On the Disintegration of the Jet in a Plunging Breaker

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ABSTRACT

An inviscid mechanism is proposed for the breakup of the jet in a plunging surface wave. Streamwise perturbations of the original surface are shown to grow rapidly owing to stretching of the thin jet and to drastic reduction in the normal pressure gradient. This converts transverse gravity waves into almost pure capillary waves. Conservation of wave action for the perturbations then implies a strong increase in the perturbation amplitude.

1. Introduction

The spontaneous growth of longitudinal striations in the jet of an overturning wave is a well-observed phenomenon; usually it is an intermediate stage in the breakup of the jet into droplets and spray. Some investigators have sought an explanation in the frictional effect of the air or in the formation of streamwise rolls in the jet, analogous to Görtler vortices. Here we shall show on the contrary that the phenomenon can be largely accounted for by the behavior of irrotational waves on a thin sheet of fluid. Two essential ingredients are the longitudinal stretching of the jet and the highly reduced pressure gradient normal to its surface.

2. Initial stages

Figure 1 illustrates the development of the jet in a typical plunging breaker, calculated numerically with the assumption that the flow is irrotational and two-dimensional. The pressure at the free surface is constant. Near time $t = t_1$, when the tangent to the surface near the crest makes a sharp right-angled turn, there is a large pressure gradient in the fluid that accelerates the fluid near the crest horizontally and propels a jet forward from the crest.

This part of the flow has been modeled analytically as a rotating Dirichlet hyperbola (see Longuet-Higgins 1980, 1983). From about $t_2 < t < t_3$ the jet narrows rapidly, and the pressure gradient within the jet diminishes drastically; the fluid is then almost in a state of free-fall in a parabolic trajectory.

Now imagine a perturbation in the form of a short surface wave of amplitude a propagated across the wave as in Fig. 2, with crests aligned in the plane of undisturbed flow. If s is measured along the surface, the normal displacement ζ will be given by

$$\zeta = a(s)\cos(ky - \sigma t), \tag{2.1}$$

where k and σ denote the wavenumber and the radian frequency. These will be related by

$$\sigma^2 = g^*k + Tk^3, \tag{2.2}$$

where T denotes surface tension and g^* is the effective value of g for the surface; thus,

$$g^* = g\cos\alpha + \ddot{n} - \kappa q^2, \tag{2.3}$$

where α is the local angle between the surface and the horizon, $\ddot{n}$ is the local normal acceleration of the undisturbed surface, κ is the curvature of the surface, and q is the particle speed in a reference frame moving with the wave. In fact, we have $\rho g^* = \partial p/\partial n$, the normal gradient of the pressure.

The wave amplitude $a(s)$ is assumed to vary relatively slowly compared to the phase $(ky - \sigma t)$, so that we may apply the principle of *conservation of wave action* where the action density A is defined by

$$A = \frac{1}{2}a^2\sigma/k. \tag{2.4}$$

Now at time $t = t_0$, when $a = a_0$, say, g^* will be of the same order as g. But at time $t = t_3$, say, the pressure gradient will be small and hence g^* will be small compared to g, and perhaps also compared to Tk^2. In that case we shall have simply

$$\sigma^2 = Tk^3; \tag{2.5}$$

that is, the waves are controlled dominantly by capillarity. Since k remains constant, it follows that

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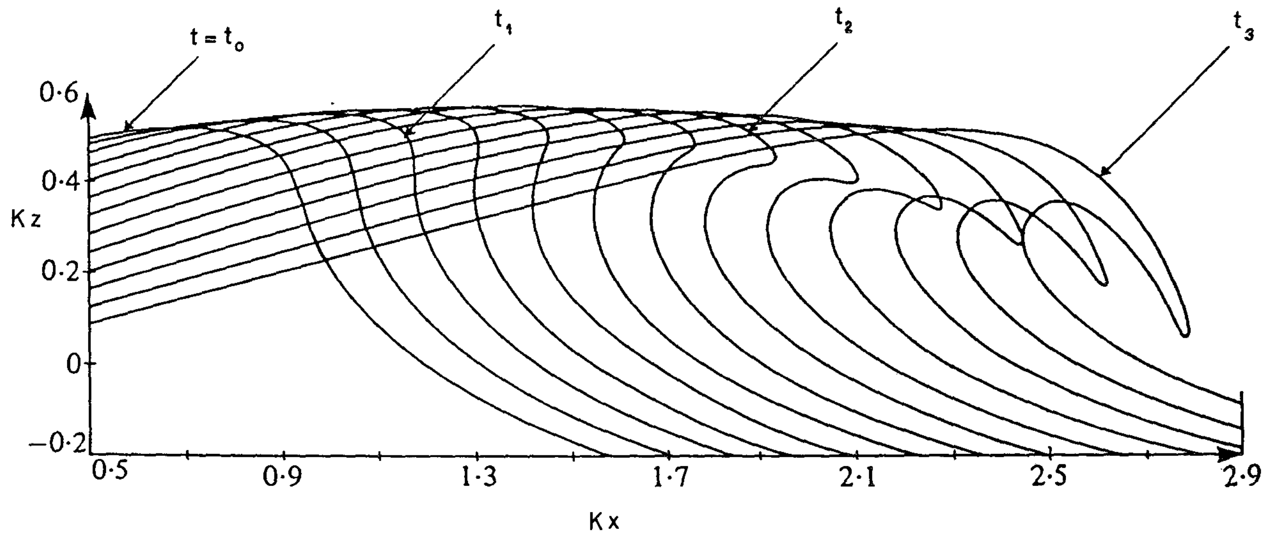


FIG. 1. Successive profiles of a plunging breaker in water of finite depth, calculated by numerical time stepping. (After Vinje and Brevig 1981.)

$$\frac{A}{A_0} = \frac{a^2\sigma}{a_0^2\sigma_0}, \tag{2.6}$$

and hence

$$\frac{a^2}{a_0^2} \sim \left(\frac{g + Tk^2}{Tk^2}\right)^{1/2} \frac{A}{A_0}. \tag{2.7}$$

Now we may apply the law of action conservation in the form

$$A\,\Delta s = \text{const} = A_0(\Delta s)_0, \tag{2.8}$$

where Δs denotes the distance along the surface between two neighboring particles moving with the fluid. At about the time $t = t_2$, $\Delta s/(\Delta s)_0$, is still of order 1, so that from (1.7) and (1.8)

$$\frac{a^2}{a_0^2} \sim \left(\frac{g + Tk^2}{Tk^2}\right)^{1/2}. \tag{2.9}$$

If the initial perturbations are dominantly gravity waves, they will, on becoming converted to capillary waves, have their amplitude increased. For example, if the wavelength of the perturbations lies between 2 and 20 cm, their amplitude will be increased by a factor between 1.23 and 3.40.

Supposing that the perturbations survive this initial increase in amplitude without breaking, let us consider their subsequent history.

3. Perturbations of a thinning jet

It was shown by Taylor in a celebrated paper (1959) that the sinusoidal small perturbations of a thin sheet of fluid of uniform thickness $2h$ are of two kinds: (a) symmetric and (b) antisymmetric (see Fig. 3). In case (a) the displacements on the two sides of the plane of symmetry are *opposite* in sign, the velocity potential ϕ being given by

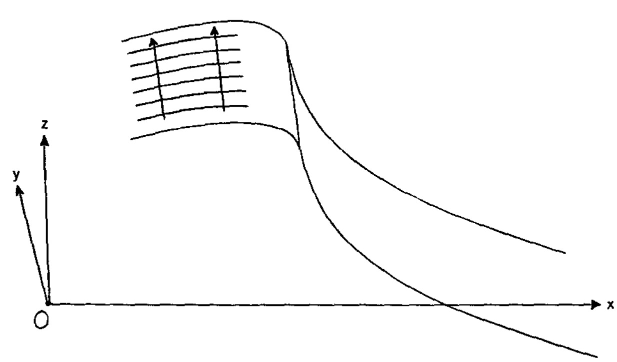


FIG. 2. Sketch of short transverse waves on a plunging breaker, at an early stage of development.

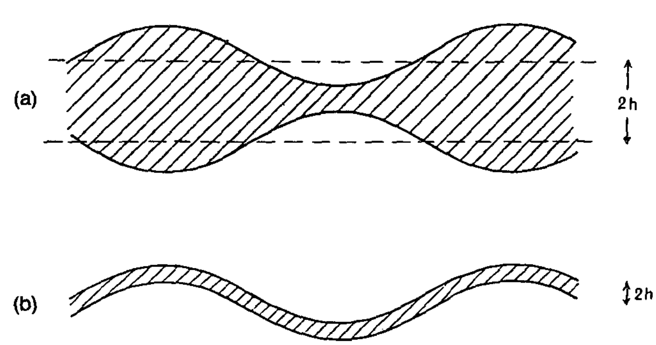


FIG. 3. Sketch of waves on a thin sheet of fluid: (a) symmetric and (b) antisymmetric (after Taylor 1959).

$$\phi = \frac{a\sigma}{k \sin hkh} \cosh kz \sin(ky - \sigma t) \quad (3.1)$$

in our notation, $z = 0$ being the central plane. The dispersion relation is

$$\sigma^2 = Tk^3 \tanh kh \quad (3.2)$$

as in pure capillary waves on water of finite depth h. In case (b) the displacements on each side are of the *same* sign, corresponding to a velocity potential

$$\phi = \frac{a\sigma}{k \cosh kh} \sinh kz \sin(ky - \sigma t) \quad (3.3)$$

and dispersion relation

$$\sigma^2 = Tk^3 \cosh kh. \quad (3.4)$$

It is obvious that an initial perturbation confined to one side of the jet will be resolved into the sum of a symmetric and an antisymmetric perturbation, which then will be propagated independently, each with its own phase and group velocity.

When the sheet is thin compared to the length of the perturbation, that is, $kh \ll 1$, the above expressions are greatly simplified. For the symmetric perturbations we have

$$\left.\begin{aligned} \phi &= \frac{a\sigma}{k^2 h} \sin(ky - \sigma t) \\ \sigma^2 &= Thk^4 \end{aligned}\right\}, \quad (3.5)$$

so that the particle velocity is entirely tangential and the phase speed c is given by

$$c = \sigma/k = (Th)^{1/2}k. \quad (3.6)$$

These waves are dispersive.

On the other hand, for the antisymmetric perturbations

$$\left.\begin{aligned} \phi &= a\sigma z \sin(kx - \sigma t) \\ \sigma^2 &= (T/h)k^2 \end{aligned}\right\}, \quad (3.7)$$

giving

$$c = (T/h)^{1/2}. \quad (3.8)$$

These waves are therefore nondispersive and, in fact, are closely analogous to the waves on a stretched membrane of tension T and thickness $2h$.

Consider now the behavior of the waves when the jet (which we assume to be already thin compared to the wavelength) is stretched in the x direction. For both types of wave the potential energy V is derived from the extension of both the upper and lower surfaces against the surface tension T. Hence we have

$$V = \frac{1}{2} T(ak)^2 \quad (3.9)$$

and

$$E = 2V = T(ak)^2. \quad (3.10)$$

The action density A is then given by E/σ. For the symmetric waves this leads to

$$A = (T/h)^{1/2}a^2. \quad (3.11)$$

We then have two relations, derived from the conservation of mass and of wave action, respectively. From mass conservation

$$h\Delta s = \text{const}, \quad (3.12)$$

and from action conservation

$$(a^2/h^{1/2})\Delta s = \text{const}. \quad (3.13)$$

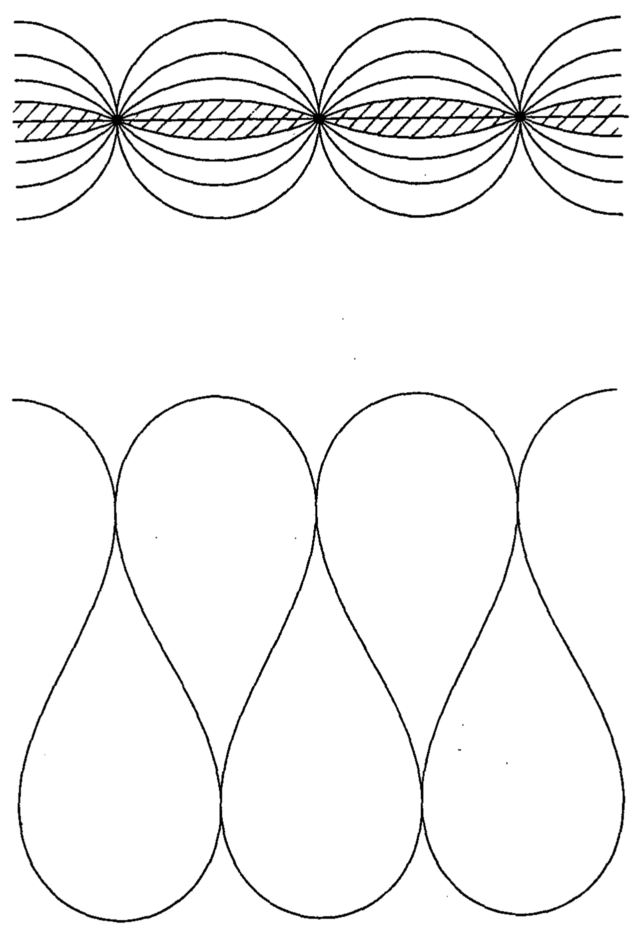
On dividing (3.13) by (3.12) we obtain

$$a^2/h^{3/2} = \text{const}, \quad (3.14)$$

so

$$a \propto h^{3/4} \quad (3.15)$$

(a)



(b)

FIG. 4. Ultimate mode of disintegration of waves on a thin sheet of fluid: (a) symmetric waves and (b) antisymmetric.

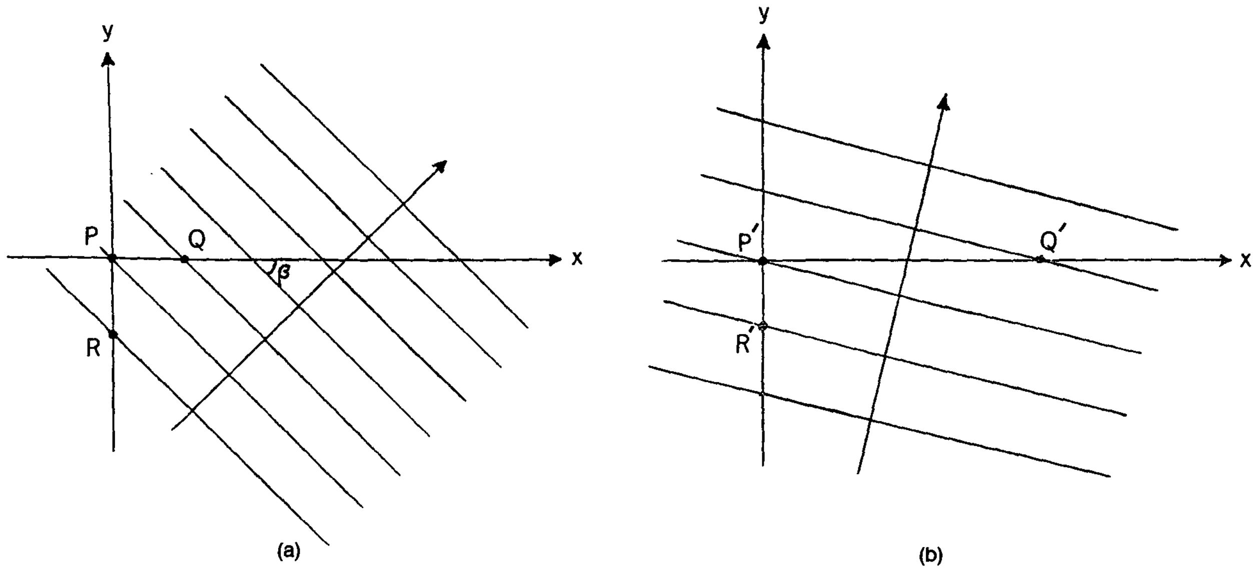


FIG. 5. Sketch showing the change in orientation of obliquely propagating waves on a sheet of water, when the sheet is stretched in the x direction.

and

$$a/h \propto h^{-1/4}. \tag{3.16}$$

From (3.15) it follows that stretching of the sheet reduces the amplitude of the symmetric perturbations, but not as fast as the thickness $2h$. From (3.16) we see that relative to the thickness h the amplitude grows, and though our analysis does not go beyond linear perturbations, it appears that the perturbations will finally pinch off the sheet into cylindrical drops.

From the nonlinear theory of capillary waves on thin sheets of water (Kinnersley 1976) we find that in the limit the surface profile is one of a family of circular arcs with centers on the line of symmetry [see Kinnersley 1976; Eq. (10)]. This is illustrated in Fig. 4a. The limiting flow in the "neck" of the wave is simple and has been discussed by Longuet-Higgins (1988, section 4).

Consider, on the other hand, the antisymmetric waves. We have for the action density

$$A = (Th)^{1/2}a^2k. \tag{3.17}$$

Therefore we have (3.12) together with

$$(a^2h^{1/2})\Delta s = \text{const}, \tag{3.18}$$

leading to

$$a \propto h^{1/4} \tag{3.19}$$

and

$$a/h \propto h^{-3/4}. \tag{3.20}$$

Hence the amplitude of the disturbance increases without limit, but because the displacements are of the same sign they do not pinch off the sheet into strips in the same way. The nonlinear theory for antisymmetric waves on thin sheets of fluid (Kinnersley 1976, section 5) indicates that the surface profile is described parametrically in terms of the elliptic integrals $E(\theta, m)$ and $F(\theta, m)$ by

$$\left.\begin{aligned} x &= \frac{a}{2m^{1/2}}(2E - F) \\ z &= a\cos\theta \end{aligned}\right\}, \tag{3.21}$$

where m denotes the modulus. The velocity potential is proportional to F. In the critical case $m = 0.7312$ when $a/L = 1.330$, the surfaces of adjacent waves touch, as shown in Fig. 4b.

4. Oblique perturbations

We have so far considered only sinusoidal perturbations that are propagated exactly transversely to the direction of the plunging wave, that is, to the direction of extension of the fluid sheet. Now consider a perturbation such that the wave crests (lines of constant phase) make a general angle β with the direction of the breaker, as in Fig. 5a. A consequence of extending the sheet in the x direction in the ratio $\epsilon > 1$ will be to distort the wave pattern so that two points P, Q, say, on the x axis are now separated by a distance $P'Q' = \epsilon PQ$, whereas as two points P, R in the transverse direction remain at the same separation as before. Hence, the wavenumber

$$(k_x, k_y) = (k\sin\beta, k\cos\beta) \tag{4.1}$$

becomes

$$(k_x', k_y') = (\epsilon^{-1}k\sin\beta, k\cos\beta). \tag{4.2}$$

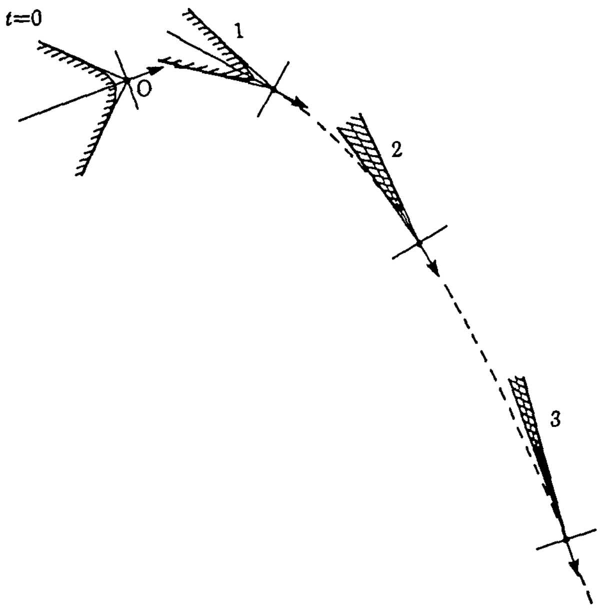


FIG. 6. The tip of a plunging breaker, modeled as a rotating Dirichlet hyperbola in free-fall (Longuet-Higgins 1980.)

The effect is to turn the direction of the perturbation more nearly transverse to the flow. In the limit, as the sheet becomes infinitely stretched, the striations become aligned with the flow, as is observed. The dynamics of the perturbation in the general case are similar to those in the case of directly transverse waves, except that some allowance must be made for propagation of the wave action, with the group velocity, normal to the wave crests.

5. Discussion

In theoretical models of plunging breakers, the plunging jet has sometimes been treated as a steady flow (see, for example, Dias and Tuck 1993; Jenkins 1994), but in reality the flow is time dependent. A possible analytic model is the rotating hyperboloid (Longuet-Higgins 1980, 1983), in which the thickness $2h$ of the sheet (following a fixed particle) diminishes like $1/t$; see Fig. 6.

In a steady waterfall, where the particles are almost in free-fall, the vertical distance of a marked particle below a given level will increase ultimately like $\frac{1}{2}gt^2$. Hence, the vertical distance separating two marked particles following each other at a time-interval of Δt will increase like $gt\Delta t$. By continuity we see that again the thickness $2h$ of the sheet forming the unperturbed waterfall must decrease like $1/t$. In other models of breaking waves (Longuet-Higgins 1980, section 10; Longuet-Higgins 1981, Fig. 12), a sharp cusp is attained in a finite time t.

The most important and drastic stage of the instability appears to be the initial transition of the perturbation from a gravity to a capillary wave. This may be responsible for the breakup of the jet at an earlier stage of its development. Once the jet has been broken up into independent strips in the manner described, the ensuing breakup of the strips into droplets can be expected to follow swiftly.

We have ignored the frictional and inertial influence of the air surrounding the jet, which may be appreciable. However, we note that if the inviscid mechanism just described is indeed significant, then an increase in the amplitude of the initial cross-wave perturbation would be expected to advance the process of disintegration. This is a prediction amenable to experimental verification.

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