t') = \text{prob}\,\{f > 0 \text{ at all points in } (t', t'')\}. \tag{2·2}$$

Now let

$$V(t''-t')\,\mathrm{d}t'' = \text{prob}\begin{Bmatrix} f > 0 & \text{at all points in } (t', t'') \\ f = 0 & \text{at some point in } (t'', t''+\mathrm{d}t'') \end{Bmatrix} \tag{2·3}$$

then

$$U(t''-t') = U(t''+\mathrm{d}t''-t') + V(t''-t')\,\mathrm{d}t'', \tag{2·4}$$

for the possibilities represented in the right-hand side are mutually exclusive and together exhaust the possibilities represented on the left-hand side. Taking the limit as $\mathrm{d}t''$ tends to zero we have

$$V(t''-t') = -\frac{\partial}{\partial t''} U(t''-t'). \tag{2·5}$$

Similarly, if we define

$$W(t''-t')\,\mathrm{d}t'\,\mathrm{d}t'' = \text{prob}\begin{Bmatrix} f = 0 & \text{at some point in } (t', t'+\mathrm{d}t') \\ f > 0 & \text{at all points in } (t'+\mathrm{d}t', t'') \\ f = 0 & \text{at some point in } (t'', t''+\mathrm{d}t'') \end{Bmatrix}, \tag{2·6}$$

we have

$$V(t''-t'-\mathrm{d}t') = V(t''-t') + W(t''-t')\,\mathrm{d}t' \tag{2·7}$$

and so

$$W(t''-t') = \frac{\partial}{\partial t'} V(t''-t'), \tag{2·8}$$

or

$$W(t''-t') = -\frac{\partial^2}{\partial t'\,\partial t''} U(t''-t'). \tag{2·9}$$

Now $p(t''-t')\,\mathrm{d}t''$ is, by definition, the probability that $f > 0$ at all points in (t'', t') and that $f = 0$ at some point in $(t'', t''+\mathrm{d}t'')$, *given* that t' is an up-crossing of f, or alternatively given that there is an up-crossing at some point in $(t', t'+\mathrm{d}t')$, no matter where. (The probability of two or more zero-crossings in $(t', t'+\mathrm{d}t')$ becomes negligible as $dt' \to 0$.) But the prior probability of an up-crossing in $(t', t'+dt')$ is $\frac{1}{2}N_0 dt'$. Hence, by the rule of inverse probabilities,

$$p(t''-t')\,dt'' = \frac{W(t''-t')\,dt'\,dt''}{\frac{1}{2}N_0 dt'}, \tag{2·10}$$

giving
$$p(t''-t') = -\frac{2}{N_0}\frac{\partial^2}{\partial t'\,\partial t''}U(t''-t'). \tag{2·11}$$

On substituting $t''-t' = \tau$, we have the relation (2·1).

The relation is proved, in the first place, only when τ denotes the interval between an up-crossing and the next down-crossing. But since $f(t)$ is symmetrical about zero, the relation holds also for the interval between a down-crossing and the next up-crossing, and so when τ denotes the interval between any two successive zeros.

The relation between $p(\tau)$ and $U(\tau)$ is sketched in figure 1, assuming a fairly narrow spectrum. $U(\tau)$ is always a positive function, tending to zero at infinity. Also, since there is an even chance that in any given small interval of time f will be positive we have $U(0) = \frac{1}{2}$. At the origin $p(\tau)$ vanishes (as will be shown), and has a finite gradient. Hence the curvature of $U(\tau)$ is zero at the origin.

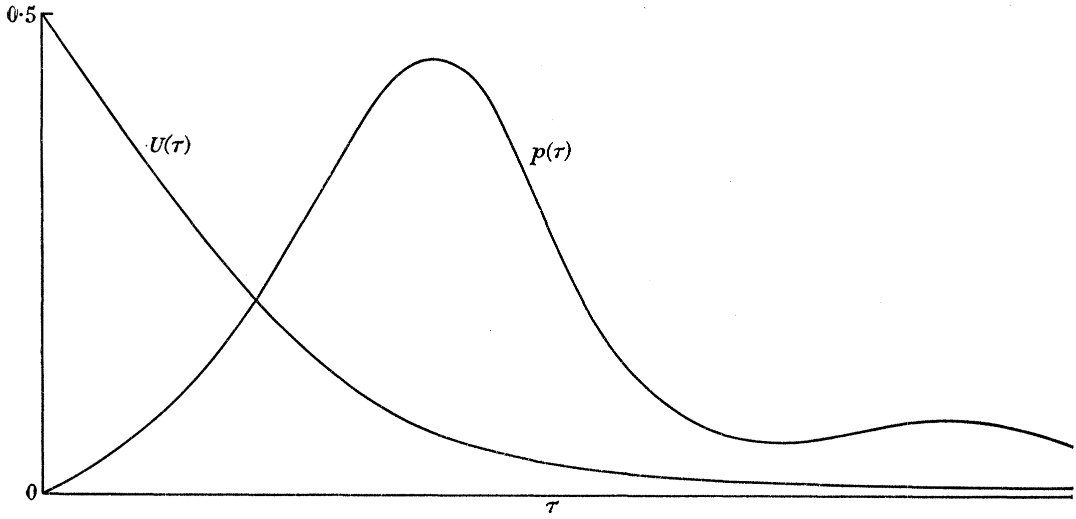

FIGURE 1. The relation between $p(\tau)$ and $U(\tau)$ for a typical random function.

Since $p(\tau)$ can never be negative it is clear from (2·1) that the curvature of $U(\tau)$ is always positive or zero.

3. An expression for $U(\tau)$

By the preceding lemma our problem is now reduced to the evaluation of $U(\tau)$. Taking n points $t_1, t_2, \ldots, t_n$ between t' and t'', with $t_1 = t'$ and $t_n = t''$, let

$$U_n(t_1, \ldots, t_n) = \text{prob}\{f(t_1) > 0, \ldots, f(t_n) > 0\}. \tag{3·1}$$

Then if n tends to infinity in such a way that the largest interval between the points tends to zero it is reasonable to assume (if $f(t)$ is continuous) that

$$\lim_{n\to\infty} U_n(t_1, \ldots, t_n) = U(t''-t'). \tag{3·2}$$

An expression for U_n may be written down immediately. For

$$U_n = \int_0^\infty \dots \int_0^\infty p(\xi_1, \dots, \xi_n)\, d\xi_1 \dots d\xi_n, \tag{3·3}$$

where ξ_i denotes $f(t_i)$. So from equation (1·18)

$$U_n = \frac{|(M_{ij})|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \int_0^\infty \dots \int_0^\infty \exp\left[-\tfrac{1}{2}\sum_{ij} M_{ij}\xi_i\xi_j\right] d\xi_1 \dots d\xi_n. \tag{3·4}$$

Since (M_{ij}) is positive-definite we may, by a real linear substitution

$$\xi_i = \sum_{j=1}^{n} a_{ij}\eta_j, \tag{3·5}$$

transform the integral into the form

$$U_n = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_V \dots \int \exp\left[-\tfrac{1}{2}(\eta_1^2 + \dots + \eta_n^2)\right] d\eta_1 \dots d\eta_n, \tag{3·6}$$

where V denotes the solid angle

$$\sum_{j=1}^{n} a_{ij}\eta_j \geqslant 0. \tag{3·7}$$

This in turn may be written

$$U_n = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_0^\infty \exp\left[-\tfrac{1}{2}r^2\right] r^{n-1} dr S_n, \tag{3·8}$$

where $r^2 = \eta_1^2 + \dots + \eta_n^2$ and S_n (or S) is the region of the unit hypersphere

$$\eta_1^2 + \dots + \eta_n^2 = 1$$

bounded by the hyperplanes $\Sigma a_{ij}\eta_j = 0$. Integration with respect to r gives

$$U_n = \frac{(\frac{1}{2}n - 1)!}{2\pi^{\frac{1}{2}n}} S_n. \tag{3·9}$$

4. Properties of S_n

When $n = 2$ or 3, S_n denotes the angle contained by two straight lines, or the solid angle contained by three given planes, respectively. For general values of n, S_n is obviously a function of the $\frac{1}{2}n(n-1)$ angles between the n bounding hyperplanes of (3·7). Denoting the *interior* angle between the ith and jth hyperplanes by θ_{ij} we have

$$\cos\theta_{ij} = -\frac{\sum_k a_{ik}a_{kj}}{(\sum_k a_{ik}^2)^{\frac{1}{2}}(\sum_k a_{kj}^2)^{\frac{1}{2}}} = -\frac{\psi_{ij}}{(\psi_{ii}\psi_{jj})^{\frac{1}{2}}} \tag{4·1}$$

and so

$$\theta_{ij} = \cos^{-1}(-\psi_{ij}/\psi_0) \qquad (0 \leqslant \theta_{ij} \leqslant \pi). \tag{4·2}$$

In the case $n = 2$,

$$S_n = S_2 = \theta_{12} \tag{4·3}$$

and when $n = 3$ the well-known formula for the area of a spherical triangle gives

$$S_n = S_3 = \theta_{23} + \theta_{31} + \theta_{12} - \pi. \tag{4·4}$$

Schlaefli (1858) has shown that in general S_n may be expressed in terms of functions of the type $S_{n-1}, S_{n-3}, \ldots, S_2$ whenever n is odd, but not when n is even. There is in fact no reduction formula for S_n which is valid for all integral values of n, although for particular values of the angles θ_{ij} it is sometimes possible to express S_n in finite terms (see Schlaefli 1860; Coxeter 1935; Anis & Lloyd 1953).

However, there exists a fundamental differential relation, first proved by Schlaefli (1858; see also Plackett 1954), namely

$$\frac{\partial S_n}{\partial \theta_{pq}} = \frac{1}{n-2} S^{(pq)} \quad (n \geq 4), \tag{4·5}$$

where $S^{(pq)}$ denotes the simplex S_{n-2} corresponding to the $(n-2)\times(n-2)$ matrix $M_{ij}^{(pq)}$ which is derived from (M_{ij}) by deleting the pth and qth rows and columns.

The relation (4·5) reduces the number of consecutive integrations to be performed in evaluating S_n to $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd. S_2 involves essentially one integration since

$$\theta_{12} = \cos^{-1}(-\psi_{12}/\psi_0) = \int_{-\psi_{12}/\psi_0}^{1} \frac{dx}{(1-x^2)^{\frac{1}{2}}}. \tag{4·6}$$

We shall make frequent use of (4·5) in the following work.

From the definition of $S^{(pq)}$ it will be seen that the (r,s)th angle of $S^{(pq)}$ is given by

$$\cos\theta_{rs}^{(pq)} = -\frac{\begin{pmatrix} p & q & {}^{r}_{s} \end{pmatrix}}{\begin{pmatrix} p & q & {}^{r}_{r} \end{pmatrix}^{\frac{1}{2}}\begin{pmatrix} p & q & {}^{s}_{s} \end{pmatrix}^{\frac{1}{2}}} \qquad (0 \leq \theta_{rs}^{(pq)} \leq \pi), \tag{4·7}$$

where $\begin{pmatrix} p & q & {}^{r}_{s} \end{pmatrix}$ denotes the 3×3 determinant

$$\begin{pmatrix} p & q & {}^{r}_{s} \end{pmatrix} = \begin{vmatrix} \psi_{pp} & \psi_{pq} & \psi_{ps} \\ \psi_{qp} & \psi_{qq} & \psi_{qs} \\ \psi_{rp} & \psi_{rq} & \psi_{rs} \end{vmatrix}. \tag{4·8}$$

5. Approximating to $p(\tau)$

From (2·1) and (2·11) it follows that

$$p(t''-t') = -\frac{2}{N_0}\frac{\partial^2}{\partial t_1 \partial t_n}\lim_{n\to\infty} U_n(t_1 \ldots t_n) \tag{5·1}$$

or

$$p(\tau) = \frac{2}{N_0}\frac{d^2}{d\tau^2}\lim_{n\to\infty} U_n(t_1 \ldots t_n). \tag{5·2}$$

Assuming that the order of differentiation and of letting n tend to infinity may be reversed, we have either

$$p(\tau) = \lim_{n\to\infty} p_n(\tau), \tag{5·3}$$

or

$$p(\tau) = \lim_{n\to\infty} p_n^*(\tau), \tag{5·4}$$

where

$$p_n(\tau) = -\frac{2}{N_0}\frac{\partial^2}{\partial t_1 \partial t_n} U_n(t_1 \ldots t_n) \tag{5·5}$$

and

$$p_n^*(\tau) = \frac{2}{N_0}\frac{d^2}{d\tau^2} U_n(t_1 \ldots t_n). \tag{5·6}$$

We have therefore two alternative approximations to $p(\tau)$, namely $p_n(\tau)$ and $p_n^*(\tau)$. In the first of these, $U_n(t_1, \ldots, t_n)$ is differentiated partially with respect to t_1 and t_n only, $t_2, \ldots, t_{n-1}$ being kept constant. In the second, each of $t_1 \ldots t_n$ is considered to be a function of τ, and in fact the most natural assumption is that the t_i are all equally spaced:

$$t_i = \frac{i\tau}{n-1}. \tag{5·7}$$

Differentiation with respect to τ then involves all the points.

We shall examine both types of approximation and compare their merits.

6. $p_n(\tau)$

From (5·5), (3·9) and (0·1) we have

$$p_n(\tau) = -\frac{(\frac{1}{2}n-1)!}{\pi^{\frac{1}{2}n-1}}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}}\frac{\partial^2 S_n}{\partial t_1\,\partial t_n}. \tag{6·1}$$

The first interesting case is $n = 3$. Substituting from (4·4) we have

$$p_3(\tau) = -\frac{1}{2}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}}\frac{\partial^2}{\partial t_1\,\partial t_3}(\theta_{23}+\theta_{31}+\theta_{12}-\pi). \tag{6·2}$$

Since θ_{ij} depends only on t_i and t_j and since

$$\theta_{13} = \cos^{-1}\left(\frac{-\psi_{13}}{\psi_0}\right) = \cos^{-1}\left(\frac{-\psi_\tau}{\psi_0}\right), \tag{6·3}$$

it follows that

$$p_3(\tau) = \frac{1}{2}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\cos^{-1}\left(\frac{-\psi_\tau}{\psi_0}\right). \tag{6·4}$$

On performing the differentiation we have

$$p_3(\tau) = \frac{1}{2}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\psi_\tau'}{(\psi_0^2-\psi_\tau^2)^{\frac{1}{2}}} \tag{6·5}$$

$$= \frac{1}{2}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}}\frac{\psi_\tau''(\psi_0^2-\psi_\tau^2)-\psi_\tau\psi_\tau'^2}{(\psi_0^2-\psi_\tau^2)^{\frac{3}{2}}}. \tag{6·6}$$

It will be noticed that this distribution is quite independent of the choice of the middle ordinate t_2. We now examine some of the properties of the distribution.

Small values of τ. By straightforward expansion of ψ in powers of τ we find

$$p_3(\tau) = \frac{1}{8}\frac{\psi_0\psi_0^{\text{iv}}-\psi_0''^2}{-\psi_0\psi_0''}\tau + O(\tau^3). \tag{6·7}$$

Thus p_3 vanishes when $\tau = 0$, and the gradient there is

$$\left(\frac{\mathrm{d}p_3}{\mathrm{d}\tau}\right)_{\tau=0} = \frac{1}{8}\frac{\psi_0\psi_0^{\text{iv}}-\psi_0''^2}{-\psi_0\psi_0''}. \tag{6·8}$$

This agrees with Rice's approximation (1945, p. 59). Now from (1·8) and (1·3) we have

$$\psi_0\psi_0^{\text{iv}}-\psi_0''^2 = m_0 m_4 - m_2^2 \tag{6·9}$$

$$= \frac{1}{2}\int_0^\infty\int_0^\infty E(\sigma_1)\,E(\sigma_2)\,(\sigma_1^2-\sigma_2^2)^2\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2 \tag{6·10}$$

$$= \frac{1}{2}\int_0^\infty\int_0^\infty E(\sigma_1)\,E(\sigma_2)\,(\sigma_1+\sigma_2)^2\,(\sigma_1-\sigma_2)^2\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2, \tag{6·11}$$

showing that, when the spectrum is narrow, the above quantity is almost proportional to δ^2, the variance of the spectrum (cf. (1·13)). Hence, the narrower the spectrum the smaller is the gradient at the origin.

In general, the gradient at the origin is closely related to the parameter ϵ^2 defined by

$$\epsilon^2 = \frac{m_0 m_4 - m_2^2}{m_0 m_4} \tag{6·12}$$

(see Cartwright & Longuet-Higgins 1956). In fact

$$\left(\frac{dp_3}{d\tau}\right)_{\tau=0} = \frac{1}{8}\frac{m_0 m_4 - m_2^2}{m_0 m_2} = \tfrac{1}{8}\pi^2 N_0^2 \frac{\epsilon^2}{1-\epsilon^2} \tag{6·13}$$

(where N_0 denotes the number of zero-crossings per unit time. It is shown, in the paper just referred to, that ϵ lies between 0 and 1, and its exact value, together with the value of m_0, determines the statistical distribution of the heights of the maxima of $f(t)$.

A narrow spectrum. We have seen in § 1 that for a narrow spectrum

$$\psi_\tau = \psi_0(1 - \tfrac{1}{2}\delta^2\bar{\sigma}^2\tau^2)\cos\bar{\sigma}\tau + O(\delta^3\bar{\sigma}^4\tau^4). \tag{6·14}$$

It follows that

$$\psi_0'' = -\psi_0\bar{\sigma}^2(1+\delta^2) \tag{6·15}$$

and therefore, from (0·1),

$$N_0 = \frac{\bar{\sigma}}{\pi}(1+\delta^2)^{\frac{1}{2}}. \tag{6·16}$$

The mean interval $\bar{\tau}$ is therefore given by

$$\bar{\tau} = \frac{1}{N_0} \doteqdot \frac{\pi}{\bar{\sigma}}, \tag{6·17}$$

neglecting δ^2. In the neighbourhood of this mean interval let us write

$$\bar{\sigma}\tau = \pi + \eta, \tag{6·18}$$

where η is of order δ. Then from (6·14) we have

$$\psi_\tau = \psi_0[1 - \tfrac{1}{2}(\eta^2 + \pi^2\delta^2)] + O(\delta^3) \tag{6·19}$$

and so (6·7) becomes

$$p_3(\tau) \doteqdot \frac{1}{2\bar{\tau}\delta}\frac{1}{(1+\eta^2/\pi^2\delta^2)^{\frac{3}{2}}} \tag{6·20}$$

(terms of order δ^2 are neglected). This expression is equivalent to the expression obtained by Rice (1945, p. 63) for a narrow spectrum and also to the results found by two independent methods (Longuet-Higgins 1957, § 2·10 and 1958). The distribution is symmetrical, about $\eta = 0$ or $\tau = \pi/\bar{\sigma}$, and it diminishes like $(\eta^2 + \pi^2\delta^2)^{-\frac{3}{2}}$.

Large values of τ. Assuming that ψ_τ and its derivatives tend to zero at infinity, we have

$$p_3(\tau) \to 0 \quad (\tau \to \infty), \tag{6·21}$$

unlike Rice's approximation, which tends to a positive value (1954, p. 60). However, neither approximation can be considered valid when τ is large.

In fact, $p_n(\tau)$ cannot be expected to give a good approximation to $p(\tau)$ when τ is greater than about $(n-1)\bar{\tau}$; for U_n is only an approximation to U provided that the probability of f being negative between two positive ordinates ξ_i, ξ_{i+1} can be neglected; and this is not so when $(t_{i+1}-t_i)$ is greater than $\bar{\tau}$.

Therefore we shall not be surprised to find $p_3(\tau)$ becoming erratic or even negative when τ is greater than about $2\bar{\tau}$.

Higher approximations

Returning to the general formula for p_n (equation (6·1)) we see that, since S_n is a function of the angles θ_{ij},

$$\frac{\partial S}{\partial t_1} = \sum_{i<j} \frac{\partial S}{\partial \theta_{ij}} \frac{\partial \theta_{ij}}{\partial t_1}. \tag{6·22}$$

Since θ_{ij} is a function of $(t_i - t_j)$ only, the above reduces to

$$\frac{\partial S}{\partial t_1} = \sum_{j=2}^{n} \frac{\partial S}{\partial \theta_{1j}} \frac{\partial \theta_{1j}}{\partial t_1}, \tag{6·23}$$

or

$$\frac{\partial S}{\partial t_1} = \frac{1}{n-2} \sum_{j=2}^{n} S^{(1j)} \frac{\partial \theta_{1j}}{\partial t_1}, \tag{6·24}$$

by (4·5). Hence

$$\frac{\partial^2 S}{\partial t_1 \partial t_n} = \frac{1}{n-2} S^{(1n)} \frac{\partial \theta_{1n}}{\partial t_1 \partial t_n} + \frac{1}{n-2} \sum_{j=2}^{n} \frac{\partial S^{(1j)}}{\partial t_n} \frac{\partial \theta_{1j}}{\partial t_1} \tag{6·25}$$

and so, from (6·1),

$$p_n(\tau) = \frac{(\frac{1}{2}n-2)!}{\pi^{\frac{1}{2}n-1}} S^{(1n)} p_3(\tau) - \frac{(\frac{1}{2}n-2)!}{2\pi^{\frac{1}{2}n-1}} \left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}} \sum_{j=2}^{n} \frac{\partial S^{(1j)}}{\partial t_n} \frac{\partial \theta_{1j}}{\partial t_1}. \tag{6·26}$$

When τ is small it can be shown (see the appendix) that the dihedral angles $\theta_{rs}^{(1n)}$ of the spherical simplex $S^{(1n)}$ all approach π; hence $S^{(1n)}$ approaches half the content of a hypersphere in $(n-2)$ dimensions, that is

$$S^{(1n)} = \frac{\pi^{\frac{1}{2}n-1}}{(\frac{1}{2}n-2)!} + O(\tau). \tag{6·27}$$

Thus the first term on the right of (6·26) becomes

$$p_3(\tau) + O(\tau^2) \tag{6·28}$$

(since p_3 itself is $O(\tau)$ at the origin). In the second group of terms we have

$$\frac{\partial \theta_{1j}}{\partial t_1} = -\frac{\psi_{1j}'}{(\psi_0^2 - \psi_{1j}^2)^{\frac{1}{2}}} = \left(\frac{-\psi_0''}{\psi_0}\right)^{\frac{1}{2}} + O(\tau^2) \tag{6·29}$$

and so

$$\sum_{j=2}^{n} \frac{\partial S^{(1j)}}{\partial t_n} \frac{\partial \theta_{1j}}{\partial t_1} = \left(\frac{-\psi_0''}{\psi_0}\right)^{\frac{1}{2}} \sum_{j=2}^{n} \frac{\partial S^{(1j)}}{\partial t_n} + O(\tau^2). \tag{6·30}$$

It is shown in the appendix that

$$\sum_{j=2}^{n} \frac{\partial S^{(1j)}}{\partial t_n} = O(\tau^2) \tag{6·31}$$

and so finally

$$p_n(\tau) = p_3(\tau) + O(\tau^2), \tag{6·32}$$

from which we conclude that $p_3(\tau)$ has both the correct value and the correct gradient at the origin.†

† Also the correct curvature. For $p_n(\tau)$ involves only odd powers of τ, all coefficients of even powers being zero.

We saw in § 4 that S_n can be evaluated in terms of known functions up to and including $n = 3$. Since $S^{(1j)}$ is of degree $(n-2)$, it follows from (6·26) that $p_n(\tau)$ may be evaluated as far as $n = 5$.

Thus for $n = 4$, for example, we find

$$p_4(\tau) = \frac{\gamma}{\pi} p_3(\tau) - \frac{1}{2\pi}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}} \left[\frac{\partial\alpha}{\partial t_4}\frac{\partial\theta_{12}}{\partial t_1} + \frac{\partial\beta}{\partial t_4}\frac{\partial\theta_{13}}{\partial t_1} + \frac{\partial\gamma}{\partial t_4}\frac{\partial\theta_{14}}{\partial t_1}\right], \tag{6·33}$$

where

$$\left.\begin{aligned} \alpha = \theta_{34}^{(12)} &= \cos^{-1}\left[-\left(1 \quad 2 \quad {3 \atop 4}\right)\Big/\sqrt{\left\{\left(1 \quad 2 \quad {3 \atop 3}\right)\left(1 \quad 2 \quad {4 \atop 4}\right)\right\}}\right], \\ \beta = \theta_{24}^{(13)} &= \cos^{-1}\left[-\left(1 \quad 3 \quad {2 \atop 4}\right)\Big/\sqrt{\left\{\left(1 \quad 3 \quad {2 \atop 2}\right)\left(1 \quad 3 \quad {4 \atop 4}\right)\right\}}\right], \\ \gamma = \theta_{23}^{(14)} &= \cos^{-1}\left[-\left(1 \quad 4 \quad {2 \atop 3}\right)\Big/\sqrt{\left\{\left(1 \quad 4 \quad {2 \atop 2}\right)\left(1 \quad 4 \quad {3 \atop 3}\right)\right\}}\right]. \end{aligned}\right\} \tag{6·34}$$

In carrying through the computation certain relations between the determinants $\left(p \quad q \quad {r \atop s}\right)$ are found useful. These arise from the fact that each determinant depends only on the correlations $\psi_{pq}, \psi_{ps}, \ldots$ and therefore on the magnitudes $|p-q|$, $|p-s|, \ldots$. Thus we have

$$\left(p \quad q \quad {r \atop s}\right) = \left(q \quad p \quad {r \atop s}\right) = \left(p \quad q \quad {s \atop r}\right) \tag{6·35}$$

and

$$\left(p \quad q \quad {r \atop r}\right) = \left(p \quad r \quad {q \atop q}\right) = \left(r \quad q \quad {p \atop p}\right). \tag{6·36}$$

If, for convenience, the points t_i are equally spaced at intervals of $\tau/(n-1)$ apart, we also have relations such as

$$\left(p \quad q \quad {r \atop s}\right) = \left(p+1 \quad q+1 \quad {r+1 \atop s+1}\right) = \left(n-p \quad n-q \quad {n-r \atop n-s}\right) \tag{6·37}$$

which greatly reduce the number of quantities to be calculated.

The computation of p_n for $n = 3, 4, 5$ has been carried through for the case of a low-pass spectrum:

$$E(\sigma) = \left\{\begin{matrix} 1 & (0 < \sigma < 1), \\ 0 & (1 < \sigma). \end{matrix}\right\} \tag{6·38}$$

The correlation function in this case assumes the simple form

$$\frac{\psi_\tau}{\psi_0} = \frac{\sin\tau}{\tau}. \tag{6·39}$$

The results for p_n are shown in figure 2. It will be seen that p_3, p_4 and p_5 all lie very close together up to about $\tau = 3{\cdot}5$, suggesting that the third approximation is accurate up to this point. In fact the numerical results show that $(p_4 - p_3)$ and $(p_5 - p_4)$ both behave like τ^5 near the origin, so that p_3 is very probably correct as far as the term in τ^4.

As we should expect, p_3 begins to differ appreciably from the next two approximations when τ exceeds π, that is, half the cut-off period 2π. When $\tau > 6$, p_3 becomes

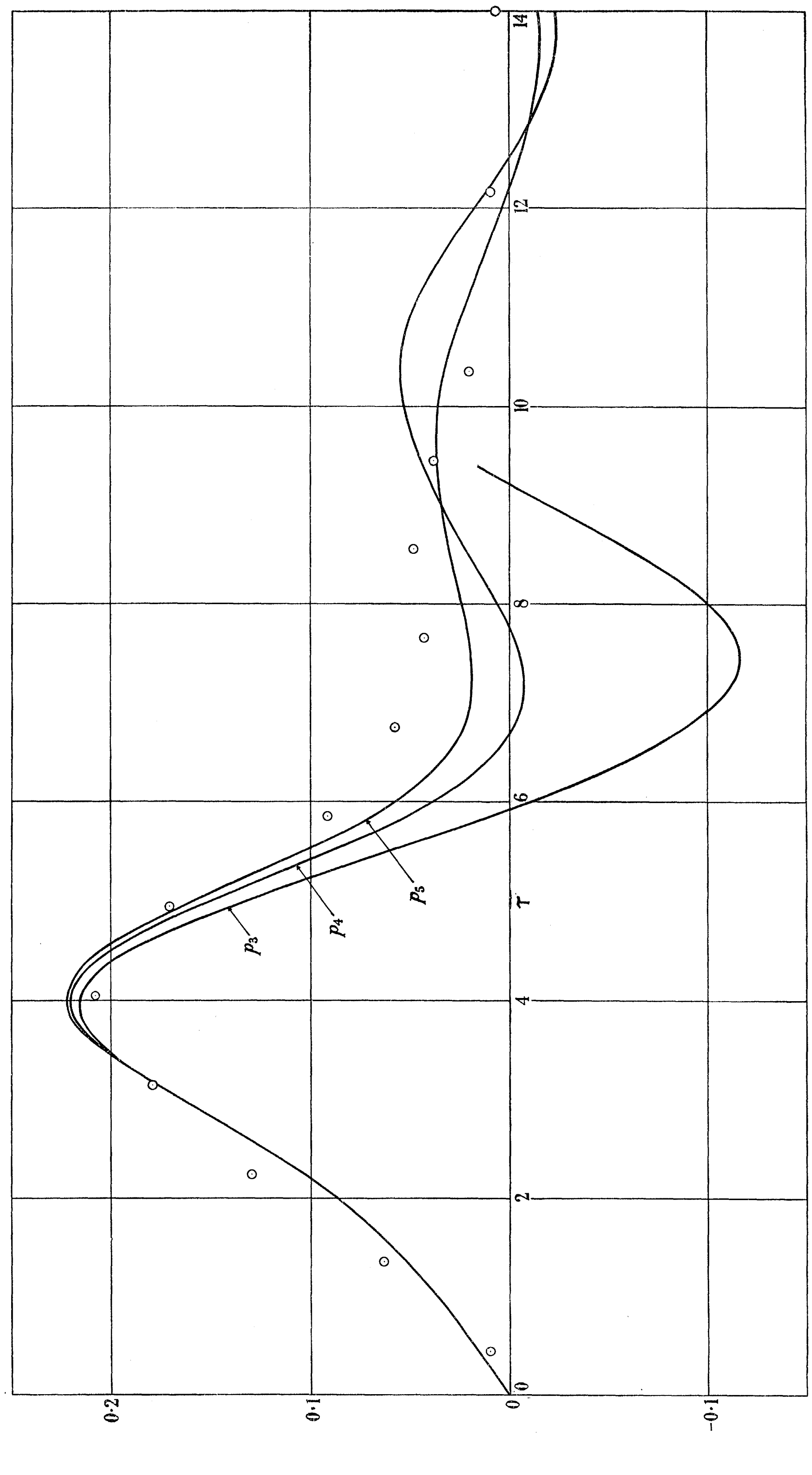

FIGURE 2. Graphs of $p_3(\tau)$, $p_4(\tau)$ and $p_5(\tau)$ for a low-pass spectrum with cut-off period 2π. Experimental data (○) are shown for comparison.

negative, which is of course impossible. p_4 also becomes negative at about $\tau = 7\cdot 3$, though to a lesser extent. However, p_5 is positive until $\tau = 12\cdot 5$ and shows the interesting phenomenon of a second maximum at about $\tau = 10$, also observed experimentally. The observations of Campbell quoted by Rice are also shown in the figure. It should be borne in mind that in the observational material the cut-off frequency was not well defined, and that this has probably affected the position of the two maxima; the presence of any additional energy beyond the theoretical cut-off frequency might be expected to increase the number of short intervals in the distribution and so to shift both observed maxima towards the left. The theoretical cut-off frequency was chosen so that the first maximum of ψ_τ coincided with the observed maximum, but the correct position of the second maximum is somewhat uncertain.

7. $p_n^*(\tau)$

From equations (5·6) we have in general

$$p_n^*(\tau) = \frac{(\frac{1}{2}n-1)!}{\pi^{\frac{1}{2}n-1}}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}}\frac{\mathrm{d}^2 S_n}{\mathrm{d}\tau^2}. \tag{7·1}$$

When $n = 3$ this becomes

$$p_3^*(\tau) = \frac{1}{2}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(\theta_{23}+\theta_{31}+\theta_{12}-\pi). \tag{7·2}$$

In contrast to the previous case, all three angles θ_{23}, θ_{31}, θ_{12} make non-zero contributions. Supposing that the points t_1, t_2, t_3 are equally spaced, then t_2-t_1 and t_3-t_2 are both equal to $\frac{1}{2}\tau$ and hence

$$p_3^*(\tau) = p_3(\tau)+\tfrac{1}{2}p_3(\tfrac{1}{2}\tau), \tag{7·3}$$

where $p_3(\tau)$ is the approximation considered in § 6.

Gradient at the origin. On differentiating with respect to τ and putting $\tau = 0$ we find

$$\left(\frac{\mathrm{d}p_3^*}{\mathrm{d}\tau}\right)_{\tau=0} = \frac{5}{4}\left(\frac{\mathrm{d}p_3}{\mathrm{d}\tau}\right)_{\tau=0}. \tag{7·4}$$

Therefore, the gradient of p_3^* at the origin is not equal to the true gradient, but exceeds it by 25 %.

A narrow spectrum. On making the same approximations as before we find, when τ is in the neighbourhood of $\bar{\tau}$, that $p_3(\frac{1}{2}\tau)$ is small; hence

$$p_3^*(\tau) \doteqdot p_3(\tau) \tag{7·5}$$

and the approximations have the same limiting form for a narrow spectrum.

At infinity, p_3^* like p_3 tends to zero.

Computation of p_3^* for a low-pass spectrum shows (figure 3) that although the approximation is not so good as p_3 near the origin, yet it is somewhat better for values of τ greater than about 4.

Higher approximations of the same type may be obtained by computing

$$\frac{\mathrm{d}S_n}{\mathrm{d}\tau} = \sum_{i<j}\frac{\partial S}{\partial\theta_{ij}}\frac{\mathrm{d}\theta_{ij}}{\mathrm{d}\tau} = \frac{1}{n-2}\sum_{i<j}S^{(ij)}\frac{-\psi_{ij}'}{(\psi_0^2-\psi_{ij}^2)^{\frac{1}{2}}} \tag{7·6}$$

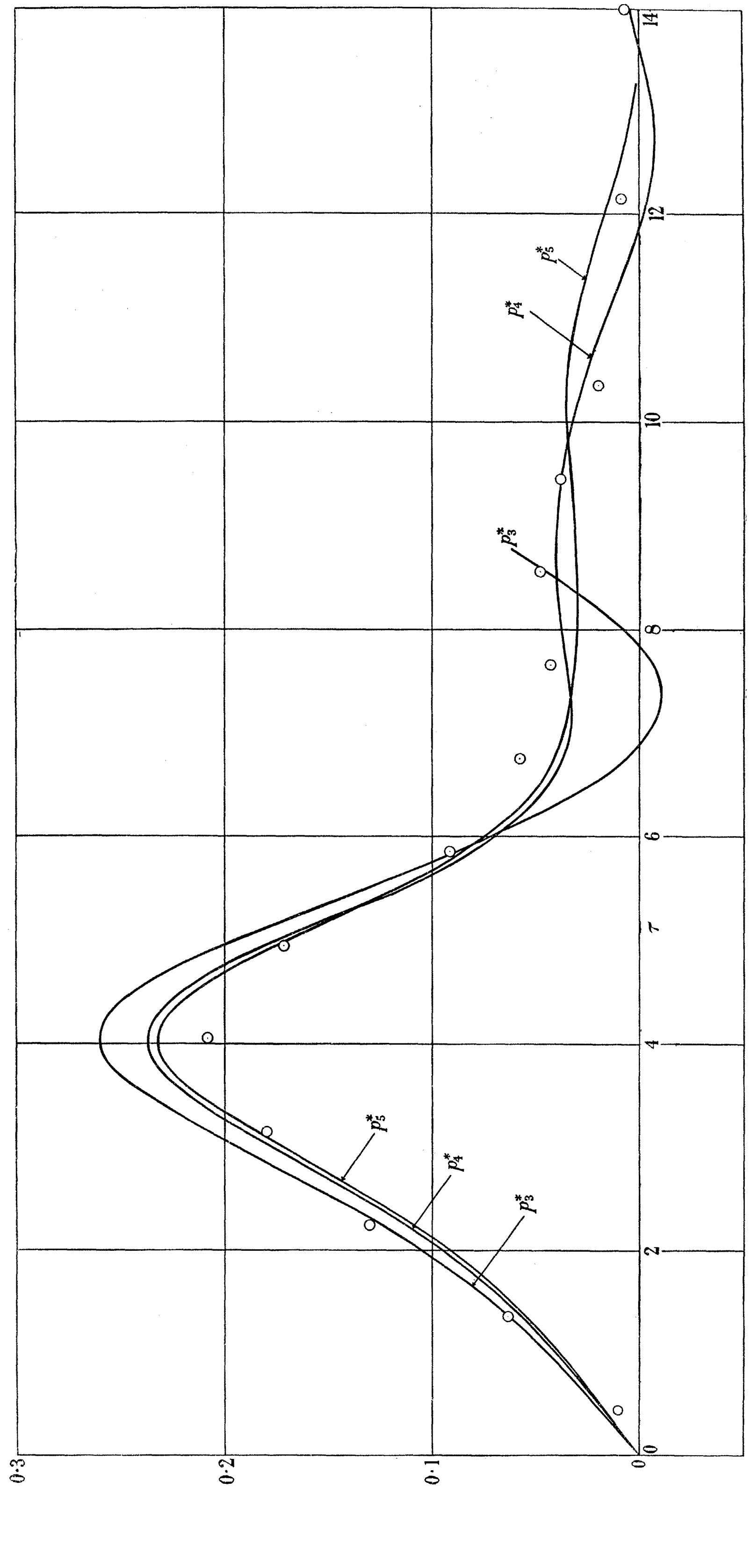

FIGURE 3. Graphs of $p_3^*(\tau)$, $p_4^*(\tau)$ and $p_5^*(\tau)$ for the same spectrum as in figure 2 (and with the same experimental data).

at regular intervals of τ and then differentiating numerically in equation (7·1). Since $S^{(ij)}$ is of degree $n-2$, this may be done up to and including $n = 5$.

The above procedure was carried out for a low-pass spectrum (given by (6·38)) and the results are shown in figure 3. It will be seen that the gradient of p_4^* and p_5^* at the origin differs from the gradient of p_3^*, and all in fact differ from the correct gradient. On the other hand, none of the approximations p_n^* becomes as negative as the corresponding approximation p_n^*, and from the observed points (which are the same as in figure 2) it appears that, for the larger values of τ, p_n^* is somewhat more accurate than p_n.

8. Conclusions

Two sequences of approximations to $p(\tau)$ have been derived, namely $p_n(\tau)$ and $p_n^*(\tau)$. Of these, the sequence $p_n(\tau)$ is the better approximation near the origin. Indeed the third approximation $p_3(\tau)$ is remarkably accurate over the lower half of the distribution, so that we have

$$p(\tau) \doteqdot \frac{1}{2\pi N_0} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \cos^{-1}\left(\frac{-\psi_\tau}{\psi_0}\right) \qquad (\tau \leqslant \tau_m), \tag{8·1}$$

where τ_m denotes the median value of τ. The alternative sequence $p_n^*(\tau)$, however, appears to be more accurate for larger values of τ, and for a low-pass spectrum p_4^* and p_5^* give secondary maxima in accordance with observation. Both types of approximation tend to the correct form when the spectrum becomes narrow.

To compute higher approximations it would be necessary to carry out numerically some further steps of integration; though rather long, this might be done on the lines suggested by Plackett (1954).

From equation (8·1) some simple conclusions may be drawn. On integrating from the limit $\tau = 0$ (that is to say over the range of τ for which the approximation is most accurate) we have

$$\int_0^\tau p(\tau)\,\mathrm{d}\tau \doteqdot \frac{1}{2\pi N_0}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\cos^{-1}\left(\frac{-\psi_\tau}{\psi_0}\right)\right]_0^\tau. \tag{8·2}$$

The expression on the right, evaluated at $\tau = 0$, is

$$\lim_{\tau\to 0} \frac{\psi'_\tau}{(\psi_0^2-\psi_\tau^2)^{\frac{1}{2}}} = \left(\frac{-\psi''_0}{\psi_0}\right)^{\frac{1}{2}} = \pi N_0 \tag{8·3}$$

and so

$$\frac{1}{2\pi N_0}\frac{\mathrm{d}}{\mathrm{d}\tau}\cos^{-1}\left(\frac{-\psi_\tau}{\psi_0}\right) \doteqdot \frac{1}{2} - \int_0^\tau p(\tau)\,\mathrm{d}\tau. \tag{8·4}$$

At the first minimum of ψ_τ, we have $\psi'_\tau = 0$, and so the left-hand side vanishes, giving

$$\int_0^\tau p(\tau)\,\mathrm{d}\tau = \tfrac{1}{2}. \tag{8·5}$$

In other words, $\tau = \tau_m$, the median of the distribution; or *the median of $p(\tau)$ is approximately at the first minimum of the correlation function ψ_τ.*

Further, from (8·4)

$$\frac{1}{2\pi N_0}\frac{\mathrm{d}}{\mathrm{d}\tau}\cos^{-1}\left(\frac{\psi_\tau}{\psi_0}\right) \doteqdot \int_{\tau_m}^\tau p(\tau)\,\mathrm{d}\tau = F(\tau), \tag{8·6}$$

where $F(\tau)$ is the distribution function of τ measured from the median. Hence

$$\psi_\tau/\psi_0 \doteqdot \cos\left[2\pi N_0\int_0^\tau F(\tau)\,\mathrm{d}\tau\right]. \tag{8·7}$$

This serves to give ψ_τ/ψ_0 very simply in terms of $F(\tau)$, which in turn may be found from the observed distribution of τ.

Thus by measuring the distribution of intervals between zeros we have a simple Monte Carlo method to determine the correlation function ψ_τ. The method is valid for values of τ less than the median of the distribution.

Appendix. The behaviour of p_n near the origin

To prove the assertions which were made in § 6 regarding the behaviour of $p_n(\tau)$ at the origin, we must examine the nature of $S^{(1n)}$ and $\Sigma\partial S^{(1j)}/\partial t_n$ for small values of τ.

From (4·7) the (r,s)th angle of $S^{(1i)}$ is given by

$$\cos\theta_{rs}^{(1i)} = -\frac{\begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{rs}\end{vmatrix}}{\begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{rr}\end{vmatrix}^{\frac{1}{2}}\begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{rs}\end{vmatrix}^{\frac{1}{2}}} \tag{A 1}$$

(where for convenience only the diagonal element of each determinant is written). Now

$$\psi_{ij} = \psi(t_i - t_j) = \psi\left(\frac{i-j}{n-1}\tau\right) \tag{A 2}$$

a function which, by hypothesis, may be expanded in even powers of τ. Thus we may apply the following lemma: If $F(x)$ is any function expansible in a power series about $x=0$, and if $x_1, \ldots, x_n, y_1, \ldots, y_n$ are proportional to τ, the first term in the expansion of

$$\begin{vmatrix} F(x_1-y_1) & F(x_1-y_2) & \ldots & F(x_1-y_n) \\ F(x_2-y_1) & F(x_2-y_2) & \ldots & F(x_2-y_n) \\ \vdots & \vdots & & \vdots \\ F(x_n-y_1) & F(x_n-y_2) & \ldots & F(x_n-y_n) \end{vmatrix} \tag{A 3}$$

is

$$\frac{\prod\limits_{i<j}(x_i-x_j)(y_j-y_i)}{[1!\,2!\,\ldots\,(n-1)!]^2}\begin{vmatrix} F(0) & F'(0) & \ldots & F^{(n-1)}(0) \\ F'(0) & F''(0) & \ldots & F^{(n)}(0) \\ \vdots & \vdots & & \vdots \\ F^{(n-1)}(0) & F^{(n)}(0) & \ldots & F^{(2n-2)}(0) \end{vmatrix}. \tag{A 4}$$

For example

$$\begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{rs}\end{vmatrix} \doteqdot \tfrac{1}{4}(t_1-t_i)(t_1-t_r)(t_i-t_r)\,.\,(t_i-t_1)(t_s-t_1)(t_s-t_i)\begin{vmatrix}\psi_0 & 0 & \psi_0'' \\ 0 & \psi_0'' & 0 \\ \psi_0'' & 0 & \psi^{\mathrm{iv}}\end{vmatrix} \tag{A 5}$$

the remainder being of order τ^8. Applying this in (A 1) we find

$$\cos\theta_{rs}^{(1i)} = -\frac{(t_1-t_r)(t_1-t_s)(t_i-t_r)(t_i-t_s)}{|(t_1-t_r)(t_1-t_s)(t_i-t_r)(t_i-t_s)|}. \tag{A 6}$$

According as i does or does not lie between r, s we have

$$\cos\theta_{rs}^{(1i)} = \pm 1; \qquad \theta_{rs}^{(1i)} = 0 \quad \text{or} \quad \pi. \tag{A 7}$$

Hence the first of our assertions is proved.

To examine $\Sigma\partial S^{(1j)}/\partial t_n$ (which we denote by Q), we have first

$$\sin^2\theta_{rs}^{(1i)} = \frac{\begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{rr}\end{vmatrix}\begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{ss}\end{vmatrix} - \begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{rs}\end{vmatrix}^2}{\begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{rr}\end{vmatrix}\begin{vmatrix}\psi_{11} & & \\ & \psi_{ii} & \\ & & \psi_{ss}\end{vmatrix}}. \tag{A 8}$$

By Jacobi's theorem on the minors of a determinant the numerator may be written

$$\begin{vmatrix}\psi_{11} & \\ & \psi_{ii}\end{vmatrix}\begin{vmatrix}\psi_{11} & & & \\ & \psi_{ii} & & \\ & & \psi_{rr} & \\ & & & \psi_{ss}\end{vmatrix} \tag{A 9}$$

and on using the lemma we find

$$\sin^2\theta_{rs}^{(1i)} \doteqdot \frac{(t_r-t_s)^2}{9}\,\frac{\psi_0}{-\psi_0''}\,\frac{\psi_0''\psi^{\text{vi}}-\psi^{\text{iv}2}}{\psi_0\psi^{\text{iv}}-\psi_0''^2}. \tag{A 10}$$

Since $\theta_{rs}^{(i1)}$ lies between 0 and π by definition, we have, assuming $r<s$ and so $t_r<t_s$,

$$\sin\theta_{rs}^{(i1)} \doteqdot G(t_r-t_s), \tag{A 11}$$

where

$$G = \frac{1}{3}\left[\frac{\psi_0(\psi_0''\psi_0^{\text{vi}}-\psi_0^{\text{iv}2})}{-\psi_0''(\psi_0\psi_0^{\text{iv}}-\psi_0''^2)}\right]^{\frac{1}{2}}. \tag{A 12}$$

Hence†

$$\theta_{rs}^{(1i)} = \begin{cases} G(t_s-t_r)+O(\tau^3) & (r<i<s), \\ \pi - G(t_s-t_r)+O(\tau^3) & (i<r<s). \end{cases} \tag{A 13}$$

Writing $s = n$ we have

$$\frac{\partial\theta_{rs}^{(1n)}}{\partial t_n} = \begin{cases} G+O(\tau^2) & (r<i), \\ -G+O(\tau^2) & (i<r); \end{cases} \tag{A 14}$$

and writing $i = n$ (so that neither r nor $s = n$) we have

$$\frac{\partial\theta_{rs}^{(1n)}}{\partial t_n} = O(\tau^2). \tag{A 15}$$

It follows from the last equation that

$$\frac{\partial S^{(1n)}}{\partial t_n} = \sum_{r<s}\frac{\partial S^{(1n)}}{\partial\theta_{rs}^{(1n)}}\frac{\partial\theta_{rs}^{(1n)}}{\partial t_n} = O(\tau^2) \tag{A 16}$$

and hence, neglecting terms of order τ^2,

$$Q = \sum_{j=2}^{n}\frac{\partial S^{(1j)}}{\partial t_n} \doteqdot \sum_{j=2}^{n-1}\frac{\partial S^{(1j)}}{\partial t_n} = \sum_{\substack{i,j=2\\ i\neq j}}^{n-1}\frac{\partial S^{(1j)}}{\partial\theta_{in}^{(1j)}}\frac{\partial\theta_{in}^{(1j)}}{\partial t_n}. \tag{A 17}$$

† A geometrical interpretation of this result is given in another paper (in preparation).

When $n = 4$ we have simply $$S^{(1j)} = \theta^{(1j)}_{i4} \tag{A 18}$$

and so $$Q = \frac{\partial\theta^{(12)}_{34}}{\partial t_4} + \frac{\partial\theta^{(13)}_{24}}{\partial t_4} = G - G = 0. \tag{A 19}$$

When $n = 5$, we have $$S^{(1j)} = \theta^{(1j)}_{kl} + \theta^{(1j)}_{k5} + \theta^{(1j)}_{l5} - \pi, \tag{A 20}$$

where i, k, l denote 2, 3, 4 in any order. Then

$$Q = \sum_{\substack{i,j=2\\ i\neq j}}^{4} \frac{\partial\theta^{(1j)}_{i5}}{\partial t_5} = 0, \tag{A 21}$$

since for every pair (i, j) with $i < j$ there is another pair (j, i) and these give contributions $\pm G$ which cancel. So again Q vanishes.

When $n \geqslant 6$ we have from (4·5) and (A 17)

$$Q = \frac{1}{n-4} \sum_{\substack{i,j=2\\ i\neq j}}^{n-1} S^{(1ijn)} \frac{\partial\theta^{(1j)}_{in}}{\partial t_n} \tag{A 22}$$

with an obvious notation. To show that this expression is of order τ^2 we may examine the dihedral angles $\theta^{(1ijn)}_{rs}$ of $S^{(1ijn)}$. These are given by

$$\cos\theta^{(1ijn)}_{rs} = -\frac{\begin{vmatrix} \psi_{11} & & & & \\ & \psi_{ii} & & & \\ & & \psi_{jj} & & \\ & & & \psi_{nn} & \\ & & & & \psi_{rs} \end{vmatrix}}{\begin{vmatrix} \psi_{11} & & & & \\ & \psi_{ii} & & & \\ & & \psi_{jj} & & \\ & & & \psi_{nn} & \\ & & & & \psi_{rr} \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} \psi_{11} & & & & \\ & \psi_{ii} & & & \\ & & \psi_{jj} & & \\ & & & \psi_{nn} & \\ & & & & \psi_{ss} \end{vmatrix}^{\frac{1}{2}}}. \tag{A 23}$$

Using the lemma, we find that when $\tau = 0$

$$\theta^{(1ijn)}_{rs} = 0 \quad \text{or} \quad \pi, \tag{A 24}$$

according as the pair (r, s) does, or does not, separate the pair (i, j). Now if any one of the angles $\theta^{(1ijn)}_{rs}$ vanishes, then $S^{(1ijn)}$ vanishes. The only cases in which this is not possible is when i, j are consecutive, or if $(i, j) = (2, n-1)$ or $(n-1, 2)$; then $S^{(1ijn)}$ equals half the surface of a hypersphere in $(n-4)$ dimensions. In all cases, interchanging i and j leaves the value of $S^{(1ijn)}$ unaltered, but reverses the sign of $\partial\theta^{(1j)}_{in}/\partial t_n$ and so the sum (A 22) vanishes, when $\tau = 0$. To the first order, therefore, this expression equals

$$Q = \sum_{k=1}^{n} Q_k t_k, \tag{A 25}$$

where $$Q_k = \frac{1}{n-4} \sum_{i,j=2}^{n-1} \frac{\partial S^{(1ijn)}}{\partial t_k} \frac{\partial\theta^{(1j)}_{in}}{\partial t_n} \tag{A 26}$$

$$= \frac{1}{n-4} \sum_{i,j=2}^{n-1} \sum_{r<s} \frac{\partial S^{(1ijn)}}{\partial\theta^{(1ijn)}_{rs}} \frac{\partial\theta^{(1ijn)}_{rs}}{\partial t_k} \frac{\partial\theta^{(1j)}_{in}}{\partial t_n}. \tag{A 27}$$

Now by considering $\sin\theta_{rs}^{(1ijn)}$ as before we find that

$$\theta_{rs}^{(1ijn)} \doteqdot \begin{cases} H(t_s - t_r) & \text{if } (r,s) \text{ separate } (i,j), \\ \pi - (t_s - t_r) & \text{if not;} \end{cases} \tag{A 28}$$

where H is a positive constant, and therefore

$$\left.\begin{aligned} \frac{\partial\theta_{rs}^{(1ijn)}}{\partial t_r} &\doteqdot \begin{cases} -H & \text{if } (r,s) \text{ separate } (i,j), \\ H & \text{if not;} \end{cases} \\ \frac{\partial\theta_{rs}^{(1ijn)}}{\partial t_s} &\doteqdot \begin{cases} H & \text{if } (r,s) \text{ separate } (i,j), \\ -H & \text{if not;} \end{cases} \end{aligned}\right\} \tag{A 29}$$

(terms of order τ^2 being neglected). Further

$$\frac{\partial\theta_{rs}^{(1ijn)}}{\partial t_k} = O(\tau^2) \qquad (k \neq r, s). \tag{A 30}$$

When $n = 6$ we have

$$S^{(1ijn)} = \theta_{rs}^{(1ijn)} \tag{A 31}$$

and so

$$Q_k = \frac{1}{2}\sum_{i,j=2}^{5}\sum_{r<s}\frac{\partial\theta_{rs}^{(1ij5)}}{\partial t_k}\frac{\partial\theta_{i5}^{(1j)}}{\partial t_5}. \tag{A 32}$$

Interchanging i and j has no effect in the first term, but reverses the sign of the second, and so on summation

$$Q_k = 0. \tag{A 33}$$

Similarly when $n = 7$. When $n \geqslant 8$ we have by (4·5)

$$Q_k = \frac{1}{(n-4)(n-6)}\sum_{i,j=2}^{n-1}\sum_{r<s} S^{(1ijrsn)}\frac{\partial\theta_{rs}^{(1ijn)}}{\partial t_k}\frac{\partial\theta_{in}^{(1j)}}{\partial t_n}. \tag{A 34}$$

By the same argument as before, the (p,q)th angle of $S^{(1ijrsn)}$ approximates to 0 or π. The only non-zero $S^{(1ijrsn)}$ are those all of whose angles are π, and these are unchanged by interchanging i and j. But $\partial\theta_{rs}^{(1ijn)}/\partial t_k$ is unaltered also, whereas $\partial\theta_{in}^{(1j)}/\partial t_n$ is reversed in sign. Therefore, the terms in the summation again cancel in pairs and

$$Q_k = 0. \tag{A 35}$$

This shows that Q is of order τ^2, as was to be proved.

I am indebted to Miss D. Greenwood and Miss D. B. Catton for assistance with the numerical computation for figures 2 and 3.

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$$p_1 = a_1 x + b_1 y + \ldots + c_1 z > 0, \quad p_2 > 0, \ldots, p_n > 0 \quad \text{and} \quad x^2 + y^2 + \ldots + z^2 < 1.$$

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