# Reflection and Refraction at a Random Moving Surface. III. Frequency of Twinkling in a Gaussian Surface

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When light is reflected or refracted at a moving Gaussian surface, the observer sees a number of moving images of the source, which appear or disappear generally in pairs; such an event is called a "twinkle." In the present paper the number of twinkles per unit time is evaluated in terms of the frequency spectrum of the surface and the distance of the source $O$ and observer $Q$, on the assumption that the surface is Gaussian and that $OQ$ is perpendicular to the mean surface level.

A solution is found first for a single system of long-crested (or two-dimensional) waves, and then extended to the case of two such systems intersecting at right angles.

The rate of twinkling is found to depend, apart from a scale factor, on two parameters of the surface, one of which, $\alpha$, increases steadily with the distance of $O$ or $Q$ from the surface; the other, $d$, discriminates between waves of standing type and waves of progressive type. Over a considerable range of $\alpha$, the rate of twinkling is almost independent of $d$, but for large values of $\alpha$ the rate is much greater for standing waves than for progressive waves; waves of intermediate type are included in the analysis.

## 1. INTRODUCTION

In two previous papers[1,2] we have studied the pattern of reflections of a point-source in a random, moving surface, and have determined the average number of distinct images seen by an observer in the case when the surface slopes and curvature have a Gaussian distribution. It was shown[1] that the specular points (that is, images of the point source) are generally created or annihilated in pairs, such an event being called a "twinkle." In this paper our purpose is to evaluate the frequency of twinkling, that is to say the average number of twinkles per unit time over the whole surface. This number $n$ is to be expressed in terms of the wave energy spectrum of the surface.

It can be shown that at a twinkle the intensity of radiation seen by the observer is greatly increased, so that the observer sees a bright flash. (This may be analogous to some sudden increases in the recorded intensity of radio waves reflected from the ionosphere.[3]) In the language of ray optics, the surface momentarily focuses the radiation at the point of observation. Corresponding to the principal radii of curvature, there are generally two focal points along a reflected ray, and the flash occurs when one of these coincides with the observer.

An exactly similar effect is produced when the surface is the boundary of a refracting medium (such as water) and the source of light is observed from a point on the far side of the surface; an observer below the water surface will see a pattern of images of the light source; these are created and destroyed in a manner analogous to the reflected images.

Another way of looking at the phenomenon is as follows. Suppose that the radiation, after passing through the surface, illuminates a horizontal plane at some fixed distance below the mean surface level—as sunlight falling on the sea bed in shallow water. If the plane is not too near to the surface it can be seen to be covered with a pattern of bright lines—the loci of those points where a twinkle may be momentarily observed. The rate of twinkling is then the average number of times that one of these lines sweeps through a fixed point in the plane.

The general problem, for a Gaussian surface with arbitrary frequency spectrum, appears to be complicated. Our approach will be to solve first the analogous problem in two dimensions (when the surface is long-crested and the light source is a line parallel to the crests); then we may deduce the solution for a surface which consists of two such long-crested systems intersecting at right angles.

It is found that apart from a scale factor the rate of twinkling depends upon two parameters of the surface. The first of these, $\alpha$, is proportional to the distance of the observer from the surface and to the rms curvature of the surface. The second, $d$, is small for waves of progressive type and increases to a maximum for waves of standing type. Over much of the range of $\alpha$ the rate of twinkling is found to be nearly independent of $d$; for larger values of $\alpha$, however, the rate of twinkling increases with $d$, and is much greater for standing waves than for progressive waves.

## 2. GEOMETRICAL CONDITIONS

If $z=\zeta(x,y,t)$ denotes the equation of the surface in rectangular coordinates, $z$ being directed vertically upward, then it can be shown[1] that the condition for a specular point, when source and observer lie on the $z$ axis, is

$$\partial f/\partial x=0, \quad \partial f/\partial y=0, \tag{2.1}$$

where

$$f(x,y,t)=\zeta(x,y,t)+\tfrac{1}{2}\kappa(x^2+y^2) \tag{2.2}$$

and $\kappa$ is a constant. In the reflection problem, if $h_1$ and $h_2$ denote the heights of source and observer above the mean surface level, then

$$\kappa=\tfrac{1}{2}[(1/h_1)+(1/h_2)]. \tag{2.3}$$

[1] M. S. Longuet-Higgins, J. Opt. Soc. Am. **50**, 838 (1960), paper I of this series.
[2] M. S. Longuet-Higgins, J. Opt. Soc. Am. **50**, 845 (1960), paper II of this series.
[3] J. D. Whitehead, J. Terrest. Atm. Phys. **9**, 269 (1956).

In the refraction problem, if $h_1$ and $h_2$ denote the distances of source and observer above and below the surface, and if $\mu_1$ and $\mu_2$ are the refractive indices of the two media, then

$$\kappa=(\mu_1h_1+\mu_2h_2)\,/[(\mu_2-\mu_1)h_1h_2]. \tag{2.4}$$

The condition for a twinkle is that, besides Eqs. (2.1),

$$(\partial^2f/\partial x^2)(\partial^2f/\partial y^2)-(\partial^2f/\partial x\partial y)^2=0 \tag{2.5}$$

shall also be satisfied; that is the vanishing of the total curvature of the surface $z=f$.

In the corresponding one-dimensional case, when the situation is independent of the $y$ coordinate the conditions are simply that

$$\partial f/\partial x=0,\quad \partial^2f/\partial x^2=0, \tag{2.6}$$

where

$$f(x,t)=\zeta(x,t)+\tfrac{1}{2}\kappa x^2. \tag{2.7}$$

## 3. TWO-DIMENSIONAL CASE: STATISTICAL MODEL

We take as a model for the surface the function

$$\zeta(x,t)=\sum_{n=1}^{n_0}c_n\cos(k_nx+\sigma_nt+\epsilon_n), \tag{3.1}$$

where the phase constants $\epsilon_n$ are supposed to be distributed randomly and uniformly over $(0,2\pi)$, the wave numbers $k_n$, $\sigma_n$ are distributed over the intervals $(-\infty,\infty)$ and $(0,\infty)$, and where $n_0$ at last tends to infinity in such a way that over any small intervals $(k, k+dk)$, $(\sigma,\sigma+d\sigma)$,

$$\sum\nolimits_n \tfrac{1}{2}c_n^2=E(k,\sigma)dkd\sigma, \tag{3.2}$$

where $E(k,\sigma)$ is a continuous function of $k$, $\sigma$ (the spectrum function). Such a model is a generalization of that employed for a time-independent surface.[2]

Under certain conditions the distribution of $\zeta$ and its derivatives at a given point $x$ and time $t$ is Gaussian. On writing, for brevity,

$$\partial\zeta/\partial x,\quad \partial^2\zeta/\partial x^2,\quad \partial^2\zeta/\partial x\partial t,\quad \partial^3\zeta/\partial x^3=\xi_1,\ \xi_2,\ \xi_3,\ \xi_4, \tag{3.3}$$

we have for the matrix of mean values of the products:

$$(\langle\xi_i\xi_j\rangle_{\mathrm{av}})=\begin{pmatrix} m_2 & 0 & 0 & -m_4\\ 0 & m_4 & m_3' & 0\\ 0 & m_3' & m_2'' & 0\\ -m_4 & 0 & 0 & m_6\end{pmatrix}, \tag{3.4}$$

where

$$m_r=\int_{-\infty}^{\infty}\int_0^{\infty}E(k,\sigma)k^rdkd\sigma,$$
$$m_r'=\int_{-\infty}^{\infty}\int_0^{\infty}E(k,\sigma)k^r\sigma dkd\sigma, \tag{3.5}$$
$$m_r''=\int_{-\infty}^{\infty}\int_0^{\infty}E(k,\sigma)k^r\sigma^2dkd\sigma.$$

The probability density of $\xi_1$, $\xi_2$, $\xi_3$, $\xi_4$ is therefore given by

$$p(\xi_1,\xi_2,\xi_3,\xi_4)=\frac{|(M_{ij})|^{\frac{1}{2}}}{4\pi^2}\exp[-\tfrac{1}{2}\sum_{i,j}M_{ij}\xi_i\xi_j], \tag{3.6}$$

where $(M_{ij})$ is the matrix inverse to (3.4). The occurrence of zeros in the matrix implies that the distributions of $\xi_1$, $\xi_4$ and of $\xi_2$, $\xi_3$ are independent. In fact,

$$p(\xi_1,\xi_2,\xi_3,\xi_4)=p(\xi_1,\xi_4)p(\xi_2,\xi_3), \tag{3.7}$$

where

$$\left.\begin{aligned} p(\xi_1,\xi_4)&=\frac{1}{2\pi(m_2m_6-m_4^2)^{\frac{1}{2}}}\times\exp\left[-\frac{m_6\xi_1^2+2m_4\xi_1\xi_4+m_2\xi_4^2}{2(m_2m_6-m_4^2)}\right],\\ p(\xi_2,\xi_3)&=\frac{1}{2\pi(m_4m_2''-m_3'^2)^{\frac{1}{2}}}\times\exp\left[-\frac{m_2''\xi_2^2-2m_3'\xi_2\xi_3+m_4\xi_3^2}{2(m_4m_2''-m_3'^2)}\right].\end{aligned}\right\} \tag{3.8}$$

It will be convenient to write

$$\begin{aligned}\eta_1,\eta_2,\eta_3,\eta_4&=\partial f/\partial x,\partial^2f/\partial x^2,\partial^2f/\partial x\partial t,\partial^3f/\partial x^3\\ &=(\xi_1+\kappa x),(\xi_2+\kappa),\xi_3,\xi_4,\end{aligned} \tag{3.9}$$

so that the probability density of $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$ at a fixed value of $x$, $t$ is given by

$$p(\eta_1,\eta_2,\eta_3,\eta_4)=p(\xi_1,\xi_2,\xi_3,\xi_4), \tag{3.10}$$

where

$$\xi_1,\xi_2,\xi_3,\xi_4=(\eta_1-\kappa x),(\eta_2-\kappa),\eta_3,\eta_4. \tag{3.11}$$

## 4. EVALUATION OF THE PROBABILITY

We wish now to find the probability that in a given small interval of time $(t, t+dt)$, and in $-\infty<x<\infty$, the conditions

$$\eta_1=0,\quad \eta_2=0 \tag{4.1}$$

shall be satisfied. Let this probability be $ndt$ (clearly, $n$ is the mean number of "flashes" per unit time). We first seek the probability of (4.1) being satisfied in a small interval $(t, t+dt)$ and in $(x, x+dx)$; if this probability is denoted by $n_xdxdt$, then clearly

$$n=\int_{-\infty}^{\infty}n_xdx. \tag{4.2}$$

The advantage of dealing first with $n_x$ is that for fixed $x$ the distributions of $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$ are invariant.[4]

Now the probability $p(\eta_1,\eta_2)d\eta_1d\eta_2$ represents the probability that $\eta_1$, $\eta_2$ lie within the limits $\eta_1$, $\eta_1+d\eta_1$ and $\eta_2$, $\eta_2+d\eta_2$ at certain fixed values of $x$, $t$. On the

[4] $n_x$ has been evaluated previously in the case when the source and observer are at infinite distance. See M. S. Longuet-Higgins, Proc. Cambridge Phil. Soc. 56, 234 (1956).

other hand if $\eta_1$, $\eta_2$ take the given values (4.1) in a given range $(x, x+dx)$ and interval $(t, t+dt)$ then $(\eta_1,\eta_2)$ at $(x,t)$ itself may vary within a region of measure

$$\left|\frac{\partial(\eta_1,\eta_2)}{\partial(x,t)}\right| = \left\|\begin{matrix}\partial^2 f/\partial x^2 & \partial^3 f/\partial x^3 \\ \partial^2 f/\partial x\partial t & \partial^3 f/\partial x^2\partial t\end{matrix}\right\| dxdt, \quad (4.3)$$

which in the neighborhood of the points (4.1) reduces to $|\eta_3\eta_4|dxdt$ simply. Hence the probability $n_x dxdt$ is given by

$$n_x dxdt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\eta_1,\eta_2,\eta_3,\eta_4)\,|\eta_3\eta_4|\,dxdtd\eta_3 d\eta_4, \quad (4.4)$$

and the required probability $ndt$ is given by

$$n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\eta_1,\eta_2,\eta_3,\eta_4)\,|\eta_3\eta_4|\,dxd\eta_3 d\eta_4. \quad (4.5)$$

Now $x$ is involved only in $\eta_1$, so that the integration with respect to $x$ may be carried out immediately. Replacing $\eta_2$ at the same time by 0, we have, from (3.10),

$$n = \frac{1}{(2\pi)^{\frac{1}{2}}\kappa m_6^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp(-\eta_4^2/2m_6)\,|\eta_4|\,d\eta_4 \times \frac{1}{2\pi(m_4 m_2''-m_3'^2)^{\frac{1}{2}}} \times \int_{-\infty}^{\infty}\exp\left[-\frac{m_2''\kappa^2+2m_3'\kappa\eta_3+m_4\eta_3^2}{2(m_4m_2''-m_3'^2)}\right]. \quad (4.6)$$

The remaining integrals present no difficulty, and give the following:

$$n = \frac{2^{\frac{1}{2}}}{\pi^{\frac{3}{2}}}\frac{m_6^{\frac{1}{2}}(m_4m_2''-m_3'^2)^{\frac{1}{2}}}{\kappa m_4}\exp(-\kappa^2/2m_4) \times\left[\exp(-\tfrac{1}{2}\phi^2)+\phi\int_0^{\phi}\exp(-\tfrac{1}{2}z^2)dz\right], \quad (4.7)$$

where

$$\phi = \kappa m_3'/[m_4^{\frac{1}{2}}(m_4m_2''-m_3'^2)^{\frac{1}{2}}]. \quad (4.8)$$

If we now define the dimensionless quantities

$$\alpha = m_4^{\frac{1}{2}}/\kappa, \quad (4.9)$$

$$d = [(m_4m_2''-m_3'^2)/m_4m_2'']^{\frac{1}{2}}, \quad (4.10)$$

then (4.7) can be written in the final form

$$n = [(m_6m_2'')^{\frac{1}{2}}/m_4]\mathfrak{f}(\alpha,d), \quad (4.11)$$

where

$$\mathfrak{f}(\alpha,d) = \frac{2^{\frac{1}{2}}}{\pi^{\frac{3}{2}}}\alpha d\exp(-2\alpha^2)^{-1} \times\left[\exp(-\tfrac{1}{2}\phi^2)+\phi\int_0^{\phi}\exp(-\tfrac{1}{2}z^2)dz\right] \quad (4.12)$$

and

$$\phi = (1-d^2)^{\frac{1}{2}}/\alpha d.$$

Since the error integral on the right-hand side is a tabulated function,[5] this completes the formal solution of the problem.

## 5. DISCUSSION

In Eq. (4.11) the first factor on the right-hand side has the dimensions of $(\text{time})^{-1}$. If the spectrum contains a single narrow band of frequencies centered on a mean wavelength $\lambda$ and period $\tau$, then

$$(m_6m_2'')^{\frac{1}{2}}/m_4 = 2\pi/\tau, \quad (5.1)$$

approximately. This factor, therefore, essentially determines the time scale.

Of the remaining two parameters $\alpha$, $d$, the first is inversely proportional to $\kappa$, and so increases linearly with the distances of $O$ and $Q$ from the surface. Also, $\alpha$ is proportional to $m_4^{\frac{1}{2}}$, the root-mean-square value of the "curvature" $\partial^2\zeta/\partial x^2$.

The parameter $d$ is a function of the frequency spectrum $E(k,\sigma)$ only. Now, since

$$2(m_4m_2''-m_3'^2) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} E(k_1,\sigma_1)E(k_2,\sigma_2) \times(k_1^4k_2^2\sigma_2^2+k_2^4k_1^2\sigma_1^2-2k_1^3k_2^3\sigma_1\sigma_2) \times dk_1d\sigma_1dk_2d\sigma_2, \quad (5.2)$$

and since the factor in the integrand is a perfect square, we have

$$m_4m_2''-m_3'^2 \geq 0,$$

whence

$$0 \leq d \leq 1.$$

The lower limit of $d$ is approached whenever the spectrum of the surface is narrow; the surface then has the appearance of a progressive wave of slowly varying amplitude and phase. The upper limit of $d$ is attained when $m_3'=0$, which occurs, for example, if the spectrum function $E(k,\sigma)$ is symmetrical with respect to $k$: $E(k,\sigma)=E(-k,\sigma)$, while $\sigma$ is an even function of $k$; the surface then has the appearance of a standing-wave pattern of varying phase and amplitude. Thus the parameter $d$ discriminates between progressive and standing waves.

The function $\mathfrak{f}(\alpha,d)$ is shown in Fig. 1 for various

[5] A. N. Lowan, "Tables of normal probability functions," Natl. Bur. Standards, Appl. Math. Ser. 23 (1953).

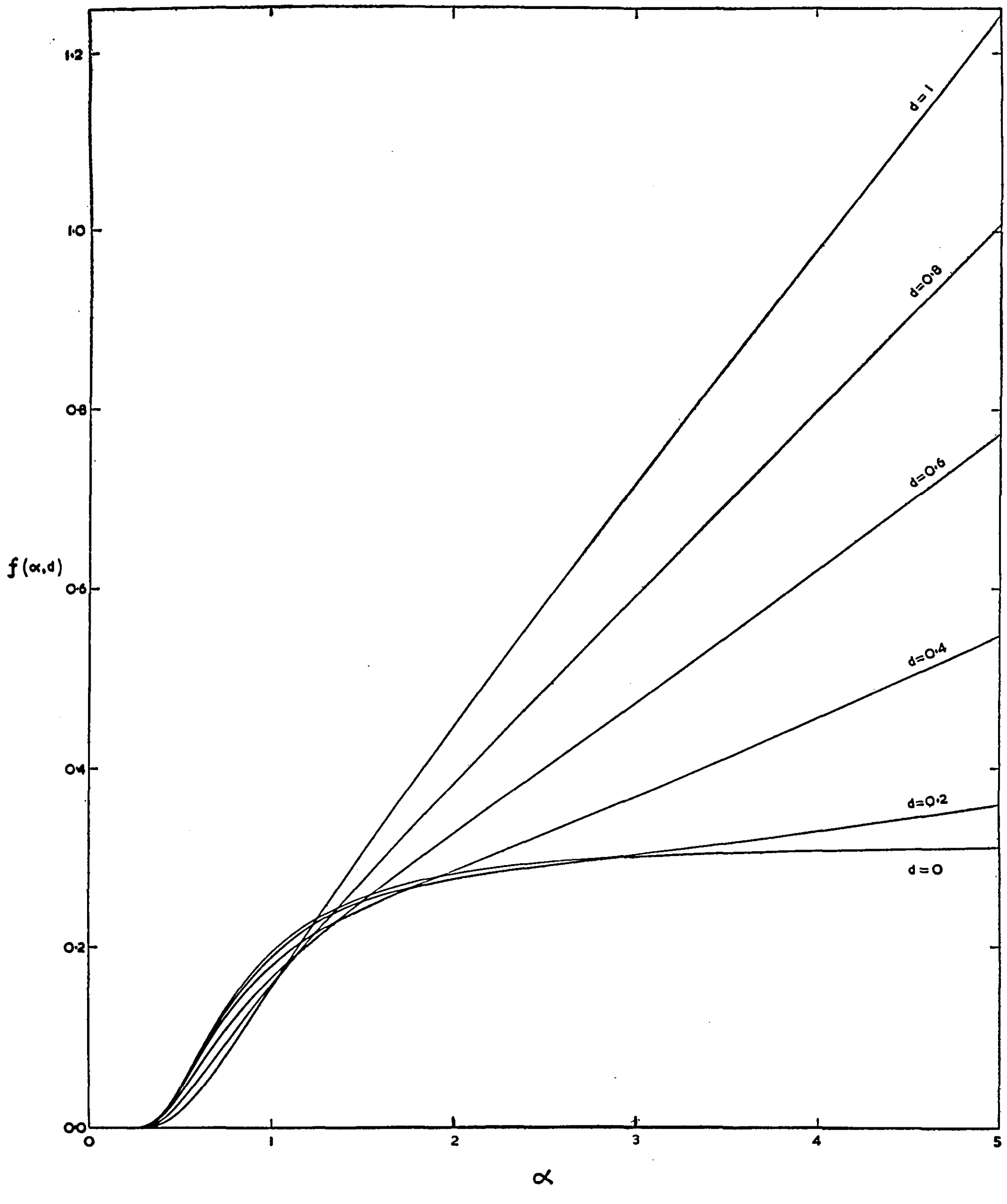


FIG. 1. Graphs of $f(\alpha,d)$ showing the rate of twinkling as a function of $\alpha$ (proportional to distance from surface) for various values of $d$.

values of the parameter $d$. Two limiting forms may be noted: as $d \rightarrow 0$

$$\mathrm{f}(\alpha,d) \rightarrow (1/\pi)\exp(-2\alpha^2)^{-1}, \tag{5.3}$$

and when $d \rightarrow 1$

$$\mathrm{f}(\alpha,d) \rightarrow (2^{\frac{1}{2}}/\pi^{\frac{3}{2}})\alpha\exp(-2\alpha^2)^{-1}. \tag{5.4}$$

Although these two functions behave very differently at infinity it is remarkable that when $\alpha<1.5$, they lie quite close together, as indeed does the whole family of functions; over the range $0<\alpha<1.5$, $\mathrm{f}(\alpha,d)$ may for some purposes be taken as independent of $d$.

All the functions have an extremely sharp cutoff at about $\alpha=0.3$, because of the factor $\exp(-2\alpha^2)^{-1}$. Hence the rate of twinkling falls off suddenly as the observer approaches the surface.

On the other hand, for large values of $\alpha$, and when $d>0$, we have

$$\mathrm{f}(\alpha,d) \sim (2^{\frac{1}{2}}/\pi^{\frac{3}{2}})\alpha d, \tag{5.5}$$

that is to say, $n$ increases linearly with distance from the surface, as we might expect. The limiting case $d=0$, when $\mathrm{f}(\alpha,d) \rightarrow 1/\pi$ as $\alpha \rightarrow \infty$, is never in fact attained, because the bandwidth of the spectrum is never quite zero.

## 6. THREE-DIMENSIONAL PROBLEM

The general problem in three dimensions, when stated analytically, involves the evaluation of multiple integrals of high order. A useful simplification, however, results whenever the surface consists of two systems of long-crested waves (both Gaussian) intersecting at right angles; for then by choosing the axes appropriately, we have

$$\zeta(x,y,t)=\zeta_1(x,t)+\zeta_2(y,t), \tag{6.1}$$

and the conditions (2.1) and (2.5) reduce to

$$\partial f_1/\partial x=0,\quad \partial f_2/\partial x=0,\quad (\partial^2 f_1/\partial x^2)(\partial^2 f_2/\partial y^2)=0, \tag{6.2}$$

where

$$\begin{aligned} f_1(x,t)&=\zeta_1(x,t)+\tfrac{1}{2}\kappa x^2,\\ f_2(x,t)&=\zeta_2(x,t)+\tfrac{1}{2}\kappa y^2. \end{aligned} \tag{6.3}$$

Equations (6.2) show that a twinkle will occur in the combined system if a specular point in the one system ($\partial f_1/\partial x=0$) is combined with a twinkle in the other ($\partial f_2/\partial y=0$, $\partial^2 f_2/\partial y^2=0$), or vice versa. Hence the total rate of twinkling is given by

$$n=n^{(1)}N^{(2)}+n^{(2)}N^{(1)}, \tag{6.4}$$

where $N^{(1)}$ and $N^{(2)}$ denote the numbers of specular points in the two systems, respectively, and $n^{(1)}$ and $n^{(2)}$ denote the rates of twinkling. Now $N^{(i)}$ has been evaluated in a previous paper,[1] in fact, for a long-crested system of waves,

$$N^{(i)}=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\alpha\exp(-2\alpha^2)^{-1}+\int_0^{1/\alpha}\exp(-\tfrac{1}{2}z^2)dz\right], \tag{6.5}$$

where $\alpha$ is the same parameter as before [Eq. (4.9)]. $N$ is a function which increases steadily from unity at small values of $\alpha$ to the asymptotic value

$$N\sim(2/\pi)^{\frac{1}{2}}\alpha \tag{6.6}$$

for large values of $\alpha$.

If the frequency characteristics of the two systems happen to be similar, then $N^{(1)}=N^{(2)}$ and $n^{(1)}=n^{(2)}$ and we have, from (6.4),

$$n=2n^{(1)}N^{(1)}. \tag{6.7}$$

This can be expressed in terms of the parameter

$$A=\tfrac{1}{2}\alpha^2=\tfrac{1}{4}\kappa^{-2}D, \tag{6.8}$$

where $D$ represents the mean-square value of the mean curvature of the surface (see footnote reference 1).

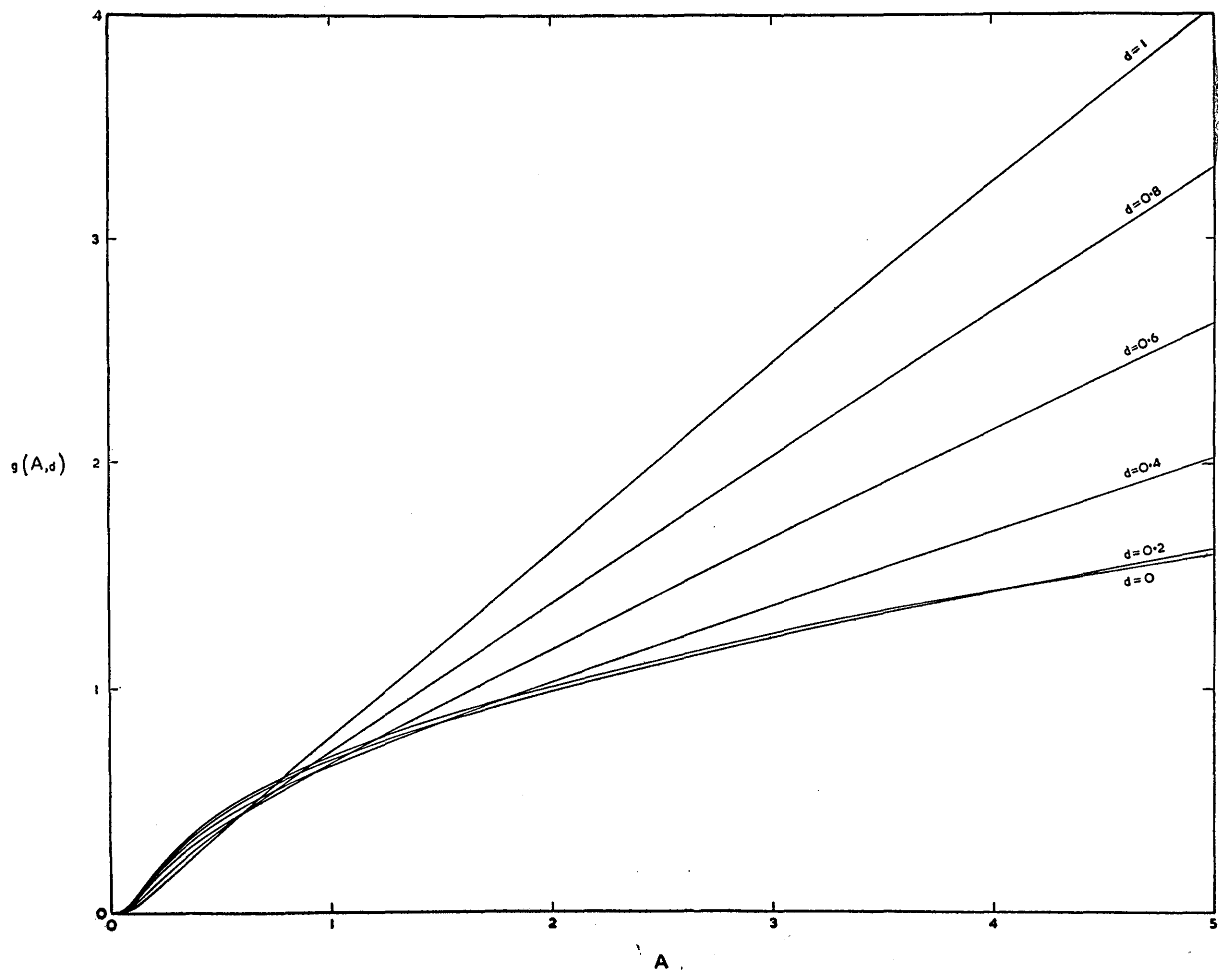


FIG. 2. Graphs of $g(A,d)$, showing the rate of twinkling for two intersecting systems as a function of $A$ (proportional to the square of the distance from the surface) for various values of $d$.

Thus,

$$n = [(m_6 m_2'')^{\frac{1}{2}}/m_4]\mathfrak{g}(A,d). \tag{6.9}$$

The function $\mathfrak{g}(A,d)$ is shown in Fig. 2.

The interpretation of this result is similar to the two-dimensional case. $A$ is proportional to the square of the distance from the origin and to the mean-square curvature, while $d$ discriminates between standing waves and progressive waves. The function $\mathfrak{g}(A,d)$ is nearly independent of $d$ for $0<A<1$, while for large values of $A$ (that is, great distances from the origin), we have

$$\mathfrak{g}(A,d) \doteqdot (8/\pi^2)Ad. \tag{6.10}$$

Thus, the rate of twinkling is more vigorous for standing waves $d=1$ than for progressive waves $d \ll 1$.

A rather general result may be deduced from Eq. (6.4). Consider the mean lifetime of a specular point. Since each specular point involves two twinkles, one at the beginning and one at the end of its life, and since two specular points are involved in a twinkle it follows that the mean lifetime of a specular point is given by

$$L = N/n, \tag{6.11}$$

and similarly in the two long-crested systems the mean lifetime of a specular *line* is given by

$$L^{(1)} = N^{(1)}/n^{(1)}, \quad L^{(2)} = N^{(2)}/n^{(2)}. \tag{6.12}$$

Now, since a specular point in the combined system is always the intersection of two specular lines, one from each of the two systems, we have

$$N = N^{(1)}N^{(2)}. \tag{6.13}$$

On dividing each side of Eq. (6.4) by the corresponding side of (6.13), we find

$$1/L = (1/L^{(1)}) + (1/L^{(2)}). \tag{6.14}$$

Hence, the lifetime of a specular point in the combined system is always less than the lifetime of a specular line in either long-crested system. When the two long-crested systems are similar, $L^{(1)} = L^{(2)}$, and hence

$$1/L = 2/L^{(1)}, \tag{6.15}$$

that is, the mean lifetime in the combined system is exactly half that in either system individually.

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# On Gabor's Expansion Theorem*

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As a generalization for optics of the well-known sampling theorem of information theory, D. Gabor proposed an expansion theorem. It relates to the number of independent solutions of the wave equation in a region defined by the object and by the aperture of an optical system.

A proof of this theorem presents formidable difficulties. In this paper, a proof relating to important cases is established, and a more accurate estimate for the number of the independent solutions in the general case is given.

## 1. INTRODUCTION

IN recent years the introduction of information theory has brought about many useful developments in optics.[1,2] In these researches the sampling theorem of Shannon plays a fundamental role. Gabor[3] and Gamo[4] formulated a theory of image formation in matrix form using this theorem, and furthermore Gabor[3] proposed the following expansion theorem as a generalization of the sampling theorem: "Assume that the object area, large compared with the square of the wavelength, is limited by a black screen. Assume also that there is a similar limitation in the aperture plane, at a great distance from the object plane. Then, in the domain limited by these two black screens, there exist $N$ independent solutions of the wave equation $\nabla^2 u + (2\pi/\lambda)^2 u = 0$, that is to say, solutions with $u=0$ immediately behind the black screens, and $N$ is $N = \lambda^{-2}\int \delta x \delta y \delta(\cos\alpha_x) \times \delta(\cos_y \alpha) = h^{-2}\int \delta x \delta y \delta \mathbf{p}_x \delta \mathbf{p}_y$. ($x,y$ are coordinates in object plane and $\cos\alpha_x$, $\cos\alpha_y$ are direction cosines of the geometrical optical rays. $\mathbf{p} \equiv h/\lambda$, where $h$ is Planck's constant.) Any progressive wave through the object area and through the aperture can be expanded in terms of these $N$ eigenfunctions with not more than $N$ complex coefficients."

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