Reflection and Refraction at a Random Moving Surface. II. Number of Specular Points in a Gaussian Surface

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The number of specular points reflected in a random Gaussian surface is determined theoretically under the following alternative conditions: (1) when the surface is perfectly long crested (two-dimensional); (2) when the surface is three-dimensional but isotropic; (3) for quite general surfaces, provided that the observer and the source of radiation are both at a great distance from the surface.

The results can be applied to the similar problem when the surface forms the boundary of two refracting media.

1. INTRODUCTION

SUPPOSE that light from a point source falls upon a wavelike surface such as the surface of a lake or sea. An observer may see many distinct images of the source reflected in the surface at the specular points. Following a previous paper[1] we shall here determine the average number of reflections seen by the observer, as a function of the wave-energy spectrum of the surface and of the positions of observer and light source.

It will be supposed in this paper that the surface is Gaussian, that is to say, the probability distribution of the surface slopes and their derivatives is jointly normal. Such an assumption is convenient mathematically and may approximate to naturally occurring surfaces under some conditions; for example, it may apply to water surfaces where the slope is not too great, so that the waves do not approach breaking point. Ocean swell or shorter wind waves passing through "slicks" may come under this heading. Very steep wind waves, however, can be markedly non-Gaussian.[2]

When both source and observer are at infinite distance from the surface, the specular points in any finite region are those points where the surface has a particular gradient. The average number of specular points per unit area in this case has been evaluated previously.[3] Here we shall treat the more general case when both the source O and observer Q may be at a finite distance from the surface; but we restrict the discussion to cases where OQ is nearly perpendicular to the surface level.

With very slight modification the solution can be applied to the case when O and Q are on opposite sides of the surface, and the latter forms the boundary between two media of different refractive index: for example, how many images can an observer above water see when a light source is situated below water level, or vice versa?

The problem is first solved in the two-dimensional case when, strictly speaking, the source O is a line source and Q is a line receiver. The full three-dimensional problem is solved formally in Sec. 3 and is explicitly evaluated in Sec. 4 for the special case when the surface is *isotropic* (its statistical properties are independent of azimuthal direction). The mean number N of images is given by Eq. (4.11), in which A is a parameter proportional to the mean-square curvature of the surface and to the square of the distances of source and observer from the surface.

In Sec. 5 the solution is given for the case when the surface consists of two sets of long-crested waves (both Gaussian) intersecting at right angles. The number N is then given by Eq. (5.7). This and the isotropic case are compared in Fig. 2.

Finally, the solution is given for large values of A (corresponding to the source and observer at great distances) and an arbitrary form of the wave spectrum. In particular it is shown that if the surface consists of two wave systems intersecting at an arbitrary angle θ_0, then the number of images is proportional to $\sin\theta_0$ [see Eq. (6.9)]. The mean number of images is equal to the mean number for an isotropic surface of the same rms curvature provided that $\theta_0 = 66° 30'$.

2. TWO-DIMENSIONAL CASE

Let (x,z) be rectangular coordinates, with z vertically upward, and let $O=(0,h_1)$ and $Q=(0,h_2)$ denote the positions of the source of light and of the observer, respectively, at heights h_1 and h_2 above the mean surface level. Further, let $P=(x,\zeta)$ denote a typical point upon the surface $z=\zeta(x)$. It is easily seen[1] that for P to be an image point we must have, at P,

$$\partial\zeta/\partial x = -\kappa x, \tag{2.1}$$

where

$$\kappa = \tfrac{1}{2}[(1/h_1)+(1/h_2)], \tag{2.2}$$

provided that $\kappa\zeta$ and $\partial\zeta/\partial x$ are both small quantities.

In the case of refraction, if h_1 and h_2 denote the distances of O and Q above and below the surface, and if μ_1 and μ_2 denote the refractive indices, then Eq. (2.1) must hold, but with

$$\kappa = (\mu_1 h_1 + \mu_2 h_2)/(\mu_2 - \mu_1) h_1 h_2. \tag{2.3}$$

[1] M. S. Longuet-Higgins, J. Opt. Soc. Am. **50**, 838 (1960).
[2] A. H. Schooley, Trans. Am. Geophys. Union **36**, 273 (1955).
[3] M. S. Longuet-Higgins, Phil. Trans. Roy. Soc. London **A249**, 321 (1957).

On writing, for brevity,

$$(1/x)(\partial\zeta/\partial x)=\xi_1, \quad \partial^2\zeta/\partial x^2=\xi_2, \tag{2.4}$$

we seek first the probability that, at some point in a given small interval $(x, x+dx)$, ξ_1 takes precisely the value $-\kappa$. Let us denote by $p(\xi_1,\xi_2)$ the joint probability density of ξ_1 and ξ_2; thus $p(\xi_1,\xi_2)d\xi_1 d\xi_2$ gives the probability that ξ_1, ξ_2 lie in given small intervals of width $d\xi_1$, $d\xi_2$. But $\xi_1=-\kappa$ in $(x, x+dx)$ if and only if ξ_1 at x lies within a range of width

$$|d\xi_1|=|\partial\xi_1/\partial x|dx=(1/|x|)|\xi_1-\xi_2|dx \tag{2.5}$$

approximately, ξ_2 being kept constant. Hence the probability of a specular point in $(x, x+dx)$ is

$$N_x dx=\int_{-\infty}^{\infty} p(\xi_1,\xi_2)\frac{1}{|x|}|\xi_1-\xi_2|dxd\xi_2, \tag{2.6}$$

and the total expectation of specular points over the whole range $-\infty<x<\infty$ is given by

$$N=\int_{-\infty}^{\infty} N_x dx. \tag{2.7}$$

As a model for the surface we may take the representation used by Rice,[4] and suppose that

$$\zeta(x)=\sum_{n=1}^{n_0} c_n \cos(k_n x+\epsilon_n), \tag{2.8}$$

where the k_n denote constant wave numbers, the phases ϵ_n are randomly distributed in $(0,2\pi)$, and where, in the end, n_0 tends to infinity and the c_n tend to zero in such a way that over any small interval of wave number $(k, k+dk)$, we have

$$\sum \tfrac{1}{2}c_n^2=E(k)dk, \tag{2.9}$$

where E is a continuous function of k, known as the energy or power spectrum of $\zeta(x)$. The function $\zeta(x)$ may also be expressed as a stochastic integral.[5]

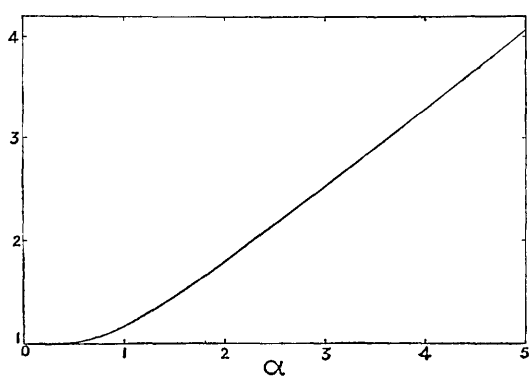

FIG. 1. The mean number of images N, as a function of the parameter α, defined by Eq. (2.15).

[4] S. O. Rice, Bell System Tech. J. 23, 282 (1944); 24, 46 (1945).
[5] J. L. Doob, *Stochastic Processes* (John Wiley & Sons, Inc., New York, 1953).

Under general conditions to be satisfied by the amplitudes c_n, the distribution of ζ and its derivatives is Gaussian: We shall assume that this is so at least as far as the second derivative ξ_2. Now the matrix of mean values for the ξ_i is[4]

$$(\langle\xi_i\xi_j\rangle_{\mathrm{av}})=\begin{pmatrix} m_2/x^2 & 0 \\ 0 & m_4 \end{pmatrix}, \tag{2.10}$$

where

$$m_r=\int_0^{\infty} E(k)k^r dr \tag{2.11}$$

and hence we have

$$p(\xi_1,\xi_2)=\frac{|x|}{2\pi(m_2m_4)^{\frac{1}{2}}}\exp[-\tfrac{1}{2}(\xi_1^2x^2/m_2+\xi_2^2/m_4)]. \tag{2.12}$$

On substituting into (2.7) and writing $\xi_1=-\kappa$, we have

$$N=\frac{1}{2\pi(m_2m_4)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\xi_2+\kappa| \times\exp[-\tfrac{1}{2}(\xi_1^2x^2/m_2+\xi_2^2/m_4)]dxd\xi_2. \tag{2.13}$$

The preceding integral is easily evaluated and we find

$$N=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\alpha\exp(-\tfrac{1}{2}\alpha^{-2})+\int_0^{1/\alpha}\exp(-\tfrac{1}{2}z^2)dz\right], \tag{2.14}$$

where

$$\alpha=m_4^{\frac{1}{2}}/\kappa. \tag{2.15}$$

From (2.2) it will be seen that when $h_1=h_2$, then $1/\kappa$ represents the distance of the observer from the mean surface [Eq. (2.2)]. Also, $m_4^{\frac{1}{2}}$ equals the root-mean-square value of the curvature ξ_2 [from (2.10)]. The nondimensional quantity is the product of these two. For small values of α (which therefore correspond to sources or observers very close to the surface), we have

$$N=1, \tag{2.16}$$

as we might expect; only one image can be seen. For large α, on the other hand, we find

$$N\sim(2/\pi)^{\frac{1}{2}}\alpha, \tag{2.17}$$

that is to say, the total number of image points increases almost linearly with distance from the surface.

The number N as a function of α is plotted in Fig. 1.

3. THREE-DIMENSIONAL CASE: GENERAL SOLUTION

The formulation of the problem in three dimensions is very similar. If $z=\zeta(x,y)$ denotes the equation of the surface, and if h_1, h_2 denote the distances of O, Q from the mean surface level, then the conditions to be satisfied by an image point $P(x,y,\zeta)$ are

$$\partial\zeta/\partial x=-\kappa x, \quad \partial\zeta/\partial y=-\kappa y, \tag{3.1}$$

where κ is defined by (2.2) or (2.3) according to the physical situation. On writing

$$(1/x)(\partial\zeta/\partial x),\quad (1/y)(\partial\zeta/\partial y)=\xi_1,\ \xi_2,$$
$$\partial^2\zeta/\partial x^2,\quad \partial^2\zeta/\partial x\partial y,\quad \partial^2\zeta/\partial y^2=\xi_3,\ \xi_4,\ \xi_5, \qquad (3.2)$$

we seek the probability that $(\xi_1,\xi_2)=(-\kappa,-\kappa)$ at some point in a given small region $dxdy$. This probability is

$$N_{xy}dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\xi_1,\cdots,\xi_5)\left|\frac{\partial(\xi_1,\xi_2)}{\partial(x,y)}\right| \times dxdyd\xi_3d\xi_4d\xi_5, \qquad (3.3)$$

where

$$\frac{\partial(\xi_1,\xi_2)}{\partial(x,y)}=\begin{vmatrix}(1/x)(\xi_3-\xi_1) & (1/x)\xi_4\\ (1/y)\xi_4 & (1/y)(\xi_5-\xi_2)\end{vmatrix} =\frac{1}{xy}[(\xi_3+\kappa)(\xi_5+\kappa)-\xi_4^2]. \qquad (3.4)$$

The total expectation of image points is then

$$N=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N_{xy}dxdy. \qquad (3.5)$$

We adopt the same model of the surface as in previous studies.[3,6] It is supposed that

$$\zeta(x,y)=\sum_{n=1}^{n_0} c_n\cos(u_nx+v_ny+\epsilon_n), \qquad (3.6)$$

where u_n and v_n denote constant wave numbers, the phases ϵ_n are randomly distributed in $(0,2\pi)$, and where, in the limit, as $n_0\to\infty$ we have, over an arbitrary small area $dudv$,

$$\sum\tfrac{1}{2}c_n^2=E(u,v)dudv. \qquad (3.7)$$

Here $E(u,v)$ is assumed to be a continuous function—the spectrum of $\zeta(x,y)$.

As before, we assume that ζ and its derivatives up to the second order are distributed normally. The matrix of mean values for the ξ_i is easily shown to be

$$(\langle\xi_i\xi_j\rangle_{\mathrm{av}})=\begin{pmatrix} m_{20}/x^2 & m_{11}/xy & 0 & 0 & 0\\ m_{11}/xy & m_{02}/y^2 & 0 & 0 & 0\\ 0 & 0 & m_{40} & m_{31} & m_{22}\\ 0 & 0 & m_{31} & m_{22} & m_{13}\\ 0 & 0 & m_{22} & m_{13} & m_{04}\end{pmatrix}, \qquad (3.8)$$

where

$$m_{pq}=\int_0^{\infty}\int_0^{\infty} E(u,v)u^pv^qdudv. \qquad (3.9)$$

Hence we have

$$p(\xi_1,\cdots,\xi_5)=p(\xi_1,\xi_2)p(\xi_3,\xi_4,\xi_5), \qquad (3.10)$$

where

$$p(\xi_1,\xi_2)=\frac{|xy|}{2\pi(m_{20}m_{02}-m_{11}^2)^{\frac{1}{2}}} \times\exp\left[-\frac{m_{02}x^2\xi_1^2-2m_{11}xy\xi_1\xi_2+m_{20}\xi_2^2}{2(m_{20}m_{02}-m_{11}^2)^{\frac{1}{2}}}\right], \qquad (3.11)$$

$$p(\xi_3,\xi_4,\xi_5) =\frac{|M_{ij}|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}}\exp\Big[-\tfrac{1}{2}\sum_{i,j=3,4,5} M_{ij}\xi_i\xi_j\Big],$$

and (M_{ij}) is the matrix inverse to

$$(\Xi_{ij})=\begin{pmatrix} m_{40} & m_{31} & m_{22}\\ m_{31} & m_{22} & m_{13}\\ m_{22} & m_{13} & m_{04}\end{pmatrix}=(M_{ij})^{-1}. \qquad (3.12)$$

On substituting the expression (3.3) into Eq. (3.5) and setting $\xi_1=\xi_2=-\kappa$, we find that the integration with respect to x, y may be carried out immediately, and hence

$$N=\frac{|M_{ij}|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}\kappa^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(\xi_3+\kappa)(\xi_5+\kappa)-\xi_4^2| \times\exp\Big[-\tfrac{1}{2}\sum_{i,j=3,4,5} M_{ij}\xi_i\xi_j\Big]d\xi_3d\xi_4d\xi_5. \qquad (3.13)$$

It is the evaluation of this integral which now concerns us.

By means of the linear transformation

$$\xi_i=\sum_{j=1}^{3} a_{ij}\eta_j,\quad i=3,4,5, \qquad (3.14)$$

it is possible to reduce the quadratic forms in (3.13) simultaneously so that

$$\sum M_{ij}\xi_i\xi_j=\eta_1^2+\eta_2^2+\eta_3^2,$$
$$\xi_3\xi_5-\xi_4^2=l_1\eta_1^2+l_2\eta_2^2+l_3\eta_3^2. \qquad (3.15)$$

The constants l_1, l_2, l_3 may be shown[3] to be the roots of the cubic equation

$$4l^3-3Hl-\Delta=0, \qquad (3.16)$$

where H and Δ are certain invariant combinations of the moments m_{pq}. Thus

$$3H=m_{40}m_{04}-4m_{31}m_{13}+3m_{22}^2,$$
$$\Delta=|(\Xi_{ij})|=|(M_{ij})|^{-1}. \qquad (3.17)$$

From (3.16) we have

$$l_1+l_2+l_3=0,$$
$$l_2l_3+l_3l_1+l_1l_2=-\tfrac{3}{4}H, \qquad (3.18)$$
$$l_1l_2l_3=\tfrac{1}{4}\Delta.$$

[6] M. S. Longuet-Higgins, Proc. Cambridge Phil. Soc. (to be published).

It can further be shown[6] that

$$(a_{31}+a_{51})^2+(a_{32}+a_{52})^2+(a_{33}+a_{53})^2=D,$$
$$(a_{31}+a_{51})^2/l_1+(a_{32}+a_{52})^2/l_2+(a_{33}+a_{53})^2/l_3=4, \quad (3.19)$$

where

$$D=m_{40}+2m_{22}+m_{04}, \quad (3.20)$$

another invariant of the surface. The first factor in the integrand of (3.13) may be written as

$$(\xi_3\xi_5-\xi_4^2)+\kappa(\xi_3+\xi_5)+\kappa^2$$
$$=(l_1\eta_1^2+l_2\eta_2^2+l_3\eta_3^2)+\kappa\sum_{j=1}^{3}(a_{3j}+a_{5j})\eta_j+\kappa^2$$
$$=\sum_{j=1}^{3}l_j\{\eta_j+\kappa(a_{3j}+a_{5j})/2l_j\}^2, \quad (3.21)$$

by the second of equations (3.19). Since the modulus of the transformation (3.14) is

$$\partial(\xi_3,\xi_4,\xi_5)/\partial(\eta_1,\eta_4,\eta_5)=|(a_{ij})|=|(M_{ij})|^{\frac{1}{2}}, \quad (3.22)$$

Eq. (3.13) becomes

$$N=\frac{1}{(2\pi)^{\frac{3}{2}}\kappa^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\sum_{j=1}^{3}l_j(\eta_j+y_j)^2|$$
$$\times\exp[-\tfrac{1}{2}\sum_{j=1}^{3}\eta_j^2]d\eta_1d\eta_2d\eta_3 \quad (3.23)$$

in which we have put, for brevity,

$$y_j=\kappa(a_{3j}+a_{5j})/(2l_i). \quad (3.24)$$

Now the corresponding triple integral without the modulus sign equals

$$\frac{1}{(2\pi)^{\frac{3}{2}}\kappa^2}\sum_{j=1}^{3}l_j(1+y_j^2)(2\pi)^{\frac{3}{2}}=1 \quad (3.25)$$

by (3.18) and (3.19). On adding this quantity to each side of (3.23), we have

$$N+1=\frac{2}{(2\pi)^{\frac{3}{2}}\kappa^2}\int\int\int[\sum_{j=1}^{3}l_j(\eta_j+y_j)^2]$$
$$\times\exp[-\tfrac{1}{2}\sum_{j=1}^{3}\eta_j^2]d\eta_1d\eta_2d\eta_3, \quad (3.26)$$

where the integration is over that region of η space for which the first factor in square brackets is positive.

Now Δ being a positive quantity it follows from (3.18) that one of the roots l_j is positive (let it be l_1, say) and that the other two are negative in general. So over the region of integration we may make the substitution

$$l_1^{\frac{1}{2}}(\eta_1+y_1)=r,$$
$$(-l_2)^{\frac{1}{2}}(\eta_2+y_2)=r\sin\theta\cos\phi, \quad (3.27)$$
$$(-l_3)^{\frac{1}{2}}(\eta_3+y_3)=r\sin\theta\sin\phi.$$

The ranges of the variables are

$$-\infty<r<\infty,\quad 0\leq\theta\leq\pi/2,\quad 0\leq\phi<2\pi, \quad (3.28)$$

and the modulus of transformation is

$$\partial(\eta_1,\eta_2,\eta_3)/\partial(r,\theta,\phi)=(l_1l_2l_3)^{-\frac{1}{2}}r^2\cos\theta\sin\theta. \quad (3.29)$$

So, on using (3.17), we have

$$N+1=\frac{2^{\frac{1}{2}}}{\pi^{\frac{3}{2}}\kappa^2\Delta^{\frac{1}{2}}}\int_{-\infty}^{\infty}dr\int_0^{\pi/2}d\theta\int_0^{2\pi}d\phi r^4\cos^3\theta\sin\theta$$
$$\times\exp[-\tfrac{1}{2}(Pr^2+2Qr+R)], \quad (3.30)$$

where we have written

$$P=l_1^{-1}-l_2^{-1}\sin^2\theta\cos^2\phi-l_3^{-1}\sin^2\theta\sin^2\phi,$$
$$Q=y_1l_1^{-\frac{1}{2}}+y_2(-l_2)^{-\frac{1}{2}}\sin\theta\cos\phi+y_3(-l_3)^{-\frac{1}{2}}\sin\theta\sin\phi, \quad (3.31)$$
$$R=y_1^2+y_2^2+y_3^2.$$

The integration with respect to r can be carried out immediately, giving

$$N+1=\frac{2e^{-\frac{1}{2}R}}{\pi\kappa^2\Delta^{\frac{1}{2}}}\int_0^{\pi/2}d\theta\int_0^{2\pi}d\phi\cos^3\theta\sin\theta$$
$$\times(3P^2+6Q^2P+Q^4)P^{-9/2}\exp(Q^2/2P). \quad (3.32)$$

4. ISOTROPIC CASE

In general, the integral (3.32) cannot be expressed in terms of known functions. We therefore specialize to the case when the surface is isotropic, that is to say, its statistical properties are independent of direction on the surface. In that case the spectrum $E(u,v)$ is a function of (u^2+v^2) only, and it has been shown[6] that

$$H=(1/16)D^2,\quad \Delta=(1/64)D^3. \quad (4.1)$$

The roots of (3.16) are then

$$l_1,l_2,l_3=\tfrac{1}{4}D,\ -\tfrac{1}{8}D,\ -\tfrac{1}{8}D. \quad (4.2)$$

Equations (3.19) then give

$$(a_{31}+a_{51})^2=D,$$
$$(a_{32}+a_{52})^2+(a_{33}+a_{53})^2=0, \quad (4.3)$$

whence it is clear that both squared terms in the second equation must be zero. So, from (3.24),

$$y_1,y_2,y_3=\kappa l_1^{-\frac{1}{2}},0,0, \quad (4.4)$$

and hence from (3.28)

$$P=4D^{-1}(1+2\sin^2\theta),$$
$$Q=4D^{-1}\kappa, \quad (4.5)$$
$$R=4D^{-1}\kappa^2.$$

We see then that the integrand in (3.32) is independent of ϕ, so that integration with respect to ϕ amounts to multiplying by 2π. On writing, for brevity,

$$D/(4\kappa^2)=A, \tag{4.6}$$

we find

$$N+1=\frac{4}{A}\int_0^{\pi/2} d\theta\cdot\cos^3\theta\sin\theta\exp\left[-\frac{\sin^2\theta}{A(1+2\sin^2\theta)}\right]$$
$$\times[3A^2(1+2\sin^2\theta)^2+6A(1+2\sin^2\theta)+1]$$
$$\times(1+2\sin^2\theta)^{-9/2}. \tag{4.7}$$

The substitution

$$1+2\sin^2\theta=s^{-2} \tag{4.8}$$

reduces this integral to the form

$$N+1=\frac{1}{A}\int_{1/\sqrt{3}}^{1}(3s^2-1)(3A^2+6As^2+s^4)$$
$$\times\exp[(s^2-1)/2A]ds, \tag{4.9}$$

which can easily be evaluated by integrating by parts. The result is

$$N+1=2+(2A/\surd 3)e^{-(\frac{1}{3}A^{-1})}, \tag{4.10}$$

so that

$$N=1+(2A/\surd 3)e^{-(\frac{1}{3}A^{-1})}. \tag{4.11}$$

The interpretation of this result is very similar to the two-dimensional case discussed in Sec. 3. The parameter A defined by (4.6) is proportional to κ^{-2}, that is to say, to the square of the distance of the observer from the surface, and also to D, which by (3.20) represents the average square of the "mean curvature"; for this is

$$\left\langle\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2\right\rangle_{\text{av}}=\langle(\xi_3+\xi_5)^2\rangle_{\text{av}}=\langle\xi_3^2\rangle_{\text{av}}$$
$$+2\langle\xi_3\xi_5\rangle_{\text{av}}+\langle\xi_5^2\rangle_{\text{av}}=m_{40}+2m_{22}+m_{04} \tag{4.12}$$

by (3.8). When the point of observation is very close to the surface (A is small), we find

$$N\sim 1 \tag{4.13}$$

as before, and at great distances

$$N\sim 2A/\surd 3, \tag{4.14}$$

that is to say, the number of points increases as the square of the distance from the surface. N is graphed against A in Fig. 2, curve (a).

5. TWO LONG-CRESTED SYSTEMS

Another special case for which a complete solution may be given is when the surface consists of two systems of long-crested waves intersecting at right angles.

If the axes of x and y are chosen to be parallel to the two systems, respectively, we have then

$$\zeta(x,y)=\zeta_1(x)+\zeta_2(y) \tag{5.1}$$

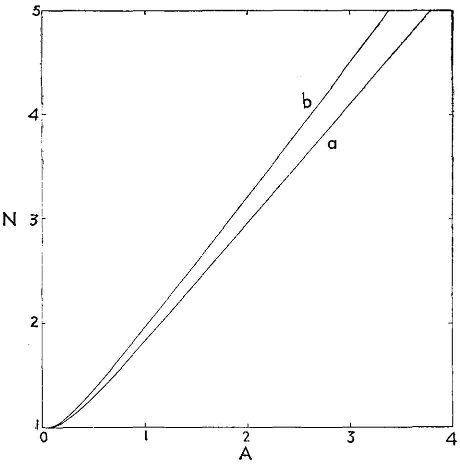

FIG. 2. The mean number of images as a function of A, defined by Eq. (4.6): (a) an isotropic surface, (b) a surface consisting of two long-crested systems at right angles.

and it is clear that the two conditions for a specular point [Eqs. (3.1)] are satisfied if and only if

$$\partial\zeta_1/\partial x=-\kappa x, \quad \partial\zeta_2/\partial y=-\kappa y. \tag{5.2}$$

Thus, specular points occur only when they would occur (in the two-dimensional sense) for each of the two long-crested systems simultaneously; whence it follows that the total number of specular points is the product of the number for the two systems individually:

$$N=N^{(1)}N^{(2)}, \tag{5.3}$$

where

$$N^{(i)}=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\alpha_i\exp[-(\tfrac{1}{2}\alpha_i^{-2})]+\int_0^{1/\alpha_i}\exp(-\tfrac{1}{2}z^2)dz\right] \tag{5.4}$$

and α_1, α_2 are the nondimensional parameters for the two systems.

The energy in the spectrum E must be regarded as being concentrated along the two axes of u, v, and negligible elsewhere. The moments m_{pq} of equation (3.9) are then zero whenever $pq\neq 0$, and the two parameters α_1, α_2 are given by

$$\alpha_1=m_{40}^{\frac{1}{2}}/\kappa, \quad \alpha_2=m_{04}^{\frac{1}{2}}/\kappa. \tag{5.5}$$

In the case when the two systems have equal mean-square curvature, i.e., $m_{40}=m_{04}$, then

$$\alpha_1=\alpha_2=(\tfrac{1}{2}D)^{\frac{1}{2}}/\kappa=(2A)^{\frac{1}{2}}, \tag{5.6}$$

where D and A are given by (3.19) and (4.6). Hence,

$$N=\frac{2}{\pi}\left[(2A)^{\frac{1}{2}}e^{-(\frac{1}{4}A^{-1})}+\int_0^{1/(2A)^{\frac{1}{2}}}\exp(-\tfrac{1}{2}z^2)dz\right]. \tag{5.7}$$

This function is shown in Fig. 2, curve (b), and it will be seen that the results are not very different from the isotropic case [curve (a)]. For large values of A, we find

$$N \sim 4A/\pi. \tag{5.8}$$

6. CASE $\kappa \to 0$

Further, it is possible to determine the behavior of N at great distances from the surface for quite general forms of the energy spectrum E. For when $\kappa \to 0$, we have, from (3.26),

$$N \sim \frac{2}{(2\pi)^{\frac{3}{2}}\kappa^2} \int\int\int [\sum_{j=1} l_j \eta_j^2] \times \exp[-\tfrac{1}{2} \sum_{j=1}^{3} \eta_j^2] d\eta_1 d\eta_2 d\eta_3. \tag{6.1}$$

The preceding integral has been evaluated previously (footnote reference 3, Sec. 2.4; the integral equals $2I'$). The result is

$$N \sim \frac{4}{\pi\kappa^2}(l_2 l_3)^{\frac{1}{2}} \left[\left(\frac{l_2 - l_1}{l_2}\right)^{\frac{1}{2}} E(k) - \left(\frac{l_2}{l_2 - l_1}\right)^{\frac{1}{2}} F(k) \right], \tag{6.2}$$

where

$$\begin{aligned} E(k) &= \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{\frac{1}{2}} d\phi \\ F(k) &= \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{-\frac{1}{2}} d\phi \end{aligned} \tag{6.3}$$

and

$$k^2 = l_1(l_3 - l_2)/l_3(l_1 - l_2). \tag{6.4}$$

This can also be written

$$N \sim (A/\pi)(l_1/D)\Phi(-l_2/l_1), \tag{6.5}$$

where

$$\Phi(\rho) \equiv \{\rho(1-\rho)\}^{\frac{1}{2}} \left[\left(\frac{1+\rho}{\rho}\right)^{\frac{1}{2}} E(k) - \left(\frac{\rho}{1+\rho}\right)^{\frac{1}{2}} F(k) \right], \tag{6.6}$$

$$\rho = -l_2/l_1, \quad k^2 = (1-2\rho)/(1-\rho^2).$$

The function Φ is plotted in Fig. 10 of footnote reference 3. It is a very slowly varying function and lies always between 0.917 and 1.

For example, when the surface consists of two equal long-crested systems of waves intersecting at an arbitrary angle θ_0, it can be shown (see footnote reference 6) that

$$H = \tfrac{1}{12} D^2 \sin^2\theta_0, \quad \Delta = 0, \tag{6.7}$$

and so from (3.16),

$$l_1, l_2, l_3 = \tfrac{1}{4} D \sin\theta_0,\ 0,\ -\tfrac{1}{4}\sin\theta_0. \tag{6.8}$$

It follows that $\rho = 0$ and $\Phi(\rho) = 1$, whence

$$N \sim (4A/\pi)\sin\theta_0. \tag{6.9}$$

When $\theta_0 = \pi/2$, i.e., the systems are perpendicular, we regain Eq. (5.8).

On the other hand, when the surface is isotropic, we see from (4.2) that $\rho = \frac{1}{2}$ and so $\Phi(\rho) = \pi/2\sqrt{3}$. Hence

$$N \sim D/2\sqrt{3}\kappa^2 = 2A/\sqrt{3} \tag{6.10}$$

in agreement with (4.14).

We may compare an isotropic surface with a surface consisting of two intersecting long-crested systems having the same mean-square curvature D. From Eqs. (6.9) and (6.10) we see that they will give equal numbers of image points provided the angle of intersection θ_0 is

$$\sin^{-1}(\pi/2\sqrt{3}) = 66°30'. \tag{6.11}$$

7. CONCLUSIONS

The average number of specular reflections seen by an observer at distance h from an isotropic Gaussian surface is given by Eq. (4.11), in which $A = \frac{1}{4}h^2 D$ and D denotes the mean-square curvature. This number increases from 1 at small distances to a value proportional to h^2 at great distances.

For two long-crested systems of waves intersecting at right angles, the number of images is given by Eq. (5.7). The two solutions are shown as functions of A in Fig. 2.

Finally a solution can always be found for large values of h; it is given by Eq. (6.5). In particular, when the surface consists of two long-crested systems of waves intersecting at an angle θ_0, the total number of images is given by Eq. (6.9).