ON THE TRANSFORMATION OF A CONTINUOUS SPECTRUM BY REFRACTION

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When waves are propagated through a medium whose velocity of propagation varies gradually from place to place, the wave direction and intensity vary according to the laws of refraction. Although the geometry of ray-paths has been well explored, and so also the laws governing the intensity of a coherent train of waves, little attention has apparently been given to the variation in intensity of an incoherent beam having a broad spectrum. The transformation of the energy spectrum is of practical importance in branches of geophysics, for example, in the study of sea waves entering shallow water, or of microseismic waves propagated through inhomogeneous regions of the earth's crust. Accordingly, it seems worth while to state and prove the rule governing the transformation of the two-dimensional spectrum function of a wave disturbance undergoing refraction.

A coherent wave propagated in a homogeneous two-dimensional region may be represented by the expression

$$\zeta = c\cos(ux+vy+\sigma t+\epsilon), \tag{1}$$

where x and y are rectangular Cartesian coordinates and t is the time coordinate. Here u and v are wave-numbers in the x and y directions:

$$u = w\cos\theta, \quad v = w\sin\theta, \tag{2}$$

where $\lambda = 2\pi/w$ is the wavelength and θ defines the direction of propagation. The frequency σ is assumed to be a function of the wavelength and the local properties of the medium only. The amplitude c and the phase ϵ are constants.

An incoherent disturbance may be represented (see Longuet-Higgins (1)) by

$$\zeta = \sum_n c_n \cos(u_n x+v_n y+\sigma_n t+\epsilon_n), \tag{3}$$

where the wave-numbers (u_n, v_n) are distributed densely over the (u,v) plane, the phases ϵ_n are randomly distributed with uniform probability in the interval $(0, 2\pi)$ and the amplitudes c_n are such that if $(u, u+du;\ v, v+dv)$ is any small region of the wave-number plane then

$$\sum_n{}^{*}\ \tfrac{1}{2}c_n^2 = E(u,v)\,du\,dv, \tag{4}$$

the summation $\sum\limits_n{}^{*}$ being over those values of (u_n, v_n) which lie in the infinitesimal region. $E(u,v)$ may be called the energy spectrum of ζ.

Now suppose that the velocity of propagation varies, but so gradually that the representation of a disturbance by (1) or (3) remains valid in any particular locality A. We shall assume that the wave-numbers (u,v) of any particular wave component in the spectrum are transformed continuously. If we suppose that the energy originates in a homogeneous region A' (say deep water in the case of sea waves) where the wave numbers have the value (u', v'), then (u, v) are functions of (u', v') and of the locality A. Assuming that the medium is linear, each part of the spectrum is transformed independently, and so the energy function $E(u, v)$ depends upon the initial energy function $E(u', v')$ and upon A. The theorem we shall prove is that

$$E(u, v) = E'(u', v'). \tag{5}$$

The result may be stated in geometrical language as follows. Imagine $E(u, v)$ as a surface, raised over the horizontal (u, v) plane. To find the height of this surface we

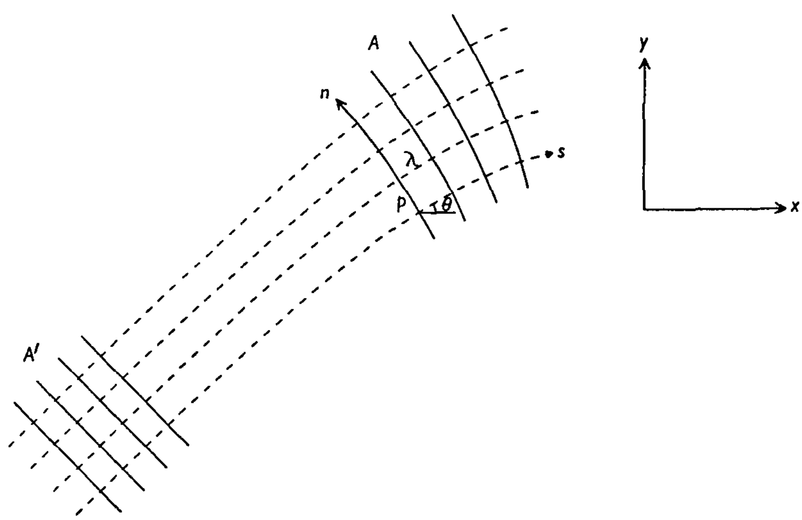

Fig. 1. Wave crests (full lines) and orthogonal trajectories (dotted lines) for a single wave component.

merely take the height of the initial surface E' at the point (u', v') of which (u, v) is the transform. In other words, to find the transformed spectrum E we simply transform the coordinates (u', v') beneath the surface E'; the shape of E' may be altered, but not its height.

Proof. Consider a component sine-wave having initial wave-number (u', v'). Let the crests and the orthogonal trajectories of this wave be drawn and let (s, n) denote coordinates measured along and perpendicular to the trajectories (see Fig. 1). Further, let p denote the perpendicular distance between two neighbouring trajectories, just as λ denotes the distance between two adjacent crests.

Now the energy corresponding to a small region $dS = du dv$ of the spectrum is equal to $E dS$, by definition. Further, it may be shown that the energy in each part of the spectrum is propagated along the corresponding trajectory with the group velocity $d\sigma/dw$. The flux of energy between two neighbouring trajectories, given by $E\, dS\, p\, d\sigma/dw$, must be constant, i.e. independent of s. Therefore

$$E\, dS\, p\, d\sigma/dw = E'\, dS' p'\, d\sigma'/dw'. \tag{6}$$

Assuming that the frequency σ of any wave-component is unaltered by refraction, we have $\sigma = \sigma'$, $d\sigma = d\sigma'$. Also since w is a function of w' only and is independent of θ', we have

$$\frac{dS}{dS'} = \frac{\partial(u, v)}{\partial(u', v')} = \frac{w}{w'}\frac{\partial(w, \theta)}{\partial(w', \theta')} = \frac{w}{w'}\frac{dw}{dw'}\frac{\partial\theta}{\partial\theta'}. \tag{7}$$

Therefore from (6)

$$\frac{E'}{E} = \frac{pw}{p'w'}\frac{\partial\theta}{\partial\theta'}. \tag{8}$$

We shall show that the right-hand side of this equation is independent of s.

Some auxiliary formulae will be required. We have

$$\frac{\partial\theta}{\partial s} = -\frac{1}{\lambda}\frac{\partial\lambda}{\partial n}, \tag{9}$$

$$\frac{\partial\theta}{\partial n} = \frac{1}{p}\frac{\partial p}{\partial s}. \tag{10}$$

The first equation expresses the fact that the curvature of an orthogonal trajectory is equal to the proportional rate at which the normals to the trajectory converge. The second equation expresses a similar relation for the curvature of the crests. Since $\lambda \propto 1/w$ equation (9) may also be written

$$\frac{\partial\theta}{\partial s} = \frac{1}{w}\frac{\partial w}{\partial n}. \tag{11}$$

Also let f denote any function (such as w or θ) which depends on both (s, n) and (w', θ'). Then if (x, y) are fixed local rectangular coordinates

$$\left.\begin{aligned} \frac{\partial f}{\partial s} &= \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}, \\ \frac{\partial f}{\partial n} &= -\sin\theta\frac{\partial f}{\partial x} + \cos\theta\frac{\partial f}{\partial y}; \end{aligned}\right\} \tag{12}$$

and so on differentiating with respect to θ'

$$\left.\begin{aligned} \frac{\partial}{\partial\theta'}\left(\frac{\partial f}{\partial s}\right) &= \left(\cos\theta\frac{\partial^2 f}{\partial\theta'\partial x} + \sin\theta\frac{\partial^2 f}{\partial\theta'\partial y}\right) + \left(-\sin\theta\frac{\partial f}{\partial x} + \cos\theta\frac{\partial f}{\partial y}\right)\frac{\partial\theta}{\partial\theta'}, \\ \frac{\partial}{\partial\theta'}\left(\frac{\partial f}{\partial n}\right) &= \left(-\sin\theta\frac{\partial^2 f}{\partial\theta'\partial x} + \cos\theta\frac{\partial^2 f}{\partial\theta'\partial y}\right) + \left(-\cos\theta\frac{\partial f}{\partial x} - \sin\theta\frac{\partial f}{\partial y}\right)\frac{\partial\theta}{\partial\theta'}; \end{aligned}\right\} \tag{13}$$

that is,

$$\frac{\partial}{\partial\theta'}\left(\frac{\partial f}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial\theta'}\right) + \frac{\partial f}{\partial n}\frac{\partial\theta}{\partial\theta'}, \tag{14}$$

$$\frac{\partial}{\partial\theta'}\left(\frac{\partial f}{\partial n}\right) = \frac{\partial}{\partial n}\left(\frac{\partial f}{\partial\theta'}\right) - \frac{\partial f}{ds}\frac{\partial\theta}{\partial\theta'}. \tag{15}$$

Now let the right-hand side of (8) be differentiated with respect to s. On multiplying by $p'w'/pw$ we have

$$\frac{p'w'}{pw}\frac{\partial}{\partial s}\left(\frac{pw}{p'w'}\frac{\partial\theta}{\partial\theta'}\right) = \left(\frac{1}{w}\frac{\partial w}{\partial s} + \frac{1}{p}\frac{\partial p}{\partial s}\right)\frac{\partial\theta}{\partial\theta'} + \frac{\partial}{\partial s}\left(\frac{\partial\theta}{\partial\theta'}\right). \tag{16}$$

By equation (10) this equals

$$\frac{1}{w}\frac{\partial w}{\partial s}\frac{\partial\theta}{\partial\theta'} + \frac{\partial\theta}{\partial n}\frac{\partial\theta}{\partial\theta'} + \frac{\partial}{\partial s}\left(\frac{\partial\theta}{\partial\theta'}\right). \tag{17}$$

On putting $f = \theta$ in equation (14) we see that the last two terms in (17) taken together can be written

$$\frac{\partial}{\partial\theta'}\left(\frac{\partial\theta}{\partial s}\right) = \frac{\partial}{\partial\theta'}\left(\frac{1}{w}\frac{\partial w}{\partial n}\right) = \frac{1}{w}\frac{\partial}{\partial\theta'}\left(\frac{\partial w}{\partial n}\right), \tag{18}$$

by (11), and since w is independent of θ'. On putting $f = w$ in equation (15), and since $\partial w/\partial\theta' = 0$, we see that (18) is equal to

$$-\frac{1}{w}\frac{\partial w}{\partial s}\frac{\partial\theta}{\partial\theta'}. \tag{19}$$

Altogether, therefore, (17) vanishes and we have

$$\frac{\partial}{\partial s}\left(\frac{E'}{E}\right) = 0. \tag{20}$$

Thus E'/E is a constant with respect to s, and by travelling back along the trajectory to the homogeneous region A' we find that this constant is unity:

$$E'/E = 1. \tag{21}$$

This proves equation (5).

It has been assumed that it is always possible to return along a trajectory to some part of the homogeneous region A', i.e. that no physical obstacles intervene to cut off part of the spectrum. If it is not possible to reach A' along a certain trajectory, then the energy function $E(u, v)$ for that particular wavelength and direction must be zero. Since trajectories passing through a particular region A may originate in different parts of the region A', it is clear that A' must cover in general an area wide compared with the area of A for which the representation (3) is valid. (In the case of sea waves A' is the open ocean.) It has also been assumed that no energy is reflected from the beach or from any other obstacle.

One consequence of equation (5) may be pointed out: contours of constant $E'(u', v')$ transform into contours of constant $E(u, v)$. In particular, if the original spectrum function E' has a maximum at the wave-number (u', v'), then the transformed spectrum E has a maximum at the corresponding point (u, v), and conversely. Thus there is a one-to-one correspondence between the maxima of E' and the maxima of E. However, the relative importance of two distinct maxima of E is not necessarily the same as that of the corresponding maxima of E'. For the total energy in the neighbourhood of (u, v) depends on $\iint E\,du\,dv$, which in turn depends on the local stretching of the coordinates.

The application of these results to the case of an incoherent beam of waves approaching a straight coast is discussed in another paper (2).

REFERENCES

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(2) Longuet-Higgins, M. S. The refraction of sea waves in shallow water. *J. Fluid Dynamics* 1 (1956), 163–176.

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