Accelerations in Steep Gravity Waves. II: Subsurface Accelerations

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ABSTRACT

It is shown that the vertical acceleration of a particle beneath the crest of a steep gravity wave does not always decrease monotonically with depth in the fluid. When the wave steepness ak exceeds 0.4, the acceleration at first increases with depth, and is a maximum at points slightly below the free surface. The result may have implications for the motions of a floating buoy.

1. Introduction

The acceleration of particles at the surface of steep gravity waves in deep water has been considered in a recent paper (Longuet-Higgins 1985), and was calculated accurately for representative values of the wave steepness ak. One of the suggestions made in the course of the discussion (section 4) was that for very steep waves, the vertical component of the real, or Lagrangian, acceleration should exceed the corresponding values at the free surface. This of course runs counter to our intuition, derived from linear theory, that the acceleration should diminish monotonically with the depth.

In this paper we present further calculations which confirm the above effect. First, we consider the acceleration beneath the crest of an almost-highest wave, according to the asymptotic theory of Longuet-Higgins and Fox (1977), and show how the acceleration increases from less than $0.4g$ at the free surface to a limiting value $0.5g$ at depths large compared to the radius. In section 3 we consider waves of arbitrary amplitude, and carry out accurate calculations for wave steepnesses ak in the range $0.1 \leq ak \leq 0.42$ (Figs. 2a to 2e). Approximate calculations for waves of limiting steepness are given in section 4. The results are collected and discussed in section 5.

2. The almost-highest wave

The fluid flow near the crest of a gravity wave approaching its limiting steepness was first calculated numerically by Longuet-Higgins and Fox (1977, 1978). A close but convenient analytical approximation was derived by Longuet-Higgins (1979). Figure 1 (right-hand curve) shows the corresponding surface profile. This has a rounded crest at the point $(x, y) = (0, -1)$, x being measured horizontally and y vertically upwards, and the scale being chosen so that $g = 1$, and q^2, the square of the speed at the wave crest, equals 2 in a frame traveling horizontally to the left with the phase speed c. It is assumed that

$$q^2/c_0^2 = \epsilon^2 \ll 1, \tag{2.1}$$

c_0 being the linear phase speed, and that at infinity the flow tends to the Stokes corner-flow, in which the free surface is inclined at 30° to the horizontal. The length scale in Fig. 1 is equal to $\epsilon c_0^2/g$, and for gravity waves in deep water it can be shown that ϵ is related to the wave steepness ak by

$$\epsilon^2 = 2.0|ak - 0.4432| \tag{2.2}$$

(see Longuet-Higgins and Fox, 1978).

The approximate profile in Fig. 1, which agrees with the exact profile to within 1 percent, corresponds to the flow

$$z = i(\alpha + i\gamma\chi)/(\beta + i\chi)^{1/3} \tag{2.3}$$

where $z = x + iy$ and $\chi = \phi + i\psi$ is the complex velocity potential. The α, β and γ are numerical constants, given in Longuet-Higgins (1979) and (1985). From this we may calculate the real (or Lagrangian) accelerations $\mathbf{a}_L$ through the general formula

$$\mathbf{a}_L = -z^*_{\chi\chi}/(z_\chi z^{*3}_\chi). \tag{2.4}$$

(Here $z_\chi = d\chi/dz$ and an asterisk denotes the complex conjugate). We shall examine only the values on the vertical line through the wave crest, on which $\phi = 0$; then from (2.3) and (2.4) we find, after some reduction,

$$a_L = 18(\psi - \beta)^3 \frac{\gamma\psi + (2\alpha - 3\beta\gamma)}{[2\gamma\psi + (\alpha - 3\beta\gamma)]^4}. \tag{2.5}$$

This is plotted on the left-hand side of Fig. 1. At the free surface ($\psi = 0$),

$$\mathbf{a}_L = -0.354g \tag{2.6}$$

as compared with the accurate value $0.386g$ computed

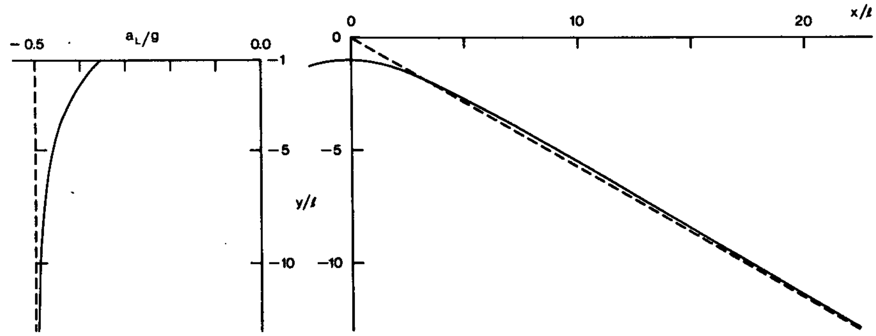

FIG. 1. The vertical acceleration a_L beneath the crest of an almost-highest wave, as given by (2.3). The free-surface profile is shown on the right.

by Longuet-Higgins and Fox (1977). On the other hand, at depths large compared to the radius of curvature at the crest ($\psi \rightarrow -\infty$) we have

$$a_L = -(9/8\gamma^3)g = -0.5g \tag{2.7}$$

since $\gamma = (3/2)^{2/3}$ by Eq. (3.3) of Longuet-Higgins (1979). So the vertical acceleration is the same as in the Stokes corner-flow, as we would expect.

Between $\psi = 0$ and $\psi = -\infty$ the acceleration increases monotonically with distance from the crest. In physical terms, the reason is that a_L is given by q^2/R where q is the local particle speed and R is the radius of curvature of the local streamline. In the Stokes corner-flow, q^2 and R are proportional: $q^2 = \frac{1}{2}gR$. But in the almost-highest wave q^2 increases with depth less rapidly than R, in general.

On the other hand, in any horizontally periodic irrotational motion, the velocity at great depths must tend to zero, in a stationary reference frame (Longuet-Higgins, 1953). So the acceleration must likewise tend to zero. To reconcile these opposing results we must now examine periodic waves of arbitrary steepness.

3. Waves of arbitrary steepness

The accurate calculation of subsurface accelerations in waves of arbitrary steepness can be carried out by precisely the same method as in Longuet-Higgins (1985) using either formula (2.4) or the equivalent result

$$a_L = -q^6 z_\chi z^*_{\chi\chi}, \tag{3.1}$$

q being the local particle speed. The only difference is that each coefficient a_n in the Fourier series for x and y must now be multiplied by $e^{n\psi}$. In (3.1) it is generally not permissible to substitute $q^2 = 2gy$, a relation which holds only at the free surface.

The results are shown in Figs. 2a to 2e, the accelerations below the crest being plotted horizontally on the left of each diagram, and the corresponding surface profiles on the right.

When $ak = 0.1$ and 0.2 (Figs. 2a and 2b, respectively) the accelerations are much as expected in linear theory, diminishing monotonically with depth in a roughly exponential manner. In Fig. 2c, corresponding to $ak = 0.3$, a point of inflexion already appears in the upper part of the acceleration curve. In Fig. 2d, when $ak = 0.4$, there is now a maximum in $|a_L|$ at a point below the free surface. When $ak = 0.42$ (Fig. 2e), the effect is even more pronounced. There the maximum acceleration occurs at a depth $0.12\ \mathrm{k}^{-1}$ below the free surface ($k = 2\pi$/wavelength), in fact at a depth comparable to the radius of curvature of the profile at the crest. The value of a_L at the crest is less than this by some 6 percent.

4. The limiting wave

A simple but fairly accurate approximation to the motion of subsurface particles in a limiting wave is given by the so-called "hexagon approximation" (Longuet-Higgins 1973, 1979). In this, the transformation

$$e^{-iz} = \zeta \tag{4.1}$$

takes six successive crests of a wave of length $L = \frac{1}{3}\pi$ into the corners of a regular hexagon in the ζ-plane. The surface profile of the wave corresponds to the sides of the hexagon, and so in the z-plane the equation of the surface in $-\frac{1}{6}\pi \leqslant x \leqslant \frac{1}{6}\pi$ is simply

$$y = \ln(\sec x) \tag{4.2}$$

which is accurate within 3 percent.

To obtain the particle velocities, the interior of the hexagon is transformed onto the interior of the unit circle $|W| = 1$ by the Schwartz-Christoffel transformation

$$\zeta = K \int_0^W \frac{dW}{(1 - W^6)^{1/3}} \tag{4.3}$$

where K is a constant. If $z = 1$ corresponds to a wave crest, then

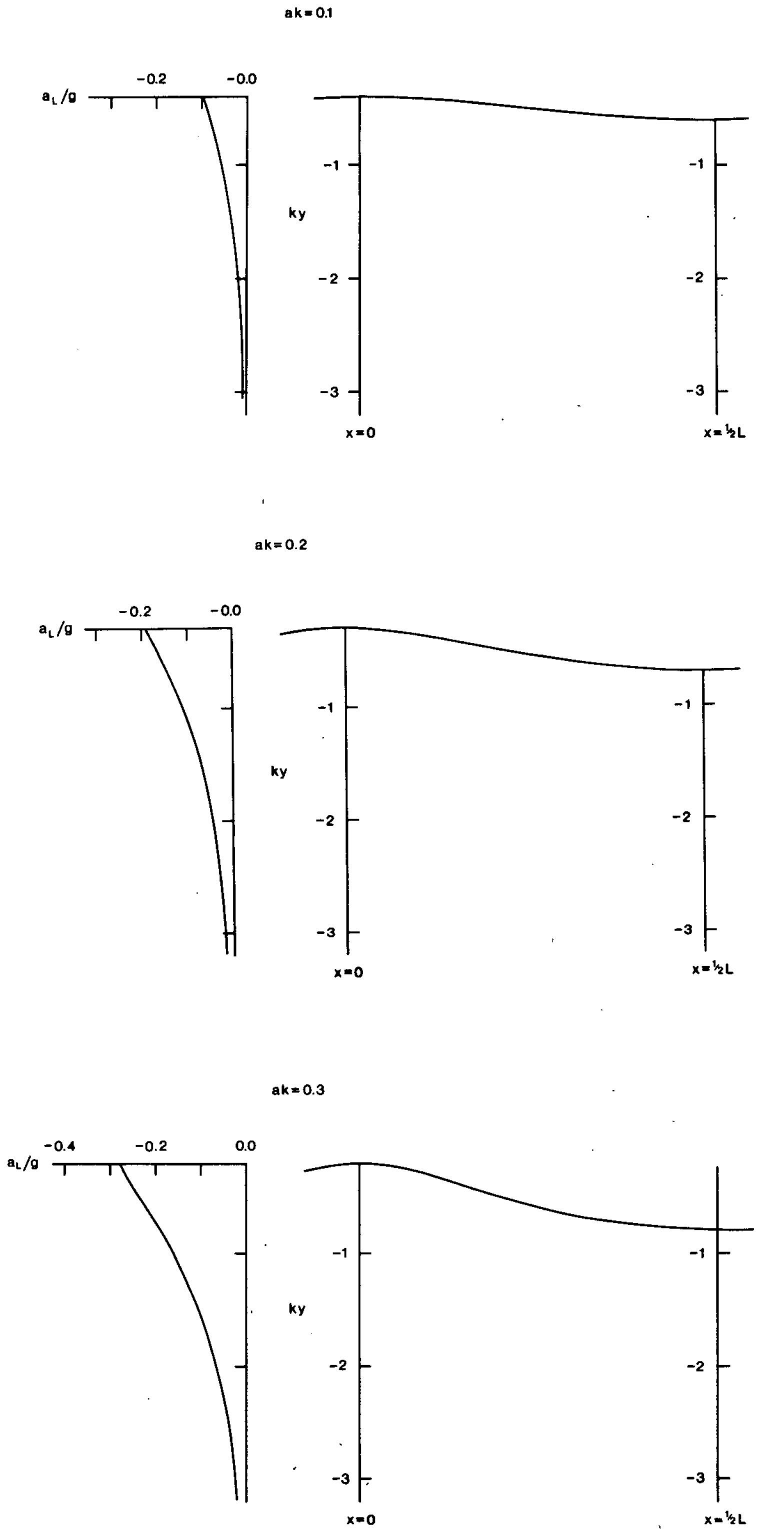

FIG. 2. Vertical accelerations a_L beneath the crest of a gravity wave with steepness parameter *ak*. The corresponding surface profile is shown on the right. (a) $ak = 0.1$, (b) $ak = 0.2$, (c) $ak = 0.3$, (d) $ak = 0.4$ and (e) $ak = 0.42$.

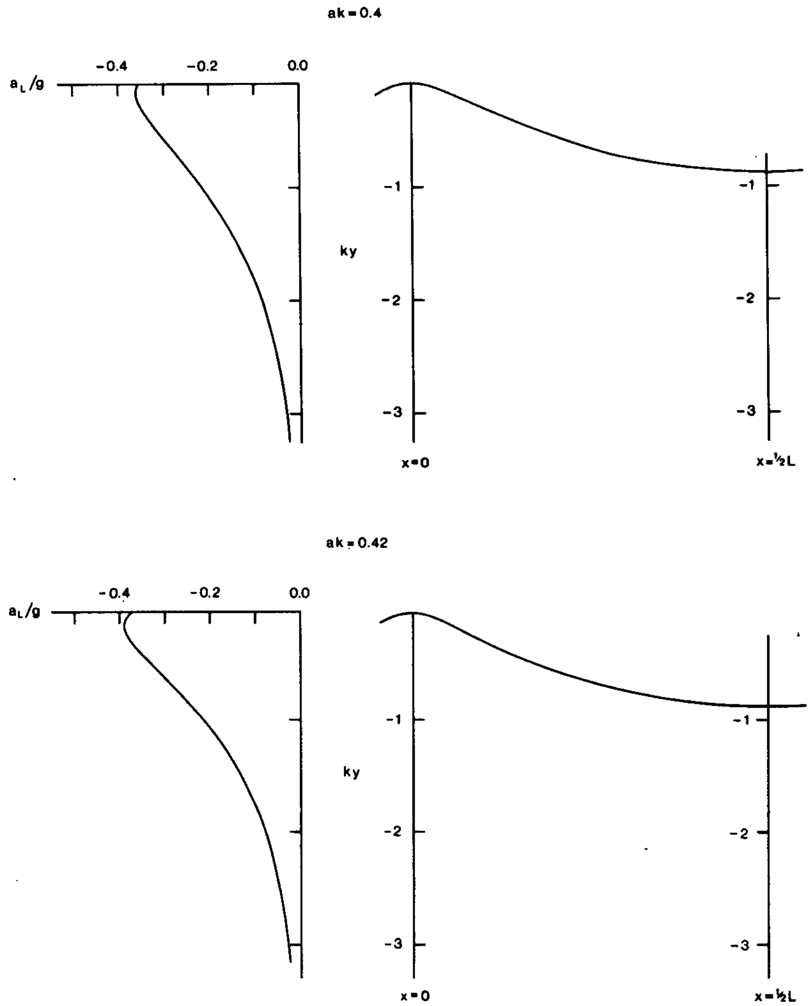

FIG. 2. (*Continued*)

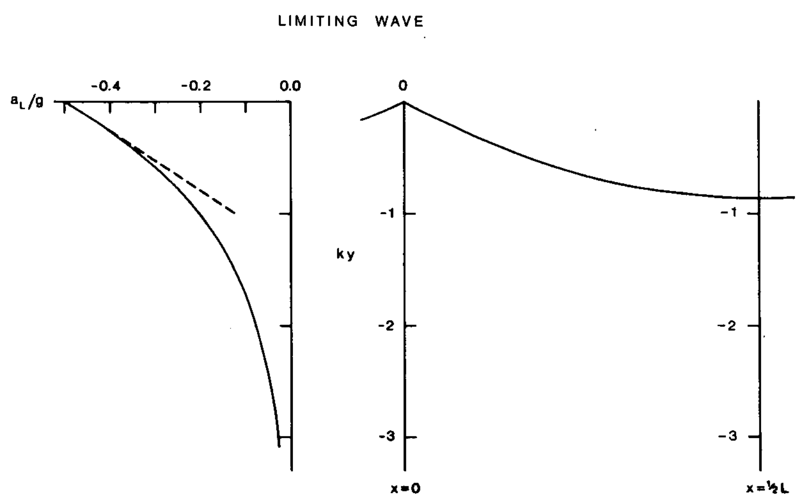

FIG. 3. The vertical acceleration beneath the crest of a limiting gravity wave, in the hexagon approximation.

$$\frac{1}{K} = \int_0^1 \frac{dW}{(1 - W^6)^{1/3}} = 1.1129. \quad (4.4)$$

The corresponding velocity potential is simply

$$\chi = -ic \ln W \quad (4.5)$$

where c is the phase speed.

To evaluate the acceleration by means of the general formula

$$\mathbf{a}_L = \chi_z \chi_{zz}^* \quad (4.6)$$

[see Longuet-Higgins, 1985, Eq. (2.3)] we have

$$\chi_z = \chi_W W_\zeta \zeta_z = -\frac{c\zeta}{K} \frac{(1 - W^6)^{1/3}}{W} \quad (4.7)$$

from (4.2), (4.3) and (4.5). Also

$$\chi_{zz} = \chi_{z\zeta}\zeta_z + \chi_{zW} W_\zeta \zeta_z \quad (4.8)$$

$$= \frac{ic\zeta}{KW}(1 - W^6)^{1/3} - \frac{ic\zeta^2}{K^2 W^2} \frac{(1 + W^6)}{(1 - W^6)^{1/3}}. \quad (4.9)$$

Beneath a wave crest, both ζ and W are real and less than 1. Hence on substitution in Eq. (4.6) we find

$$a_L = A^2 c^2 [(1 - W^6)^{2/3} - A(1 + W^6)] \quad (4.10)$$

where

$$A = \zeta / KW. \quad (4.11)$$

Together with

$$y = \ln \zeta \quad (4.12)$$

[from Eq. (4.2)], this enables us to plot a_L against ky ($k = 6$). We note that some numerical values of ζ/K are already tabulated in Table 4 of Longuet-Higgins (1979, p. 512).

To explore the neighborhood of the wave crest itself, we write

$$W = 1 - \eta^3/6, \quad \eta \ll 1, \quad (4.13)$$

and expand these expressions in powers of η. To lowest order we obtain

$$\left.\begin{aligned} \chi &= ic\eta^3/6 \\ \zeta &= 1 - K\eta^2/4 \\ ky &= -\tfrac{3}{2} Ky^2 \end{aligned}\right\} \quad (4.14)$$

since $k = 2\pi/L = 6$. Hence

$$\mathbf{a}_L = -\frac{2ic^2}{K^3}\left(1 - \frac{9}{8} Ky^2\right), \quad (4.15)$$

that is

$$-a_L/g = \frac{2c^2}{K^3}\left(1 - \frac{3}{4} ky\right) \quad (4.16)$$

by (4.14). To satisfy the condition of constant pressure at the free surface we must choose

$$c^2 = K^3/4 \quad (4.17)$$

[see Longuet-Higgins 1973, Eq. (4.13)], giving

$$-a_L/g = \frac{1}{2}\left(1 - \frac{3}{4} ky\right). \quad (4.18)$$

The acceleration at the surface ($y = 0$) is now $-\frac{1}{2}g$ in agreement with the Stokes corner-flow, and the vertical gradient of the acceleration is given by

$$\frac{da_L}{dy} = \frac{3}{8} gk. \quad (4.19)$$

This gradient is represented by the broken line in Fig. 3.

5. Discussion

Figure 4 compares the acceleration curves for different wave steepnesses ak, and shows how these approach the curve for the limiting wave. Clearly the limit is approached nonuniformly, so that for all wave steepnesses greater than 0.4 the maximum acceleration below the surface exceeds that at the surface itself.

The occurrence of this maximum appears to be a property of the local flow in the neighborhood of the crest, related closely to the "almost-highest wave" discussed in section 2.

It is tempting to associate the subsurface maximum with the known instability of steep gravity waves at high wave steepnesses (Tanaka 1983). However, the

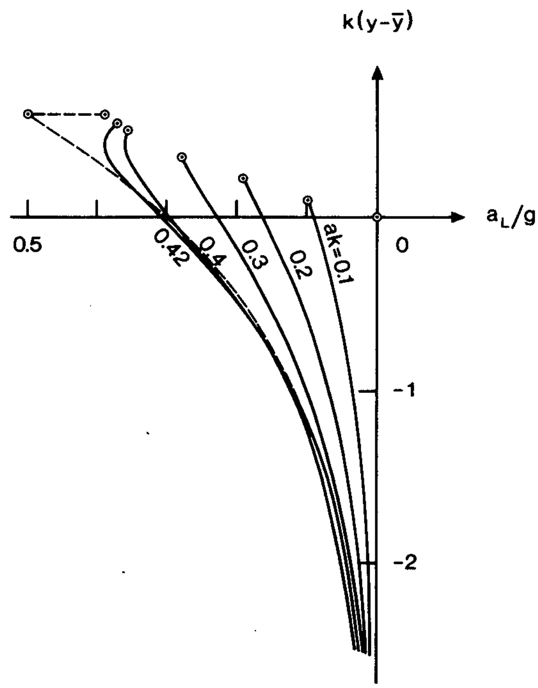

FIG. 4. Comparison of the vertical accelerations beneath the crest, for waves of different steepness ak. The broken line corresponds to the limiting wave ($ak = 0.4432$).

first superharmonic instability does not occur until $ak > 0.4292$—a value somewhat exceeding the first appearance of the acceleration maximum ($ak < 0.42$). In no case does the vertical acceleration exceed g, or even $0.5g$, so that the vertical pressure gradient never changes sign.

Wave buoys or other floating objects whose vertical dimension is comparable to the radius of curvature at a steep wave crest may experience a greater vertical acceleration than others with a shallower draught. An experimental investigation might prove interesting.

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