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Action of a Variable Stress at the Surface of Water Waves

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A boundary-layer argument shows that, paradoxically, a variable tangential stress which is greatest at the wave crests and least in the wave troughs produces a thickening of the boundary layer on the *rear* slopes of the waves and a thinning on the forward slopes. In deep water, a variable tangential stress τ is precisely equivalent to a normal stress $i\tau$ in quadrature with the tangential stress. The corresponding rate of growth of the waves is calculated.

A problem which is of interest in the theory of wave generation by wind is the following: A tangential stress is supposed to act on the surface of already existing waves in water of constant depth. The stress is applied unequally over the surface of the waves, being greatest at the wave crests and least in the wave troughs. What is its effect on the rate of growth of the waves?

If the flow is purely laminar, the problem may be treated by the methods of classical hydrodynamics[1]; the rate of growth is given simply by the imaginary part of the complex frequency. However, this solution does not provide a satisfactory physical explanation of the wavegrowth, nor does it cover the case when the flow is turbulent.

Clearly the tangential stress must create, in the first place, a shearing motion in a thin boundary-layer close to the surface. How then is it possible for this shear to increase the energy of the potential flow in the interior of the fluid?

In a recent review[2] Stewart has intuitively seen that the explanation lies in the convergence of the tangential motion in the surface boundary layer producing a small additional component of velocity normal to the free surface. Unfortunately, however, he has given an analytical solution which is certainly incorrect since it does not satisfy the requirement of energy conservation (see below). In the following we give a boundary-layer discussion differing significantly from Stewart's. We then show that this boundary-layer solution is consistent with the classical solution, and so satisfies the conservation of energy. Thirdly, we indicate how Stewart's analysis may be modified so as to bring it into agreement with the other two approaches.

Consider a progressive wave in which the surface elevation is approximately given by

$$\zeta = a \exp [i(kx - \sigma t)], \qquad \sigma/k = c, \tag{1}$$

where a denotes the amplitude, and the wavenumber k and frequency σ are related by the dispersion relation for free waves in water of finite depth h:

$$\sigma^2 = gk \tanh kh. \tag{2}$$

A small tangential stress of the form

$$\tau = \bar{\tau} + \tau_1 \exp [i(kx - \sigma t)], \tag{3}$$

having a maximum at the wave crests and minimum in the troughs, is now applied to the upper surface of the water; the normal stress remaining constant. In time, the mean stress $\bar{\tau}$ will produce a mean current in the direction of wave propagation. We are not concerned with this here. On the other hand, the fluctuating part of the wind stress $\tau_1 \exp [i(kx - \sigma t)]$ which we denote by τ', can be expected to produce a thin boundary layer whose

[1] H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, England, 1932), 6th ed. See especially Sec. 349.
[2] R. W. Stewart, Phys. Fluids Suppl., **10**, S54 (1967).

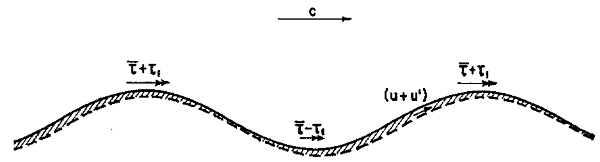

FIG. 1. The boundary layer at the free surface induced by a tangential stress in phase with the surface elevation. The boundary layer is thickest on the rear slope of the wave.

thickness is of order $(\nu/\sigma)^{1/2}$, as described, for example by Lamb.[1]

We need not enter into the details of the boundary-layer solution but deal only with the integrated properties of the motion. Let u' denote the additional velocity in the boundary layer produced by the tangential stress and define the mass flux M in the boundary layer by[3]

$$M = \int \rho u' \, dz, \tag{4}$$

the integral being taken across the layer. If at first we neglect the tangential stress beneath the layer, then by the conservation of momentum parallel to the boundary we have simply

$$\frac{\partial M}{\partial t} = \tau'. \tag{5}$$

Now, if D denotes the local thickness of the boundary layer, conceived as always consisting of the same marked particles, and if w' denotes the additional component of velocity normal to the boundary, we have

$$\frac{\partial D}{\partial t} = [w'] = \int \frac{\partial w'}{\partial z} dz = -\int \frac{\partial u'}{\partial z} dz \tag{6}$$

by continuity. But since the motion is progressive, $\partial/\partial x \sim -(1/c)\,\partial/\partial t$. Hence,

$$\frac{\partial D}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\int u' \, dz = \frac{1}{\rho c}\frac{\partial M}{\partial t} = \frac{\tau'}{\rho c} \tag{7}$$

by Eqs. (4) and (5). Since τ' is proportional to $\exp\,[i(kx - \sigma t)]$ it follows on integrating with respect to time that

$$D = \frac{\tau'}{-i\sigma\rho c} + \text{const.} \tag{8}$$

Thus, the boundary layer is thinnest on the forward slopes of the waves, and thickest on the rear slopes.

We may interpret this result physically (see Fig. 1) by remarking that the greatest acceleration in the boundary layer is where the stress is greatest, that is, on the crests of the waves. Hence, the forward velocity is greatest just *after* the crests have passed, that is, on the rear slope of the waves, and is least on the forward face. Hence, the rate of convergence of the horizontal velocity u', which coincides with the rate of thickening of the layer, is greatest between these two positions, that is to say, at the wave crests. Lastly, the layer is thickest just after the crest has passed, that is, on the rear slopes, in accordance with the analysis.

Now, the pressure at the free surface, or more strictly the normal component of the stress, is assumed to be constant. The thickening of the layer is equivalent, in its effect on the waves, to an additional pressure δp on the upper surface of the wave, given by

$$\delta p = \rho g D = \frac{g\tau'}{-i\sigma c} + \text{const.} \tag{9}$$

Neglecting the constant, whose significance is irrelevant here, and using Eq. (2) we find

$$\delta p = i\tau' \coth kh \tag{10}$$

or in deep water ($e^{kh} \gg 1$) simply

$$\delta p = i\tau'. \tag{11}$$

In other words *a fluctuating tangential stress τ applied at the free surface is dynamically equivalent to a normal pressure fluctuation $i\tau'$ lagging in space 90° behind the tangential stress.*

Note that the mean rate of working by the tangential stress on the waves is given by

$$W = \overline{\tau'(u + u')}, \tag{12}$$

where u and w denote the components of the orbital velocity in the wave at the surface. If the waves are already well developed, we may assume that $u' \ll u$. Using the relation that $u = a\sigma \coth kh \exp\,[i(kx - \sigma t)]$ we find

$$W = \tfrac{1}{2}\tau_1 a\sigma \coth kh. \tag{13}$$

Likewise, the work done on the waves by the additional normal pressure δp is given by

$$W' = \overline{\delta p \frac{\partial \zeta}{\partial t}} = \tfrac{1}{2}\tau_1 a\sigma \coth kh, \tag{14}$$

from (1) and (10). Clearly, $W' \doteqdot W$, implying that the loss of energy in the boundary layer is negligible. This conclusion depends directly on our assumption that $u' \ll u$.

Now consider the rate of growth of the wave

[3] Here x and z denote horizontal and vertical coordinates; more exactly they may be taken as tangential and normal to the surface. See M. S. Longuet-Higgins, Phil. Trans. Roy. Soc. (London) **A245**, 535 (1953).

amplitude. We fix our attention on the deep-water case when the boundary layer at the bottom can be neglected. The mean density of energy per unit horizontal area being given by

$$E = \tfrac{1}{2}\rho g a^2; \tag{15}$$

clearly we must have

$$\frac{dE}{dt} \leq W, \tag{16}$$

that is,

$$\rho g a \frac{da}{dt} \leq \tfrac{1}{2}\tau_1 a \sigma \tag{17}$$

or

$$\frac{da}{dt} \leq \frac{\tau_1}{2\rho c}. \tag{18}$$

If the work done by the surface stress is much larger than the internal dissipation due to viscosity, then in (16)–(18) equality signs are appropriate.

It will be noted that we have neglected the stress on the boundary layer due to the shear associated with the wave motion in the interior of the fluid. This stress is given by

$$\tau'' = -\rho\nu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{19}$$

where u and w are the components of the orbital velocity in the interior. Since the motion in the interior is irrotational, we have $\partial u/\partial z = \partial w/\partial x$ and so

$$-\frac{\tau''}{\rho\nu} = 2\frac{\partial w}{\partial x} = 2\frac{\partial^2 \zeta}{\partial x\, \partial t} = 2\sigma k \zeta \tag{20}$$

from Eq. (1). To include this effect in the previous analysis we simply have to replace τ' by $(\tau' + \tau'')$ giving, instead of Eq. (11),

$$\delta p = i(\tau' - 2\rho\nu\sigma k \zeta). \tag{21}$$

To the same order, we must include the viscous part of the normal stress component p_{zz} at the surface. This is given by

$$2\rho\nu \frac{\partial w}{\partial z} = 2\rho\nu k w = -2i\rho\nu\sigma k\zeta \tag{22}$$

[see Ref. (2) Sec. 348]. Altogether then the applied stress τ' and the action of viscosity are equivalent to an additional pressure

$$\delta p = i(\tau' - 4\rho\nu\sigma k\zeta) \tag{23}$$

applied at the free surface. Instead of (18) we now have

$$\frac{da}{dt} = \frac{\tau_1}{2\rho c} - 2\nu k^2 a. \tag{24}$$

When the applied stress τ' vanishes, we have $\tau_1 = 0$ and so Eq. (24) reduces to

$$\frac{da}{dt} = -2\nu k^2 a \tag{25}$$

giving the classical law of viscous decay

$$a \propto \exp(-2\nu k^2 t) \tag{26}$$

[see Ref. 1, p. 624]. In this case the tangential stress *beneath* the boundary layer acts to produce a thickening on the *forward* slopes of the waves, which combines with the normal stress to produce the wave damping.

By adopting a boundary-layer approximation we have implied that the thickness $(\nu/\sigma)^{1/2}$ of the layer is small compared with the wavelength $2\pi/k$, and hence that $(\nu k^2/\sigma) \ll 1$. However, detailed solutions of the full (linearized) equations of motion and boundary conditions including an applied tangential stress at the upper surface, can readily be obtained by the techniques implicit in Lamb's treatment of the problem,[1] not only for small values of $(\nu k^2/\sigma)$ but for all nonzero values. Thus, if ψ denotes the stream function, satisfying the vorticity equation

$$\left(\nabla^2 - \nu\frac{\partial}{\partial t}\right)\nabla^2\psi = 0, \tag{27}$$

a solution satisfying the condition that $\psi \to 0$ as $z \to -\infty$ is of the form

$$\psi = \{A \exp(kz) + B \exp[(i\sigma/\nu)^{1/2} z]\} \cdot \exp[i(kx - \sigma t)], \tag{28}$$

where A and B are complex constants, which can be chosen so as to satisfy the conditions

$$p_{zz} = 0, \qquad p_{xz} = \tau \tag{29}$$

when $z = \zeta = -\int (\partial\psi/\partial x)\, dt$. For a given k the solution to this problem yields a value of the frequency σ which is, in general, complex giving a rate of wave growth (or decay) in agreement with (24) when $(\nu k^2/\sigma) \ll 1$.

On the other hand, Stewart, in the paper referred to previously,[2] found, instead of (18), the result

$$\frac{da}{dt} = \frac{\tau_1}{\rho c}. \tag{30}$$

This is clearly impossible, for by Eq. (18) it would imply a rate of growth of the wave energy in excess of that supplied by the wind. The explanation appears to lie in the fact that Stewart's solution does not satisfy the requirement of constant pressure at the free surface. To his expression for the potential ϕ in the interior must be added another term, in quadrature with the first, which can be determined by applying Bernoulli's theorem. This gives an additional term to his second expression for the vertical velocity W_1 (his notation), so that on equating it to his first expression and comparing coefficients of cos kx and sin kx one obtains

$$\frac{da}{dt} - \frac{\tau_1}{\rho c} = -\frac{da}{dt} \tag{31}$$

in place of (30). Equation (31) now agrees substantially with (18) above.

Because of the integrated boundary-layer argument used here and by Stewart[2] it can be seen that the simple results (11) and (18) are quite insensitive to the actual value of the viscosity or of the eddy viscosity, if the flow is turbulent. Therefore, they should be very useful in discussing certain aspects of wave generation.[4]

One may easily generalize the conclusions so as to include surface tension by noting that the boundary layer produces an additional normal stress $-T\ \partial^2 D/\partial x^2$ in quadrature with the surface elevation, where T is the surface tension constant.

In addition, one can consider the effect of an applied stress τ which is not necessarily sinusoidal in space, acting on a wave field that contains more than one wave component. To find the work W_k done by the stress τ on a particular wave component having wavenumber k it will be seen by Fourier decomposition that, provided the wave motions are linear and superposable, the rule

$$W_k = \overline{\tau u_k} \tag{32}$$

is always valid, where u_k denotes the tangential velocity corresponding to that particular wave component. Thus, even if τ were independent of u_k, the work done by the stress on a particular Fourier component would depend on the energy already present in that component.

[4] M. S. Longuet-Higgins, Proc. Roy. Soc. (London) (to be published).

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