JOURNAL OF THE OPTICAL SOCIETY OF AMERICA VOLUME 50, NUMBER 9 SEPTEMBER, 1960

Reflection and Refraction at a Random Moving Surface.
I. Pattern and Paths of Specular Points

M. S. LONGUET-HIGGINS
National Institute of Oceanography, England
(Received January 25, 1960)

Light falling from a point source on a ruffled surface produces a pattern of images, which move about over the surface. The image points correspond to the maxima, minima, and saddle points of a certain function. It is shown that the images are generally created in pairs, a maximum with a saddle point or a minimum with a saddle point, and that the total numbers of maxima, minima, and saddle points satisfy the relation

$$N_{ma}+N_{mi}=N_{sa}+1.$$

The process of creation or annihilation of images is studied in detail, and also the tracks of the image points, in certain special cases. It is shown that closed tracks may be common. This is confirmed by photography of the sea surface.

1. INTRODUCTION

When light from a fixed source falls upon a wave-like surface, such as the surface of a lake when ruffled by the wind, an observer may see a number of dancing images of the source reflected at different points in the surface; these points are sometimes called the "specular points."[1] Similarly, an observer beneath the surface would also see a number of moving images, depending upon the refractive index and the positions of the source and observer.

The number of images seen by the observer is not constant. The images move, and two specular points may come together and disappear or, on the other hand, two such points may suddenly appear where there were none before. Such an event, namely, the creation or the annihilation of two specular points may be called a "twinkle."[2]

It can be shown that at a "twinkle" the intensity of the image becomes exceptionally bright; the light is partially focused on the observer, so that the latter sees a bright flash. Correspondingly, if the reflected or refracted light is allowed to illuminate a fixed surface parallel to the mean wave surface, the intensity of illumination on the fixed surface fluctuates, and lines of especially bright illumination may be seen, for example, on the bottom of a shallow lake or sea. At the instant when one such line sweeps across a point Q in the plane, an observer at Q will see a "twinkle." A particular case of this phenomenon when the water surface is perfectly regular and sinusoidal and the source is at infinite distance has been considered by Shenck.[3]

It is well known, however, that water waves generated by wind are not perfectly regular but have a certain degree of randomness arising from the character of their origin. For example, the slopes of wind waves are known to have a statistical distribution which is approximately Gaussian.[1] It is sometimes convenient to assume that the water surface is the sum of an infinite number of long-crested waves of different wavelengths and directions, whose phases have been chosen at random from the interval $(0,2\pi)$; under suitable conditions, this leads to a Gaussian distribution of the elevation, slopes, and higher derivatives.[4]

The purpose of the present paper is to study the pattern of specular points in a random surface, to show how specular points may be added or subtracted at a "twinkle," and to examine the paths of specular points such as would be revealed by a time exposure of the surface. It is not here assumed that the surface is Gaussian but only that it has a certain degree of randomness so that special and unlikely cases (with probability zero) can be ignored. In subsequent papers the Gaussian assumption will be explicitly made, and the average numbers of specular points, as well as the mean number of twinkles per unit time, will be determined in terms of the spectrum of the surface.

First, in Sec. 2, we consider the pattern of specular points on the surface at a typical instant. Some of these points are "maxima," some are "minima," and some "saddle points." A simple relation between the number of each kind, namely,

$$N_{ma}+N_{mi}=N_{sa}+1 \tag{1.1}$$

is established.

It is then shown that as the surface moves, the specular points are generally created in pairs—a maximum with a saddle point or else a minimum with a saddle point. The way in which these fit into the previous pattern is also considered.

Ordinarily, a specular point, as it moves about on the surface, has a finite velocity; but we find that at the beginning and end of its life (that is to say when it is created or destroyed with another specular point), the

[1] C. Cox and W. Munk, J. Opt. Soc. Am. 44, 838 (1954).
[2] M. S. Longuet-Higgins, Proc. Cambridge Phil. Soc. 56, 234 (1956).
[3] H. Shenck, J. Opt. Soc. Am. 47, 653 (1957).
[4] M. S. Longuet-Higgins, Phil. Trans. Roy. Soc. London A247, 321 (1957).

velocity becomes infinite—in such a way, however, that the total distance traveled by the point is finite.

Typical tracks of specular points are considered in Sec. 6. When the surface consists of certain kinds of wave systems, it is shown that closed tracks will be common. A photograph of such tracks on the sea surface is reproduced in Fig. 9.

2. CONDITIONS AT A SPECULAR POINT

Let the equation of the surface in rectangular coordinates be

$$z=\zeta(x,y,t), \tag{2.1}$$

where the z axis is directed vertically upward. If the light source O and the point of observation Q are at $(0,0,h_1)$ and $(0,0,h_2)$, respectively (both above the surface), then the conditions for a point P at (x,y,ζ) to be a specular point are

$$\partial\zeta/\partial x=-\kappa x, \quad \partial\zeta/\partial y=-\kappa y, \tag{2.2}$$

where

$$\kappa=\tfrac{1}{2}[(1/h_1)+(1/h_2)], \tag{2.3}$$

it being supposed that $\kappa\zeta$ and $\partial\zeta/\partial x$, $\partial\zeta/\partial y$ are all small quantities. Similarly, if Q is situated at a distance h_2 below the surface and μ_1, μ_2 are the refractive indices for the two media above and below, we again have Eq. (2.2), but with

$$\kappa=(\mu_1 h_1+\mu_2 h_2)/(\mu_2-\mu_1)h_1 h_2. \tag{2.4}$$

It follows from (2.2) that the specular points correspond to the solutions of the equations

$$\partial f/\partial x=0, \quad \partial f/\partial y=0, \tag{2.5}$$

where

$$f(x,y,t)=\zeta(x,y,t)+\tfrac{1}{2}\kappa(x^2+y^2), \tag{2.6}$$

that is to say, they are the stationary points of the function f.

Let us first consider the surface as "frozen" at one particular time t, so that f is a function of x, y only. The form of the surface in the neighborhood of a specular point is well known. Shifting the origin of x, y to the point P and assuming that $\zeta(x,y)$ is differentiable up to the second order, we have

$$f(x,y)=\tfrac{1}{2}(a_{20}x^2+2a_{11}xy+a_{02}y^2)+R, \tag{2.7}$$

where R is a remainder of higher order than the second. We may write

$$\Omega_f=(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2)-(\partial^2 f/\partial x\partial y)^2 = a_{20}a_{02}-a_{11}^2 \tag{2.8}$$

for the discriminant of the quadratic form in (2.7); Ω_f also equals the "total" curvature of the surface $z=f$ at P. There are generally two distinct cases: either

(1) $\Omega_f>0$; the quadratic form in (2.7) is always of the same sign, and f has a maximum or a minimum according as $a_{20}\lessgtr 0$; the contours $f=\text{constant}$ are ellipses as in Figs. 1(a) and 1(c). Alternatively,

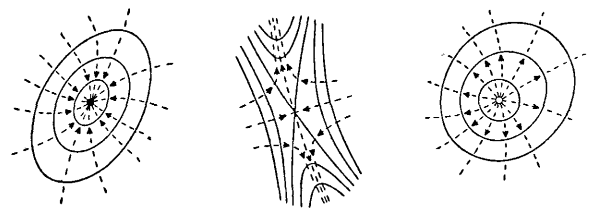

FIG. 1. The full lines indicate contours of $f(x,y,t)$ in the neighborhood of an ordinary specular point: (a) a maximum, (b) a saddle point, (c) a minimum. The broken lines and arrows indicate directions of steepest ascent.

(2) $\Omega_f<0$; the quadratic form is indefinite, and the contours of f are hyperbolic, as in Fig. 1(b).

Of special interest to us are the paths of steepest ascent on the surface; these are the orthogonal trajectories of the contour lines shown in Fig. 2. It is evident that in case (1) a path of steepest descent may either leave or enter P in any direction whatsoever, and there is a continuous family of such paths. In case (2) on the other hand, the orthogonal trajectories are rectangular hyperbolas with center P and so can never pass through P itself, with the exception of the hyperbola of zero "radius," that is, the line pair which forms the asymptotes of all the other paths. Thus, at a saddle point only two pairs of directions exist from which a path of steepest ascent may enter or leave the point, compared with a continuous family of directions for a maximum or minimum.

We have purposely not investigated the special case $\Omega_f=0$ at present, because if the surface is "frozen" the probability of such points occurring is nil; only when the surface is allowed to move, that is to say, it is given an extra degree of freedom, is there a finite probability that Ω_f will pass through zero in a given length of time.

3. PATTERN OF SPECULAR POINTS

It is sufficient, then, so long as the surface is "frozen," to suppose that the stationary points are either maxima, minima, or saddle points; any other cases have a total probability zero.

We now give a chain of reasoning which suggests that *all the minima on the surface may be joined by a network of paths so that each mesh contains one maximum and each segment contains one saddle point.*

Consider the form of $z=f(x,y)$ as the radius $r=(x^2+y^2)^{\frac{1}{2}}$ tends to infinity. In Eq. (2.6) the constant κ is positive. If we suppose that $\zeta(x,y)$ is Gaussian, so that the probability of large negative values is exponentially small, then it will follow that as $r\rightarrow\infty$, $f(x,y)$ almost always tends to infinity also.

Further, if the first and second derivatives of ζ are also Gaussian (and certainly if the slopes are bounded) we may expect that the paths of steepest ascent on the surface will, outside a circle of given radius r_0, all tend to infinity, except for a set of surfaces having probability

ϵ, where ϵ tends to 0 as $r_0 \to \infty$. We assume, therefore, that beyond a given radius all paths are directed outward to a "maximum at infinity," it being understood that we are neglecting a set of cases of vanishing total probability.

We shall also assume that there is only a finite number of stationary points at any time throughout the whole plane.

Starting from a typical point P on the surface (not a stationary point), let us follow the path of steepest ascent from P; this will climb until it reaches a stationary point or else goes to infinity. Generally, the path will not encounter a saddle point, since to each saddle point there are only two paths of steepest ascent; therefore, it will generally reach a maximum A (which may be the "maximum at infinity"). Moreover, if P' is a point in the neighborhood of P, the path of steepest ascent from P' will generally arrive at the same maximum A. Hence P lies in a continuous region, all points of which are connected to A by paths of steepest ascent. From Fig. 1(a), every maximum is surrounded by such a region.

In this way the whole plane, with the exceptions of the minima and of the paths passing through the saddle points, is divided into regions, one region for each maximum.

Let us now go to a typical minimum B and follow the line of steepest ascent, starting out from B in an arbitrary direction θ. This path, for the same reason, generally leads to a maximum $A(\theta)$. Moreover, all paths adjacent to the first path, that is starting in slightly different directions $\theta+d\theta$, generally arrive at A also.

Suppose now there exist two different directions θ_1 and θ_2 for which the paths of steepest ascent arrive at different maxima A_1 and A_2 [Fig. 2(a)]. By varying θ continuously from θ_1 to θ_2 we must encounter a direction θ_{13} for which the path bifurcates, one branch going to A_1 and one to A_3 (where A_3 may be the same as A_2). The point of bifurcation cannot be an ordinary point or a maximum or minimum; it must therefore be a saddle point C, say.

Now the path from B to C must form a part of the boundary of the region surrounding A_1 (for a slight variation of θ produces a path leading to A_1 on the one side or to A_3 on the other). Further, if the path BC is continued beyond the saddle point C and down the other side it must eventually reach a stationary point, which is either a minimum or another saddle point. A saddle point is ruled out, as being of vanishing probability. So in almost all cases the path ends in another minimum B', say.

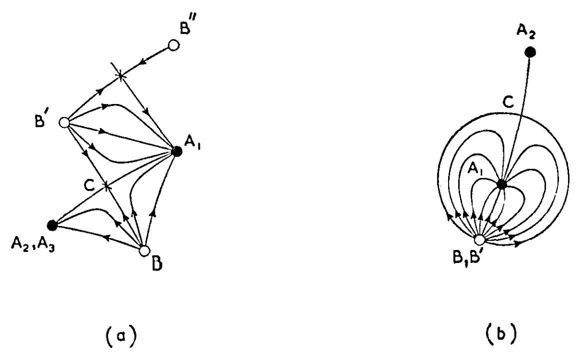

FIG. 2. Configurations of stationary points. (● = maximum, ○ = minimum, × = saddle point)

On continuing in this way round the maximum A_1 we have a succession of minima B, B', B'', and we eventually arrive back at B, having toured A_1 just once. It is quite possible for B' to coincide with B, as in Fig. 2(b).

Proceeding to the contiguous region which surrounds A_3, say, we may make a similar circuit. So eventually we fill up the whole plane with a network of paths, each mesh of the net containing just one maximum. The minima lie at the corners of the mesh, and along each segment between two adjacent minima there is one saddle point.

In fact, the network of minima may be considered as the Schlegel diagram of a polyhedron[5] in which the faces correspond to the maxima, the vertices correspond to the minima, and the edges correspond to the saddle points—with the difference, however, that it is allowable to have one "vertex" joined to the rest of the network by a single "edge," as in Fig. 2(b).

The dual network, formed by lines joining the *maxima*, and passing through the saddle points, is easily constructed.

Both the original network and its dual satisfy Euler's theorem[6,7]:

$$N_{\text{faces}}+N_{\text{vertices}}=N_{\text{edges}}+2 \tag{3.1}$$

(where N_{faces} denotes the number of faces, etc.). One "face" in the original network corresponds to the maximum at infinity. On omitting this, we have

$$N_{ma}+N_{mi}=N_{sa}+1, \tag{3.2}$$

where N_{ma}, N_{mi}, and N_{sa} denote the total numbers of maxima, minima, and saddle points, respectively.

It may be noted that the surface can be divided in another way, into regions where Ω_f is positive (elliptic regions) on the one hand and regions where Ω_f is negative (hyperbolic regions) on the other. The maxima and minima all lie in elliptic regions, and saddle points in hyperbolic regions. The boundaries between these, that is to say, the loci of points for which $\Omega_f=0$, are called the *parabolic lines*.

4. CONDITIONS AT A TWINKLE

From now on we shall allow the surface to be in motion, so that individual specular points move about

[5] H. S. M. Coxeter, *Regular Polytopes* (Methuen and Company, Ltd., London, 1948), p. 321.

[6] L. Euler, Nov. Comment. Acad. Sci. Imp. Petropol. 4, 109 (1752–1753).

[7] D. M. Y. Sommerville, *An Introduction to the Geometry of n Dimensions* (Methuen and Company, Ltd., London, 1929), p. 196.

on the surface. Let us follow one such point. Its coordinates are given by the conditions that

$$\partial f/\partial x=0, \qquad \partial f/\partial y=0, \tag{4.1}$$

and on taking the differential of these equations with respect to both x,y and t we have

$$\begin{aligned}(\partial^2 f/\partial x^2)dx+(\partial^2 f/\partial x dy)dy+(\partial^2 f/\partial x\partial t)dt=0,\\ (\partial^2 f/\partial x\partial y)dx+(\partial^2 f/\partial y^2)dy+(\partial^2 f/\partial y\partial t)dt=0.\end{aligned} \tag{4.2}$$

Equations (4.2) may be solved uniquely for the ratios dx/dt and dy/dt, provided that

$$\Omega_f, =(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2)-(\partial^2 f/\partial x dy)^2\neq 0. \tag{4.3}$$

In other words, after a short time dt, each specular point will move to a well-defined new position, provided that Ω_f is not zero. A necessary condition, therefore, for the creation or annihilation of a specular point (which we call a "twinkle") is the vanishing of Ω_f.

Let us shift the origin of coordinates to the position of the twinkle, at which time we also take $t=0$. If the surface is continuous and differentiable up to the third order, $f(x,y,t)$ may be expanded in the Taylor series

$$f(x,y,t)=\sum_{i,j,k=0}^{3}\frac{a_{ijk}}{i!j!k!}x^i y^j t^k+R, \tag{4.4}$$

where

$$a_{ijk}=(\partial^{i+j+k}f/\partial x^i\partial y^j\partial t^k)_{x=y=t=0} \tag{4.5}$$

and R is a higher-order remainder. Since the origin is at

$$a_{000}=0, \tag{4.6}$$

and the conditions (4.1) at $t=0$ give also

$$a_{100}=a_{010}=0. \tag{4.7}$$

Further, by a rotation of the axes of x,y we may make

$$a_{110}=0. \tag{4.8}$$

The condition that Ω_f shall vanish now gives

$$a_{200}a_{020}=0, \tag{4.9}$$

whence either a_{200} or a_{020} must vanish also. By naming the axes appropriately we make

$$a_{200}=0. \tag{4.10}$$

Lastly, the terms independent of x,y do not alter the form of the surface near P, except to raise or lower it bodily by a small amount. So without loss of generality, we assume

$$a_{001}=a_{002}=a_{003}=0. \tag{4.11}$$

The resulting expression for f in the neighborhood of the twinkle is

$$\begin{aligned}f(x,y,t)=\tfrac{1}{2}a_{020}y^2+\tfrac{1}{6}(a_{300}x^3+3a_{210}x^2y+3a_{120}xy^2+a_{030}y^2)\\ +(a_{101}x+a_{011}y)t+\tfrac{1}{2}(a_{201}x^2+2a_{111}xy+a_{021}y^2)t\\ +\tfrac{1}{2}(a_{102}x+a_{012}y)t^2+R.\end{aligned} \tag{4.12}$$

The coordinates of a specular point in the neighborhood are found by substituting this expression in (4.1), which gives

$$\begin{aligned}\tfrac{1}{2}(a_{300}x^2+2a_{210}xy+a_{120}y^2)+a_{101}t+\cdots=0,\\ a_{020}y+\tfrac{1}{2}(a_{210}x^2+2a_{120}xy+a_{030}y^2)+a_{011}t+\cdots=0,\end{aligned} \tag{4.13}$$

whence it is clear that x is of order $|t|^{\frac{1}{2}}$ and y of order t. In fact, on retaining only the lowest powers of t in each case, we have

$$x=\pm\left(\frac{-2a_{101}t}{a_{300}}\right)^{\frac{1}{2}}, \quad y=\frac{a_{210}a_{101}-a_{300}a_{011}}{a_{300}a_{020}}t. \tag{4.14}$$

The interpretation is interesting. If a_{100}/a_{300} is positive, two solutions exist when $t<0$ and none when $t>0$; hence, two specular points are simultaneously annihilated. If, on the other hand, a_{100}/a_{300} is negative, no solution exists for $t<0$ and two solutions exist for $t>0$; therefore, two specular points are simultaneously created.

The path of the points is found by eliminating t from (4.14):

$$(a_{210}a_{100}-a_{300}a_{010})x^2+2a_{100}a_{020}y=0, \tag{4.15}$$

which is a parabola with axis $y=0$. The velocity of the specular points near the vertex of the parabola is given by

$$\frac{dx}{dt}=\pm\left(\frac{-a_{101}}{2a_{300}t}\right)^{\frac{1}{2}}, \quad \frac{dy}{dt}=\frac{a_{210}a_{101}-a_{300}a_{011}}{a_{300}a_{020}}, \tag{4.16}$$

showing that the x component of velocity tends to infinity as $|t|\rightarrow 0$, as was expected.

Consider now the locus of "parabolic points," that is to say, points for which the total curvature, given by

$$\Omega_f, =(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2)-(\partial^2 f/\partial x\partial y)^2 \tag{4.17}$$

vanishes. Substitution from (4.12) gives this locus as

$$a_{300}x+a_{210}y+a_{201}t=0 \tag{4.18}$$

(terms of higher order being neglected). This is a straight line making an angle

$$\tan^{-1}(a_{300}/a_{210}) \tag{4.19}$$

with the path of the specular points, and passing within a distance of order t from the origin. But the x coordinates of the specular points are of order $t^{\frac{1}{2}}$. Hence, the two specular points lie generally on either side of the parabolic line $\Omega_f=0$.

Now the parabolic line is a boundary separating points for which $\Omega_f<0$ from those for which $\Omega_f>0$. It follows that one of the two specular points is a saddle point and the other is a maximum or a minimum.

In other words, specular points are generally created or annihilated in pairs; a maximum together with a saddle point or a minimum together with a saddle

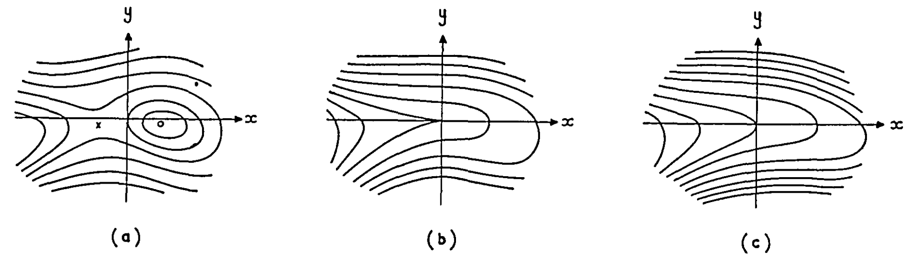

FIG. 3. Contours of the function $f(x,y,t)=\frac{1}{2}y^2+\frac{1}{6}(x^3+3x^2y)+xt$ when (a) $t=-0.01$; (b) $t=0$; (c) $t=0.01$.

point. It is evident that this process preserves the relation (3.2).

The form of the surface at the twinkle itself (time $t=0$) is found from (4.12):

$$f(x,y,0)=\tfrac{1}{2}a_{020}y^2+\tfrac{1}{6}(a_{300}x^3+3a_{210}x^2y+3a_{120}xy^2+a_{030}y^3)+\cdots. \tag{4.20}$$

By the linear transformation

$$\begin{aligned} x+(a_{210}/a_{300})y&=\xi \\ y&=\eta \end{aligned} \tag{4.21}$$

(a change to oblique axes), the equation becomes

$$f(x,y,0)=\tfrac{1}{6}a_{300}\xi^3+\tfrac{1}{2}a_{020}\eta^2(1+A\xi+B\eta)+\cdots, \tag{4.22}$$

where A and B are constants; or, since ξ and η are small near the origin,

$$f=\tfrac{1}{2}a_{020}\eta^2+\tfrac{1}{6}a_{300}\xi^3+\cdots. \tag{4.23}$$

The contour through the origin ($f=0$) is thus a semicubical parabola with a cusp lying to the left or right of the origin, according as a_{020}/a_{300} is positive or negative. The tangent at the cusp is the line $\eta=0$, i.e., the x axis.

The essential features of the surface before and after a twinkle are illustrated by the function

$$f(x,y,t)=\tfrac{1}{2}y^2+\tfrac{1}{6}(x^3+3x^2y)+xt, \tag{4.24}$$

whose contours are plotted in Fig. 3 for $t=-0.01$, 0, and 0.01. Two specular points—a minimum and a saddle point—are shown in the process of annihilation.

A geometrical interpretation may be given as follows. At each point on the surface there are two principal curvatures, κ_a and κ_b, say, and the total curvature Ω_f is the product of these. At a maximum or minimum, κ_a and κ_b are of the same sign, while at a saddle point they are of opposite signs. At a twinkle, when Ω_f vanishes, one of the principal curvatures also vanishes (in the foregoing example this curvature is in the x direction). That is to say, one of the principal sections of the surface has a point of inflexion. It is not difficult to see, by considering the corresponding two-dimensional problem[2] that, at a point of inflexion, two specular points must coincide and that their velocities become infinite.

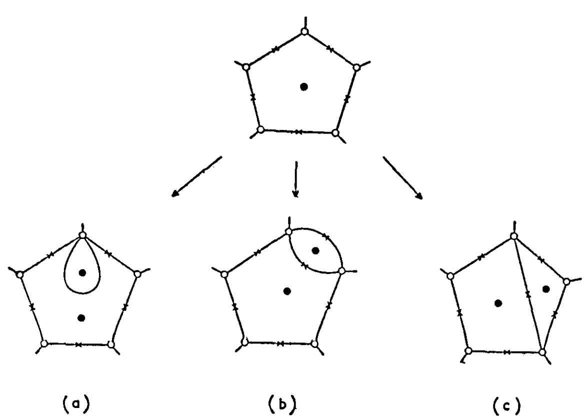

FIG. 4. Modifications of the pattern of stationary points by the addition of a maximum and a saddle point.

If the source of light is of small but still finite dimensions, each image on the surface covers a small area. It can be shown that as the two specular points approach each other, the images become elongated along their direction of travel (that is to say in the x direction). During this process the area of the image is greatly enlarged, so that an observer sees a bright flash.[8] However, the brighter the image, the faster it is moving, and it can be shown that the total intensity of light (integrated with respect to time) which is received from any small part of the track remains finite. Hence a time exposure of the whole track shows no particular increase in brightness at the twinkle itself.

In what has been said we have purposely ignored the possibility of such special cases as $a_{020}=0$ or $a_{300}=0$. These situations, besides being of zero probability, may be considered as coincidences of the kind of twinkle just described. For example, if $a_{020}=0$, then we have for the coordinates of the specular points the equations

$$\begin{aligned} \tfrac{1}{2}(a_{300}x^2+2a_{210}xy+a_{120}y^2)+a_{101}t+\ldots&=0 \\ \tfrac{1}{2}(a_{210}x^2+2a_{120}xy+a_{030}y^2)+a_{011}t+\ldots&=0, \end{aligned} \tag{4.25}$$

which represent two concentric conics. Generally, there are either four real intersections or none, giving four specular points in the neighborhood or none. If either conic is real when $t<0$ it will be imaginary when $t>0$. So we may distinguish the following cases: (1) both conics are simultaneously real and intersecting: then four specular points are simultaneously created or annihilated; (2) both conics are simultaneously real but nonintersecting: this gives an isolated flash at $t=0$; (3) one conic is real, the other imaginary: again there is an isolated flash at $t=0$. In case (1) the event can be regarded as the simultaneous creation of two

[8] At an ordinary point the total brightness is proportional to $|\Omega_f|^{-1}$, but when Ω^f vanishes this approximation breaks down.

pairs of specular points (or their simultaneous destruction). In cases (2) and (3) the event can be regarded as the simultaneous creation and annihilation of the same pair of specular points; their life ends as soon as it has begun.

5. CHANGING THE PATTERN OF SPECULAR POINTS

Let us now consider how two new specular points may be fitted into an already existing pattern.

We have seen that specular points are generally born in pairs at a parabolic line. Let us consider first the addition of a saddle point and a *maximum*.

The saddle point must lie on a path joining two minima. Since the minima are to be preserved, the only way to create a new path is to join up two already existing minima—these must therefore belong to the same mesh. The mesh being thus divided into two parts, a new maximum is created at the same time.

Three possible ways of dividing the mesh are illustrated in Figs. 4(a)–4(c). These ways correspond to the

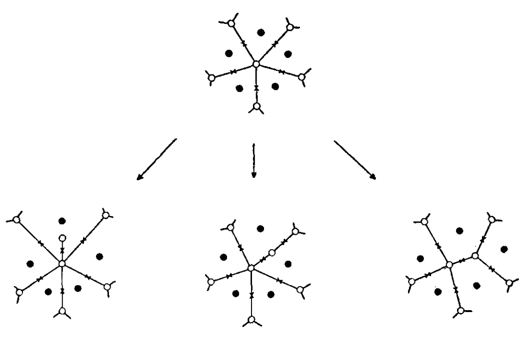

FIG. 5. Modifications of the pattern of stationary points by the addition of a minimum and a saddle point.

joining of one minimum to itself, to an adjacent minimum, or to one of the other minima of the same mesh.

The addition of a new *minimum* may be regarded in a precisely similar way but from the point of view of the dual network (see Sec. 3). Modifying the dual as in Figs. 4(a)–4(c) and then returning to the original we obtain the three types of division shown in Figs. 5(a)–5(c).

The destruction of two specular points consists of any such step in reverse.

Since a complete network may be built up from a single minimum or may be reduced to a single minimum by a combination of such steps, it follows that any pattern of specular points may be converted into any other by the steps described.

6. PATHS OF SPECULAR POINTS

If $z=\zeta(x,y,t)$ is a Gaussian surface, the tracks of the specular points are generally complicated. However, in

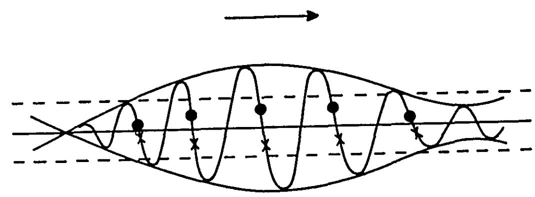

FIG. 6. The formation of specular lines on a moving waveform.

some special cases purely qualitative considerations may help in understanding certain features of the observed tracks.

Consider the special case when the surface consists of two systems of long-crested waves crossing at right angles. We have

$$\zeta(x,y,t)=\zeta_1(x,t)+\zeta_2(y,t), \tag{6.1}$$

and the conditions for a specular point reduce to

$$\partial\zeta_1/\partial x=-\kappa x, \quad \partial\zeta_2/\partial y=-\kappa y, \tag{6.2}$$

which is to say that a specular point in the combined system is the intersection of two *specular lines*, one from each of the long-crested systems individually.

Let us further suppose that each of the systems ζ_1 and ζ_2 consists of a fairly narrow band of wavelengths, and that the distances of the source and observer from the surface are great compared with the mean wavelength λ. Then the condition for a specular line in the system ζ_1 (say) is that the gradient $\partial\zeta_1/\partial x$ shall take the value $-\kappa x$, which value is almost constant over a few wavelengths.

Consider now a progressive train of waves in a dispersive medium such as water (Fig. 6). The envelope of such a wave train will move forward with the group velocity of the waves, and if, as in water, the phase velocity exceeds the group velocity,[9] the individual waves will grow at the rear of the group, move forward through the group and eventually die out at the front. At the instant when the wave amplitude rises through the value $\kappa|x|\lambda/2\pi$, two specular lines suddenly appear, and when the amplitude falls below this value, they disappear together. The specular lines are thus carried along through a distance comparable to the length

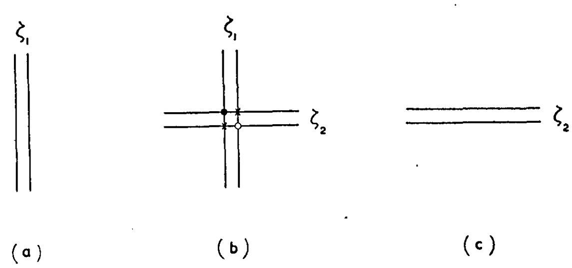

FIG. 7. The formation of specular points by two intersecting wave systems.

[9] This is for gravity waves. For surface-tension waves the reverse is true, but a similar argument applies.

of the group, which equals $n\lambda$, where n is the number of waves in the group.

Consider on the other hand a standing wave train. The wavelength is nearly uniform but the amplitude fluctuates rapidly, twice per complete cycle. Specular lines will appear (in pairs) and disappear again within half a cycle. The distance that they traverse is, by contrast with the previous case, only a fraction of λ.

Figure 7 illustrates the combined effect of the two intersecting systems. In Fig. 7(a) a pair of specular lines exists in system ζ_1 but not in system ζ_2; then (b) a pair appears also in the system ζ_2; this generates simultaneously two pairs of specular points (of which one pair is a maximum and a saddle point, the other a minimum and a saddle point). The pairs of points quickly separate in the y direction. Then either (c) the specular lines of ζ_1 vanish first or (a) the specular lines of ζ_2.

Typical tracks of the points are shown in Fig. 8. In Figs. 8(a) and 8(b), both systems ζ_1 and ζ_2 are progressive. In case (c), ζ_1 is a progressive wave but ζ_2 a standing wave; in case (d), both ζ_1 and ζ_2 are standing waves. The directions of movement are shown by arrows.

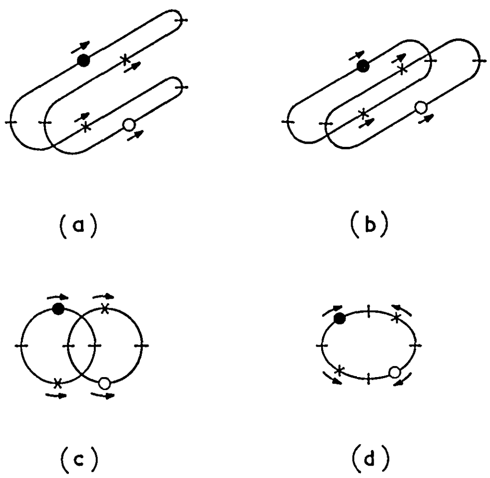

Fig. 8. Possible tracks of specular points (the arrows indicate directions of motion).

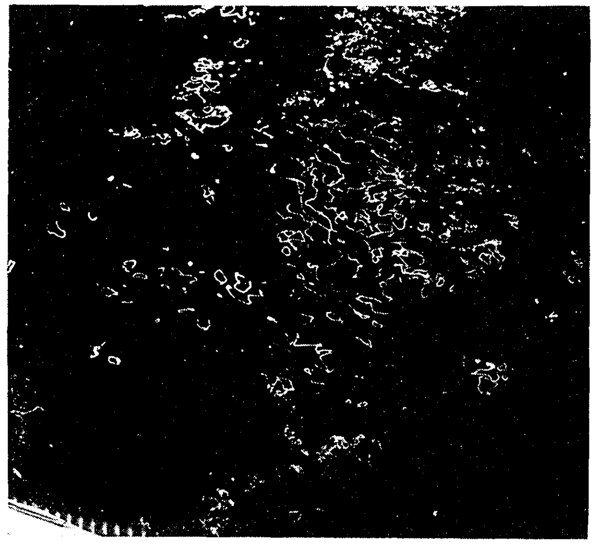

Fig. 9. A time exposure of the sea surface, showing tracks formed by images of the sun. The photograph was taken at midday, the camera being inclined at about 45° to the horizontal. (Triex XXX plate film was used, with a red filter.)

A time exposure (0.2 sec) of the pattern of sunlight reflected in the sea surface, taken a few feet above the water, is shown in Fig. 9. It seems from the photograph that the existence of closed tracks is quite common. Probably some waves were being reflected from the structure in the foreground, thus producing standing waves.

In Figs. 8(c) and 8(d) we saw that a closed track may correspond to two or four nearly simultaneous twinkles. Thus the closed tracks will enhance strongly the glittering appearance of the sea surface.

ACKNOWLEDGMENT

I am much indebted to Timothy Rhodes for a vacation on Beach Island, Maine, which first stimulated consideration of this problem. For the photograph in Fig. 9, I am indebted to Harlow G. Farmer of the Woods Hole Oceanographic Institution.