A Fractal Approach to Breaking Waves

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ABSTRACT

Fractal models of breaking waves in a random surface should preferably describe dynamical as well as geometrical properties. This becomes feasible if there is a wide separation between the length scales of component waves. Using this idea, a simple model of breaking waves is constructed, which shows that whereas the downward acceleration of particles at a wave crest is limited to g, the upward accelerations in a wave trough are unbounded. Owing to tangential stretching or contraction, certain phases of a progressive or standing wave can be identified as being stable or unstable. The most striking instabilities are expected on the forward slopes of progressive waves, and in the troughs of steep waves meeting a vertical wall.

1. Introduction

Because of the involvement of breaking waves in many important transfer processes at the sea surface, much attention has been given in recent years to understanding the various types and mechanisms of wave breaking in deep water. For reviews, see Longuet-Higgins (1988) or Banner and Peregrine (1993). Here we shall consider some aspects of a breaking sea state from a fractal point of view.

The idea of self-similarity in a wind-driven sea state is not new. Beginning with Phillips (1958), various power-law spectra, which imply a dynamical self-similarity, have been suggested. To lowest order in the surface slopes, such sea states are generally considered as Gaussian, so that the phases of the various harmonic components are uncorrelated. Correlation between the phases occurs only at higher orders.

A fractal representation, however, involves generally not only self-similarity with respect to different length scales, but also a nonrandom relation between the phases. Recently, Huang et al. (1992) showed that the "unwrapped phase" of an ocean wave record has a fractal property, but no clear dynamical explanation is forthcoming. A rather thorough analysis of photographic data of the sea surface in terms of a multifractal representation has been performed by Kerman (1990) (see also Kerman 1993). While successfully separating out a "breaking wave" or fractal component of the data from a "nonbreaking" or Gaussian component, this analysis was essentially that of a static surface, frozen in time. The dynamical equations of the surface, or their immediate consequences, did not enter directly. Such a task is indeed very difficult.

The dynamical discussion of superimposed trains of finite-amplitude surface waves is much simplified if we assume that 1) the waves are all pure gravity waves on deep water and 2) there is a large proportional difference in scale between each component wave and the next largest wave. In the present paper we use these assumptions to prove certain properties of breaking waves. Thus, whereas the maximum downward acceleration in an upward-point crest is equal to g (section 3), the maximum *upward* acceleration is shown to be unlimited; see section 4. The very high upward acceleration may be realized in the smooth interaction of random waves with a vertical wall, as has been found in the numerical calculations by Cooker and Peregrine (1991). In sections 5 and 6 we discuss the reasons why certain phases of a progressive or standing wave may be more or less stable to short-wavelength disturbances. A discussion follows in section 7.

The dynamical mechanisms that will be described are fractal in character, insofar as they involve amplification of the tendency toward breaking when wave components of different scales are added in a self-similar manner.

2. Superposing self-similar waves

In the simplest case, imagine a short surface wave of shape given in rectangular coordinates (x, y) by $y = f(x)$, as in Fig. 1. On this is superposed a similar wave smaller by a factor ρ, and on top of this a still smaller wave reduced by a factor ρ^2, and so on to $(N + 1)$ waves where N is finite. Analytically, the surface elevation η is given by

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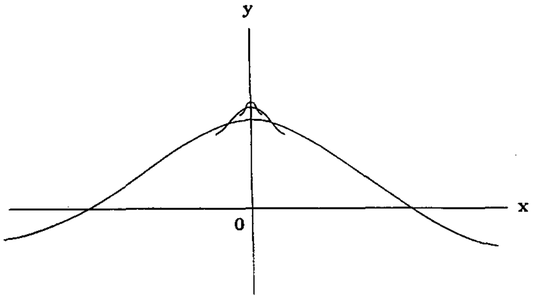

FIG. 1. Superposition of wave profiles $y = f(x)$ having differing length scales, with zero phase shift at $x = 0$.

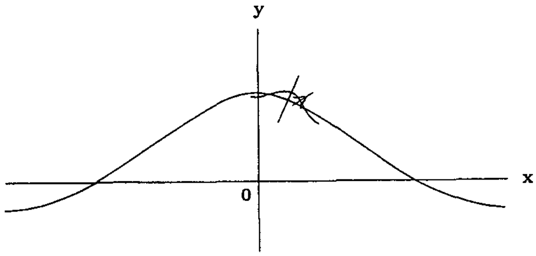

FIG. 2. Superposition of profiles $n = f(s)$ with nonzero phase shift ϵ.

$$\eta = f(x) + \rho f(x/\rho) + \rho^2 f(x/\rho^2) + \cdots + \rho^N f(x/\rho^N). \quad (2.1)$$

At the origin $x = 0$ the elevation is given by

$$\eta = f(0)(1 + \rho + \cdots + \rho^N) = f(0)\frac{1 - \rho^{N+1}}{1 - \rho}, \quad (2.2)$$

which is finite as $N \to \infty$. The surface gradient $\eta_x(0)$ is given by

$$\eta_x = f_x(0)(N + 1), \quad (2.3)$$

which vanishes in the case of a profile symmetric about $x = 0$. The curvature at the origin is given by

$$\eta_{xx} = f_{xx}(0)(1 + \rho^{-1} + \rho^{-2} + \cdots + \rho^{-N}), \quad (2.4)$$

which tends to infinity with N. Thus, in the limit the surface must develop a sharp point.

A more interesting case is when the normal displacement n at the rth stage of the superposition is given as a scaled function $\rho^r f(\epsilon + s/\rho^r)$ where s is the arc length measured along the surface after the rth wave is added, ϵ being some phase constant. Then by a similar argument, the coordinates (x, y) of the crest will remain finite as $N \to \infty$, but the surface curvature will become infinite. If ϵ is independent of r, the angle between the tangent and a fixed line will also increase indefinitely so that locally the surface will take the form of a tight spiral (see Fig. 2).

3. Acceleration of a particle at a wave crest

Returning to the original case $\epsilon = 0$, as in Fig. 1, let us suppose that the original profile $y = f(x)$ represents the surface of a Stokes gravity wave in deep water. The downward acceleration a of a particle of water as it passes through the wave crest will be some fraction of g, the acceleration of gravity, say

$$a = \lambda g, \quad 0 < \lambda < 0.39 \quad (3.1)$$

(see Longuet-Higgins and Fox 1977). A second gravity wave whose length is short compared to the radius of curvature in the first wave will behave locally as if it were propagated on a steady stream parallel to the surface of the first wave, but in a field of gravity

$$g' = g - a = (1 - \lambda)g. \quad (3.2)$$

Accordingly a particle at the crest of the second wave will have a downward acceleration

$$a' = \lambda' g' = \lambda'(1 - \lambda)g \quad (3.3)$$

relative to the first wave. A third wave riding on the second will therefore experience a gravitational field

$$g'' = g' - a' = (1 - \lambda')(1 - \lambda)g, \quad (3.4)$$

and so on. After the $(N + 1)$th wave is added, the effective gravity will be

$$g^{(N)} = (1 - \lambda)(1 - \lambda') \cdots (1 - \lambda^{(N+1)})g. \quad (3.5)$$

If the waves are all self-similar, then clearly $g^{(N)} \to 0$ as $N \to \infty$. Hence the total downward acceleration of a particle, which is given by

$$a + a' + a'' + \cdots = (g - g') + (g' - g'') + \cdots = g - g^{(N)}, \quad (3.6)$$

will tend to g (cf. Longuet-Higgins 1985).

This result is in accordance with our expectation that at any sharp point in the free surface of a fluid the pressure gradient, which always has a zero component tangential to the free surface, should be identically zero. Hence, if the ordinary momentum equation

$$\mathbf{a} = -\nabla \mathbf{p} + \mathbf{g} \quad (3.7)$$

applies there, we might expect $\mathbf{a} = \mathbf{g}$.

However, this expectation fails in the next example.

4. Acceleration of a particle in a wave trough

Consider on the other hand the acceleration of a particle in a wave trough (Fig. 3). In a single Stokes wave, the *upward* acceleration has a magnitude a given by

$$a = \lambda g, \quad 0 < \lambda < 0.30; \quad (4.1)$$

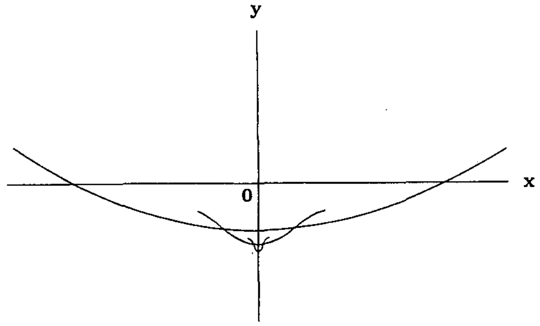

FIG. 3. Superposition of waves in the trough of a gravity wave.

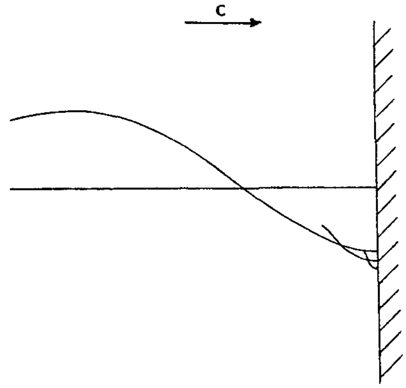

FIG. 4. Random waves impinging on a vertical wall.

see Longuet-Higgins (1985). A similar, much shorter wave near the trough of a first wave then experiences an effective gravity

$$g' = a + g = (1 + \lambda)g \tag{4.2}$$

in contrast to Eq. (3.2). After adding N such waves in succession, we see by a similar argument that the effective gravity $g^{(N)}$ is given by

$$g^{(N)} = (1 + \lambda)(1 + \lambda')\cdots(1 + \lambda^{(N-1)})g. \tag{4.3}$$

If the waves are all self-similar, or even if only $\sum \lambda^{(r)} \rightarrow \infty$, it follows that

$$g^{(N)} \rightarrow \infty. \tag{4.4}$$

Hence the particle acceleration in the trough of the composite wave *is indefinitely large.*

We remark that the demonstrations given in this and the previous section apply equally well both to progressive and to standing waves (or even to combinations of progressive and standing waves). In each case, however, it is necessary that the *phases* of the component waves on different scales be suitably related. In the next section we shall see under what circumstances this is most likely to occur. Here, however, we may note that in the special situation where random waves are impinging on a vertical wall (see Fig. 4), all the wave components are in effect standing waves with an antinode in surface elevation at the wall itself. Thus, the phases at the wall are either 0° or 180° and hence have a greater probability of being precisely correlated.

The above result helps to show that the very high accelerations (20 g and higher) found by Cooker and Peregrine (1991) in numerical studies of waves impinging on a vertical wall are not in themselves unreasonable.

5. Stability of the free surface

Those phases of a surface wave that are most liable to instability may be investigated very simply by considering the *radiation stress* associated with short waves riding on longer waves. It was shown by Longuet-Higgins and Stewart (1960) that short waves riding on a horizontal current U interact with it so as to produce an additional momentum flux associated with the short waves. The additional horizontal flux of horizontal momentum is given by the leading component S_{xx} of the *radiation stress* tensor S_{ij}. In deep water

$$S_{xx} = \frac{1}{2}E, \tag{5.1}$$

where E is the mean energy density of the short waves. If the current U is varying in space, work is done by the current on the short waves, so as to produce a change in the short-wave energy. Thus, if U varies only in the x direction,

$$\frac{\partial E}{\partial t} + (c_g + U)\frac{\partial U}{\partial x} = -S_{xx}\frac{\partial U}{\partial x}, \tag{5.2}$$

where c_g is the group velocity of the short waves. For short waves propagating on long waves, we may take U to be the orbital motion in the long waves. The term $(c_g + U)\partial U/\partial x$ will then be relatively small and we shall have roughly

$$\frac{\partial E}{\partial t} = -\frac{1}{2}E\frac{\partial U}{\partial x}. \tag{5.3}$$

Thus the parts of the long wave that are most unstable to short-wave disturbances are those for which $\partial U/\partial x < 0$; that is to say, where the surface is contracting horizontally.

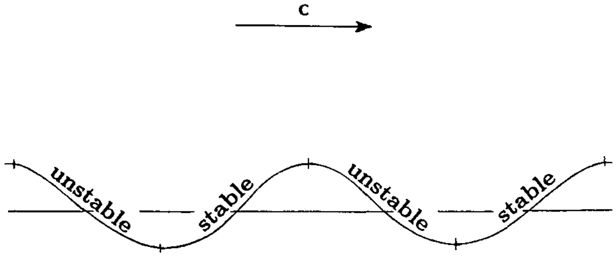

FIG. 5. Stable and unstable phases in a progressive gravity wave.

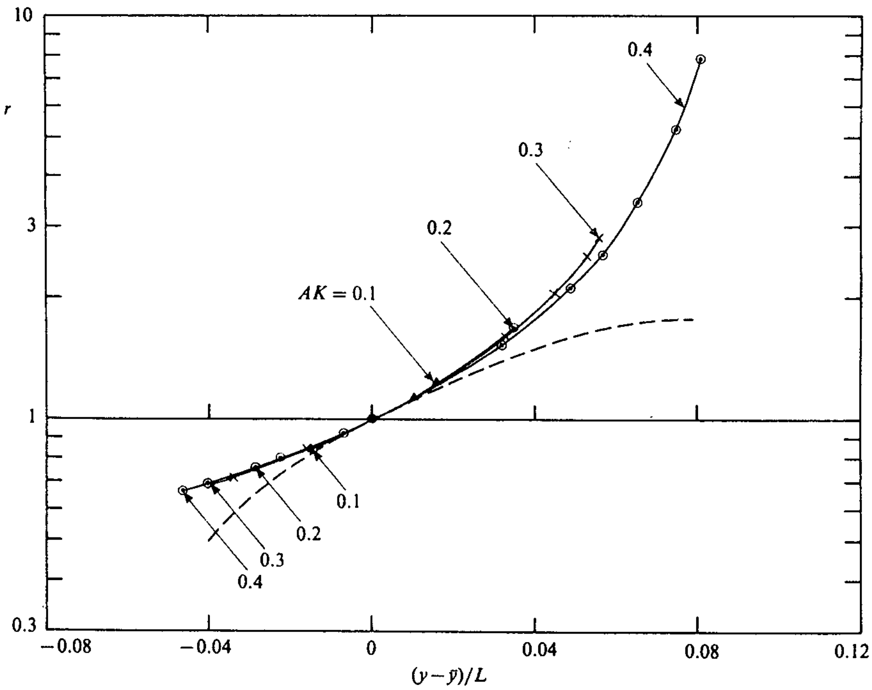

FIG. 6. Relative steepening factor r for short waves riding on a long wave of finite steepness AK, shown as a function of the surface elevation y in the long wave (from Longuet-Higgins 1987).

In a *progressive* wave (see Fig. 5) the surface is contracting on the *forward* face of the wave, and stretching horizontally on the rear face. Thus, the forward face of the wave is *unstable* to short-wave disturbances, and the rear face is stable. This is reflected in the common observation that the forward face of a steep wind wave is generally rough, while the rear face is smooth.

The wavelength of the short waves is also affected by the contraction of the free surface, and the amplitude of the short waves is further increased by the relatively smaller value of the apparent gravity g' on the upper part of the long waves (Longuet-Higgins and Stewart 1960).

The extent of the short-wave steepening on long waves of finite amplitude is, in fact, greater than would be expected on a linear theory for the long waves. Exact calculations (see Fig. 6) show that even on a long wave with steepness parameter $AK = 0.3$, the short waves steepen between the long-wave trough and long-wave crest by a factor as great as 4. When $AK = 0.4$ the steepening factor exceeds 10.

Thus, it is most probable that the steepest short waves will be found near the crests of the long waves, and preferably on their forward face. For, if the short waves break ahead of the long-wave crests, they will not regain enough energy from the wind to break on the rear face (cf. Longuet-Higgins 1991).

Still shorter waves riding on the short waves will be subject to the similar processes of steepening by contraction of the free surface due to the short waves, work done by radiation stress, and reduction in the effective gravity. All these effects will be cumulatively greater on the forward face of the longest waves.

6. Surface stability in standing waves

Consider on the other hand a standing surface wave (Fig. 7). In such a wave the amplitude of the crests and troughs are either increasing simultaneously as in

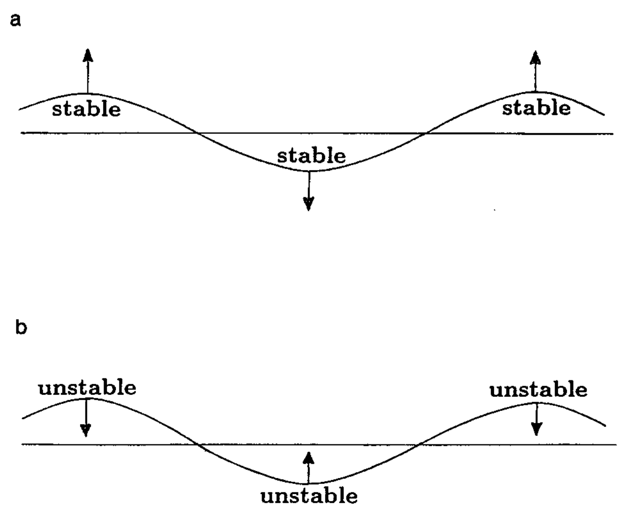

FIG. 7. Stable and unstable phases in a standing gravity wave: (a) crests and troughs increasing in amplitude; (b) crests and troughs diminishing.

Fig. 7a or decreasing simultaneously (Fig. 7b). In the first case the free surface is everywhere being stretched. Hence, it is everywhere stable to short-wavelength perturbations. In the second case the free surface is everywhere contracting and so it is *unstable* to short-wavelength perturbations. The latter case, when the wave trough is rising, corresponds to the situations found by Cooker and Peregrine (1991) to give rise to very high upward accelerations.

7. Discussion

We have shown how, by incorporating dynamical considerations into a simple fractal model of surface waves it is possible to understand some observed features of the sea surface. The model is highly simplistic and is made possible only by the assumption of a wide separation of scales between the component waves. This enables us to assume that the short waves are in a sense controlled by the longer waves on which they ride and to ignore the reaction of the short waves on the much more energetic long waves.

Nevertheless it is worth noting that the short-wave steepening shown in Fig. 6, for example, is nearly independent of the wavelength of the short waves when the ratio of the wavelengths exceeds about 2 (see Longuet-Higgins 1987, Figs. 4 and 5). Thus the aspects of wave steepening discussed here may not be very dependent on the assumption concerning the component length scales.

It should also be noted that the horizontal contraction, which destabilizes certain portions of the free surface, is merely the mechanism that initiates the breaking. Another such mechanism is the instability of the crest of a steep wave (see Longuet-Higgins and Cleaver 1994). It appears the latter is mainly a local phenomenon. But even there the local contraction of the free surface in the unperturbed flow is a significant factor in the instability.

A satisfying fractal model of a breaking sea that fully incorporates the dynamics of a free surface under the action of gravity still awaits development. The present paper draws attention to this challenge and makes a modest start.

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