Longshore Currents Generated by Obliquely Incident Sea Waves, 2

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The profile of the longshore current, as a function of distance from the swash line, is calculated by using the concept of radiation stress (introduced in an earlier paper) together with a horizontal eddy viscosity μ_e of the form $\mu_e = \rho N x (gh)^{1/2}$, where ρ is the density, x is the distance offshore, g is gravity, h is the local mean depth, and N is a numerical constant. This assumption gives rise to a family of current profiles whose form depends only on the nondimensional parameter $P = (\pi/2)(sN/\alpha C)$, where s denotes the bottom slope, α is a constant characteristic of breaking waves ($\alpha \doteqdot 0.41$), and C is the drag coefficient on the bottom. The current profiles are of simple analytic form, having a maximum in the surf zone and tending to zero at the swash line. Comparison with the laboratory experiments of Galvin and Eagleson (1965) shows remarkably good agreement if the drag coefficient C is taken as 0.010. The theoretical profiles are insensitive to the exact value of P, but the experimental results suggest that P never exceeds a critical value of 2/5.

1. INTRODUCTION

In the companion paper (hereafter referred to as paper 1) a new theory for the generation of longshore currents by sea waves was developed; it is based on the concept of the radiation stresses associated with the incoming waves. The theory was found to be consistent with observed currents at the breaker line, in both model experiments and field observations, provided that the friction coefficient on the bottom was of order 0.010 and that the horizontal mixing length was of the same order, but less than, the distance between breaker line and shoreline.

To make further progress in predicting the longshore current, one must make some further detailed assumption about the horizontal mixing in the surf zone. This we propose to do by adopting a certain form for the coefficient μ_e of the horizontal eddy viscosity, as a function of distance from the shoreline.

It is fairly clear that μ_e must tend to zero as the shoreline is approached, since the dimensions of the turbulent eddies responsible for horizontal mixing can hardly be greater than the distance to the shoreline. For comparison, one can consider the analogous situation of turbulent flow over a rough plate, in which μ_e is proportional to height above the plate [e.g., *Prandtl,* 1952]. However, the present flow differs from flow over a plate in that horizontal driving forces (in the form of the gradient of the radiation stresses) are also present throughout the surf zone. Thus, although μ_e should tend to zero, it does not necessarily do so linearly.

In fact, we assume in the following that μ_e is proportional to the offshore distance x multiplied by a typical velocity $(gh)^{1/2}$, where h denotes the local depth. When the bottom slope s is uniform, this particular form for the eddy viscosity μ_e yields a very simple analytical form for the longshore current profile, which is found to be in remarkably good quantitative agreement with the detailed laboratory measurements by *Galvin and Eagleson* [1957]. In particular, the position and magnitude of the maximum current appear to be correctly predicted.

While this paper was in preparation, the author's attention was drawn to a then unpublished paper by *Bowen* [1969] in which the concept of radiation stress was also applied to the same problem. Bowen also takes into account both bottom friction and horizontal mixing, though in a somewhat different way. Although in general agreement with Bowen's approach we should like to point out two primary differences. The first is that he has assumed a bottom friction proportional to the longshore current v, whereas it was shown in section 5 of paper 1 that the bottom friction is proportional to uv, where u is the amplitude of the local orbital

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velocity (normal to the coastline). This is the form adopted in the present paper. Second, in Bowen's model the coefficient of horizontal eddy viscosity μ_e is taken to be a constant, not tending to zero at the shoreline. This apparently simpler assumption leads in fact to a more complicated analytical form for the velocity profile, which is at variance in some respects with the velocity profiles as measured by *Galvin and Eagleson* [1957]. It appears then that the present formulation of the theory is both more plausible on physical grounds and better in agreement with the observations now available.

2. Equations of Motion

We take axes $0x$, $0y$ normal and parallel to the coastline, with the origin 0 at the coastline (which may differ from the still-water line because of wave setup). The local mean depth $h(x)$ will be taken as including the change in level due to wave setup, or 'set down,' so that $h(0) = 0$ exactly.

If the longshore current v is steady and independent of y, then, as was shown in section 6 of paper 1, the momentum balance in the y direction can be expressed by the equation

$$0 = \tau_y + \frac{\partial}{\partial x}\left(\mu_e h \frac{\partial v}{\partial x}\right) - \langle B_y \rangle \qquad (1)$$

in which τ_y denotes the driving force due to the radiation stresses, which is given in shallow water by

$$\tau_y = \tfrac{5}{4}\alpha^2 \rho (gh)^{3/2} s \left(\frac{\sin\theta}{c}\right) \quad \text{or} \quad 0 \qquad (2)$$

as $x \lesseqgtr x_B$, the breaker distance (see section 4 of paper 1). In (2) α is a constant, about 0.41, ρ denotes the density, g is the acceleration of gravity, $s = dh/dx$ is the local depth gradient, θ is the local angle of incidence ($\theta^2 \ll 1$), and c is the local velocity of shallow-water waves where $c = (gh)^{1/2}$. By Snell's law $(\sin\theta)/c$ is a constant independent of x. Also in (1) the mean stress $\langle B_y \rangle$ on the bottom is given by

$$\langle B_y \rangle = \frac{2}{\pi}\alpha C \rho (gh)^{1/2} v \qquad (3)$$

where C is the drag coefficient on the bottom. The middle term on the right of (1) represents the effect of horizontal mixing. Now μ_e has the dimensions of ρLU, where L is a typical length scale and U is a typical velocity. Following the reasoning outlined in the introduction we take $L \propto x$ and $U \propto (gh)^{1/2}$, where h is the local depth. As the simplest possible assumption, we take

$$\mu_e = N\rho x (gh)^{1/2} \qquad (4)$$

where N is a dimensionless constant. Since L is not likely to exceed Kx, where K is von Kármán's constant, 0.40, and since the turbulent velocities are not likely to exceed $0.1u_{max}$ at most, where $u_{max} = \alpha(gh)^{1/2}$, the probable limits of N can be set as

$$0 < N < 0.016 \qquad (5)$$

(for $N\rho x(gh)^{1/2} = \mu_e = \rho LU \leq \rho(Kx)\ 0.1\ \alpha(gh)^{1/2}$).

We are particularly interested in a constant (or almost constant) beach gradient. We shall therefore suppose that

$$h = sx \qquad (6)$$

where $s = dh/dx$ is a constant that is nearly but not exactly equal to the bottom gradient m. Then (1) can be written in the form

$$p\frac{\partial}{\partial x}\left(x^{5/2}\frac{\partial v}{\partial x}\right) - qx^{1/2}v = \begin{cases} -rx^{3/2} & 0 < x < x_B \\ 0 & x_B < x < \infty \end{cases} \qquad (7)$$

where p, q, and r are constants, independent of x, given by

$$p = N\rho g^{1/2} s^{3/2}$$
$$q = \frac{2}{\pi}\alpha C \rho g^{1/2} s^{1/2} \qquad (8)$$
$$r = \tfrac{5}{4}\alpha^2 \rho g^{3/2} s^{5/2} \frac{\sin\theta_B}{(gh_B)^{1/2}}$$

In the expression for r the quantities θ_B and h_B signify the values of θ and h at the breaker line, but the values at any other particular location might also be chosen.

Now let us introduce the nondimensional variables

$$X = x/x_B \qquad V = v/v_0 \qquad (9)$$

where v_0 is the velocity defined by equation 55 of paper 1:

$$v_0 = \frac{5\pi}{8}\frac{\alpha}{C}(gh_B)^{1/2}s \sin \theta_B \qquad (10)$$

Then, noting that $h_B = sx_B$, we find that (7) reduces to the simple form

$$P \frac{\partial}{\partial X}\left(X^{5/2}\frac{\partial V}{\partial X}\right) - X^{1/2}V = \begin{cases} -X^{3/2} & 0 < X < 1 \\ 0 & 1 < X < \infty \end{cases} \qquad (11)$$

where

$$P = (\pi/2)(sN/\alpha C) \qquad (12)$$

Thus P is a nondimensional parameter representing the relative importance of the horizontal mixing.

If there is no horizontal mixing, $P = 0$ and we obtain the simple solution

$$V = \begin{cases} X & 0 < X < 1 \\ 0 & 1 < X < \infty \end{cases} \qquad (13)$$

noted in section 6 of paper 1. That is to say, the current increases linearly from the shoreline to the breaker line. Beyond the breaker line it is zero. At the breaker line itself the current velocity is discontinuous.

For general values of P equations 11 are to be solved subject to the boundary conditions that V is bounded when $0 < X < \infty$ and that at the breaker line $X = 1$ both V and $\partial V/\partial X$ are to be continuous. (It is not necessary for V to vanish at $X = 0$, but we shall see that in fact it does.)

A particular integral of equations (11) in the region $0 < X < 1$ is given by

$$V = AX \qquad 0 < X < 1 \qquad (14)$$

where

$$A = 1/(1 - \tfrac{5}{2}P) \qquad P \neq \tfrac{2}{5} \qquad (15)$$

To this we must add a complementary function satisfying, in both regions, the homogeneous equation

$$P \frac{\partial}{\partial X}\left(X^{5/2}\frac{\partial V}{\partial X}\right) - X^{1/2}V = 0 \qquad (16)$$

The above equation has a solution of the form

$$V = BX^p \qquad (17)$$

where B is a constant, provided that

$$P(p + 3/2)p - 1 = 0 \qquad (18)$$

In other words p must be a root of the quadratic equation

$$p^2 + 3/2p - 1/P = 0 \qquad (19)$$

Denoting these roots by p_1 and p_2, we have

$$p_1 = -\frac{3}{4} + \left(\frac{9}{16} + \frac{1}{P}\right)^{1/2} \qquad p_2 = -\frac{3}{4} - \left(\frac{9}{16} + \frac{1}{P}\right)^{1/2} \qquad (20)$$

Clearly $p_1 > 0$ and $p_2 < 0$. Hence the complete solution to (11) is of the form

$$V = \begin{cases} B_1X^{p_1} + AX & 0 < X < 1 \\ B_2X^{p_2} & 1 < X < \infty \end{cases} \qquad (21)$$

The boundary conditions at $X = 1$ are then satisfied by taking

$$B_1 = \frac{p_2 - 1}{p_1 - p_2}A \qquad B_2 = \frac{p_1 - 1}{p_1 - p_2}A \qquad (22)$$

It is useful to note that from (19)

$$p_1 + p_2 = -3/2 \qquad p_1p_2 = -1/P \qquad (23)$$

and so

$$(p_1 - 1)(p_2 - 1) = p_1p_2 - (p_1 + p_2) + 1 = \frac{5}{2} - \frac{1}{P} = \frac{-1}{AP} \qquad (24)$$

Then we have also

$$B_1 = [P(1 - p_1)(p_1 - p_2)]^{-1} \qquad B_2 = [P(1 - p_2)(p_1 - p_2)]^{-1} \qquad (25)$$

Equation 21, together with (22) or (25), represents the solution to the problem, for general values of P.

For $P = 2/5$ the particular integral (14) no longer applies. Instead we have a different particular integral

$$V = -\tfrac{5}{7}X \ln X \qquad 0 < X < 1 \qquad (26)$$

Since $p_1 = 1$ and $p_2 = -5/2$ for $P = 2/5$, we obtain, as before,

$$V = \begin{cases} \tfrac{10}{49}X - \tfrac{5}{7}X \ln X & 0 < X < 1 \\ \tfrac{10}{49}X^{-5/2} & 1 < X < \infty \end{cases} \qquad (27)$$

The constants multiplying X and $X^{-5/2}$ are chosen to satisfy the continuity of V and $\partial V/\partial X$ at $X = 1$. Equation 27 also represents the limit of the solution (21) when $P \rightarrow 2/5$.

3. Discussion

The current profiles given by equation 20 have been calculated and plotted in Figure 1 for various values of the horizontal mixing parameter P. These current profiles have the following properties.

1. *Velocity near the breaker line.* As $P \rightarrow 0$, the profile tends to the triangular form (13) appropriate to zero mixing. There is a single maximum velocity $V_{max} \rightarrow 1$ just to the left of the breaker line. To the right of the breaker line we have $V \rightarrow 0$.

2. *Velocity at the breaker line.* When $X = 1$ we have from (21) and (25)

$$V_B = [P(1 - p_2)(p_1 - p_2)]^{-1} \qquad (28)$$

On using the values of p_1 and p_2 given by (20) we find that in the limit, as $P \rightarrow 0$, $V_B \rightarrow 0.5$. In other words, the velocity at the breaker line is the mean of the limiting velocities on either side. This was foreshadowed in paper 1, section 6. Now as P increases from zero to infinity, V_B decreases monotonically from 0.5 to 0. At large values of P we find that asymptotically

$$V_B \sim 4/15P \qquad \text{as} \quad P \rightarrow \infty \qquad (29)$$

Values of V_B for some representative values of P are given in Table 1. Also, when $P = 2/5$, we have

$$V_B = 10/49 = 0.2041 \qquad (30)$$

3. *Maximum velocity.* The velocity profile generally has a single maximum value V_{max} lying within the surf zone ($0 < X < 1$). To find the position X_m of this maximum, we differentiate (21) and obtain

$$0 = B_1 p_1 X_m^{p_1 - 1} + A \qquad (31)$$

Therefore from (22)

$$X_m = \left[\frac{p_1 - p_2}{p_1(1 - p_2)}\right]^{1/(p_1 - 1)} \qquad (32)$$

From (21) and (31) the corresponding velocity is given by

$$V_{max} = \left(1 - \frac{1}{p_1}\right) A X_m \qquad (33)$$

Using the values of p_1 and p_2 given by (20), we can show that, as $P \rightarrow 0$, $X_m \rightarrow 1$ and $V_{max} \rightarrow 1$, and, as $P \rightarrow \infty$, both X_m and V_{max} tend to

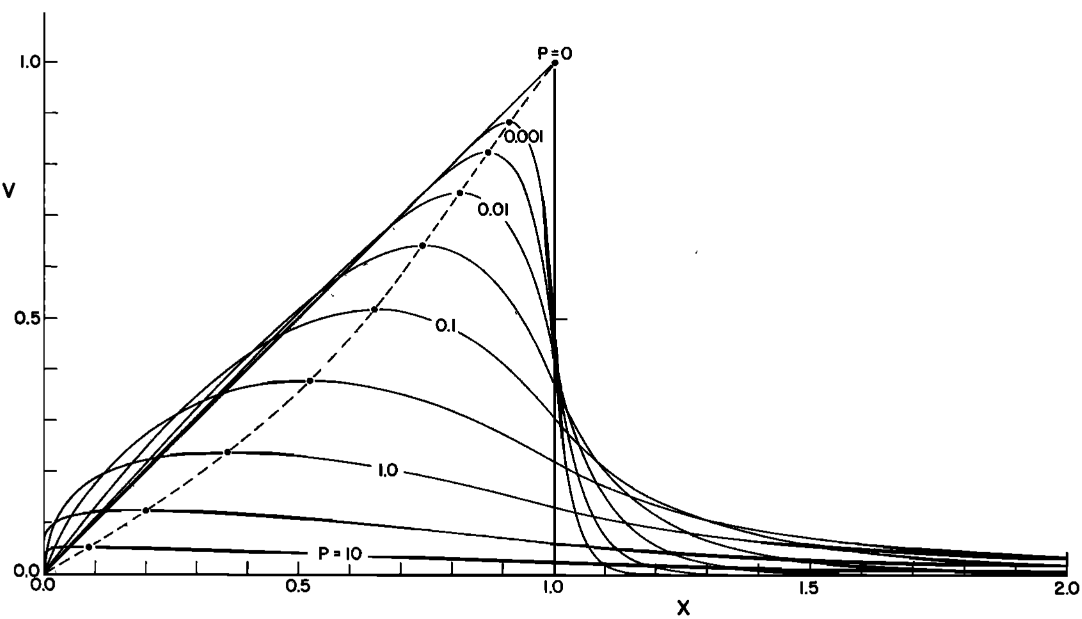

Fig. 1. The form of the current profiles as given by (21) for a sequence of values of the mixing parameter P.

TABLE 1. Parameters of the Velocity Profile (21) for Various Values of P

$\log_{10}P$	V_B	$\langle V \rangle$	V_{max}	X_m	Q_1	Q_2	Q
$-\infty$	0.5000	0.5000	1.0000	1.0000	1.0000	0.0000	1.0000
−3.0	0.4735	0.4847	0.8835	0.9108	0.9542	0.0467	1.0000
−2.5	0.4544	0.4733	0.8254	0.8699	0.9208	0.0824	1.0032
−2.0	0.4233	0.4542	0.7456	0.8148	0.8654	0.1447	1.0101
−1.5	0.3754	0.4230	0.6422	0.7422	0.7780	0.2546	1.0327
−1.0	0.3077	0.3736	0.5173	0.6466	0.6496	0.4615	1.1111
−0.5	0.2226	0.2992	0.3786	0.5198	0.4803	0.9822	1.4625
0.0	0.1333	0.2000	0.2400	0.3600	0.2933	∞	∞
0.5	0.0628	0.1024	0.1246	0.1984	0.1399	∞	∞
1.0	0.0240	0.0408	0.0524	0.0861	0.0537	∞	∞
1.5	0.0081	0.0141	0.0191	0.0316	0.0183	∞	∞
2.0	0.0026	0.0046	0.0064	0.0107	0.0059	∞	∞

0 in such a way that

$$V_{max} \sim \tfrac{3}{5} X_m \tag{34}$$

Thus X_m covers the entire range of X values between 0 and 1. For $P = 2/5$ we find from (27)

$$X_m = e^{-5/7} = 0.4895 \qquad V_{max} = \tfrac{35}{49} X_m = 0.3496 \tag{35}$$

The values of V_{max} and X_m corresponding to some representative values of P are shown in Table 1. It appears that, as P increases from 0 to ∞, both V_{max} and X_m decrease steadily from 1 to 0.

Interpreting this result physically, we can say that the effect of increasing the horizontal mixing is to redistribute the momentum so that the fluid near the shoreline is dragged along at a faster speed by the fluid farther offshore, but farther offshore the fluid is slowed down by the mass beyond the breaker line.

4. *Gradient of velocity profile at the shoreline.* As $X \rightarrow 0$, we see from (21) that

$$\partial V/\partial X \sim B_1 p_1 X^{p_1 - 1} + A \qquad P \neq 2/5 \tag{36}$$

So long as $p_1 > 1$, the horizontal velocity gradient remains finite and equal to A. However, when $p_1 < 1$ the gradient at $X = 0$ becomes infinite. The critical case $p_1 = 1$ corresponds to $P = 2/5 = 0.4$. Thus we have

$$\lim_{X \to 0} \frac{\partial V}{\partial X} = \begin{cases} 1/(1 - \frac{5}{2}P) & 0 \leq P < 2/5 \\ \infty & 2/5 \leq P < \infty \end{cases} \tag{37}$$

5. *Total transport.* In the longshore direction the total transport can easily be found by integration of vh with respect to the offshore distance x. Without horizontal mixing the total transport Q_0 was shown in paper 1, section 6 to be given by

$$Q_0 = \tfrac{1}{3} v_0 h_B x_B \tag{38}$$

where v_0 is given by (10). This follows from the fact that vh is proportional to x^2. When $P > 0$, the transport within the surf zone is given by

$$Q_1 = v_1 h_B x_B \int_0^1 VX\, dX = \left(\frac{3B_1}{2 + p_1} + A\right) Q_0 \tag{39}$$

On the other hand, the transport beyond the surf zone is given by

$$Q_2 = v_0 h_B x_B \int_1^\infty VX\, dX = \begin{cases} \dfrac{-3B_2}{2 + p_2} Q_0 & p_2 < -2 \\ \infty & p_2 \geq -2 \end{cases} \tag{40}$$

For $p_2 < -2$, which corresponds to $P < 1$, the total transport Q, which equals $(Q_1 + Q_2)$, is given by

$$Q = \left(\frac{3B_1}{2 + p_1} - \frac{3B_2}{2 - p_2} + A\right) Q_0 \tag{41}$$

which after some reduction becomes simply

$$Q = Q_0/(1 - P) \tag{42}$$

When $P \geq 1$, the transport outside the surf zone (and hence the total transport) becomes infinite. This must mean that a steady state cannot be established in a finite time, over the whole field. However, the flow within any finite distance of the shoreline can still be established effectively within a finite time.

6. *Mean current* $\langle V \rangle$. The mean current in the surf zone, given by

$$\langle V \rangle = \int_0^1 V\, dX = \begin{cases} \dfrac{B_1}{p_1 + 1} + \dfrac{A}{2} & P \neq \frac{2}{5} \\ 55/196 & P = \frac{2}{5} \end{cases} \tag{43}$$

is plotted in Figure 2 with V_B and V_{max} as functions of the mixing parameter P. Each is a monotonically decreasing function of P. The corresponding ratios $V_B/\langle V \rangle$, V_B/V_{max}, and $\langle V \rangle / V_{max}$ (Figure 3) can be seen to vary between somewhat narrower limits. In particular, V_B/V_{max} lies always between 0.4 and 0.6.

The most remarkable feature of Figure 1 is that, even when the mixing parameter P varies by a factor of three orders of magnitude (from 0.001 to 1.0), the corresponding value of the velocity V_{max} changes by a factor of less than 4. This is in striking contrast to the dependence of the velocity on the drag coefficient C on the bottom. Since v_0 is inversely proportional to C (but P depends also on C), we see that v itself is nearly inversely proportional to C.

4. Comparison with Observation

The most careful laboratory studies of longshore currents along a plane beach appear to be those of *Galvin and Eagleson* [1965]. Their model beach was 22 feet wide and had a gradient of about 0.11. Some care must be taken, even with these experiments, in comparing theory and observation since, as the authors themselves emphasize, the measured currents were not uniform along the beach but were being accelerated downstream from one end of the beach. This effect is probably present, but unrecognized, in many other laboratory measurements. One result of the acceleration must be to entrain fluid from beyond the surf zone into the surf zone itself, which may have an effect similar to a horizontal exchange of momentum by eddy viscosity.

The parameters for Galvin and Eagleson's experiments are summarized in Table 2. Their measurements for which the angle of incidence differed from zero were in series II, III, and IV, the deep-water angles of incidence for these series being 10°, 20°, and 51°, respectively. In

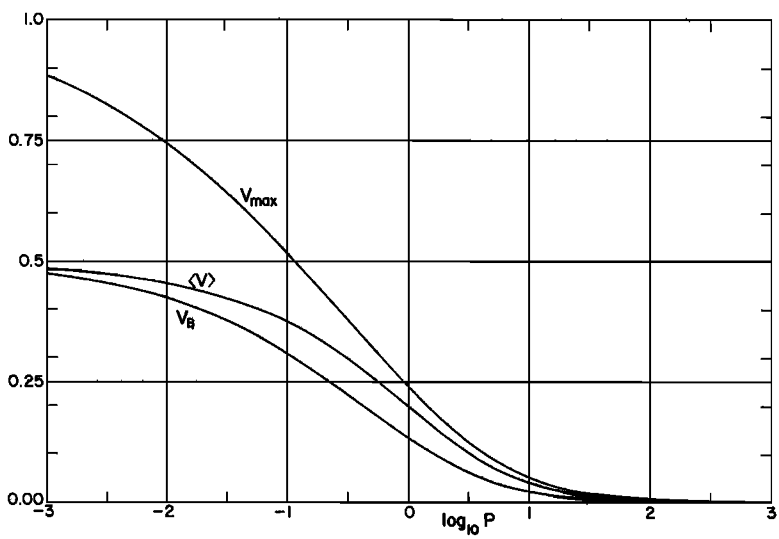

Fig. 2. The theoretical values of V_B (the velocity at the breaker line), V_{max} (the maximum velocity) and $\langle V \rangle$ (the mean velocity in the surf zone $0 < X < 1$) as functions of the mixing parameter P.

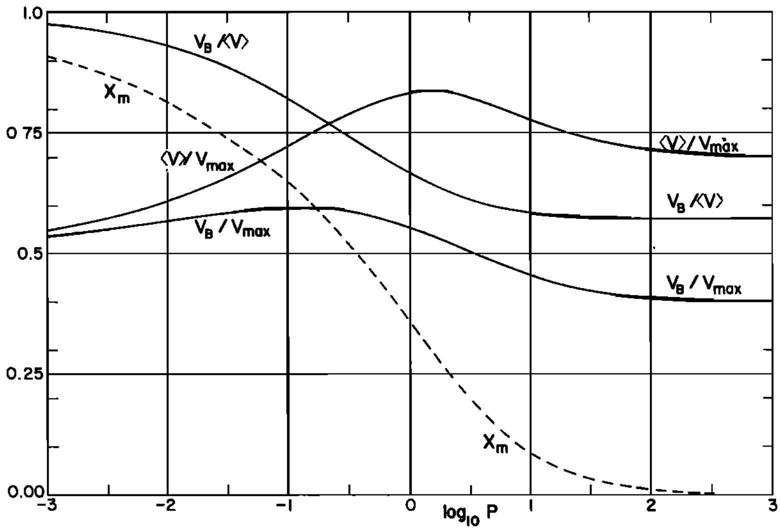

Fig. 3. The ratios $V_B/\langle V\rangle$, V_B/V_{max}, and $\langle V\rangle/V_{max}$ as functions of the mixing parameter P (full lines). The dashed line gives the coordinate X_m of the position of maximum velocity.

runs 2 to 6 of each series the wave period was varied, and in runs 7 to 11 the period was kept constant and the amplitude was varied.

The measurements thought most likely to represent steady unaccelerated conditions were those at a distance of 15 feet, or about ⅔ the width of the beach, from the upstream end. For each velocity profile we have assumed as origin of X (in our notation) the mean position of the swash zone, taken at a distance $(r - W/2)$ from the still water line. Here r, as in *Galvin and Eagleson* [1965], denotes the runup distance and W denotes the width of the swash zone. No measurements being available, we have assumed W to be given closely enough by $\frac{1}{8}sgT^2$, which is the distance through which a particle would slide down the beach under gravity in a time equal to half the wave period T. The distance b of the breaker line offshore is tabulated by Galvin and Eagleson, so that altogether we have

$$X = \frac{\xi + r - W/2}{b + r - W/2} \tag{44}$$

where ξ is the horizontal distance offshore from the still water line ($\xi = Y$, in the notation of Galvin and Eagleson).

To normalize the measured velocity v, we define v_0 by (10), in which h_B is assumed to be very nearly equal to sb (if allowance is made for wave setup, the appropriate value of s is reduced by about 10%; see appendix 1); θ_B is the measured angle of incidence at the breaker line (see Table 2). In series IV, however, the present approximate theory in which θ_B^2 is negligible cannot very well be justified. Accordingly a rough correction has been made in Figure 4 by replacing the value of v_0 in Table 2 by $v_0 \cos \theta_0$, where θ_0 is the angle of incidence in deep water. Though not rigorous, this correction factor can be justified on the grounds that the total longshore force exerted by the waves is equal to $\frac{1}{2} E_0 \cos \theta_0 \sin \theta_0$, where E_0 is the wave energy density in deep water (see section 3 of paper 1). Since the total longshore thrust is proportional to $\cos \theta_0$, we expect the longshore current v will, on average, be reduced by approximately this amount.

In the experiments, the parameters r, b, and θ_B were found to fluctuate to some extent both with time and with distance along the shore. However, rather than use the local values of these quantities at the position of the current profile, we have adopted the longshore averages r_{av}, b_{av}, and $(\theta_B)_{av}$ given in Table A3 of *Galvin and Eagleson* [1965]. The purpose of doing so is to help reduce the statistical variability. In addition, we note that the longshore currents are, in fact, affected not only by the local values of h_B and θ_B but also by conditions along the

TABLE 2. Parameters for the Model Experiments of *Galvin and Eagleson* [1965]

Run	θ_B, deg	H_B, feet	T, sec	r_{av}, feet	b_{av}, feet	$W/2$, feet	v_0, ft/sec	V_{max}	$\log_{10}P$
Series II									
2	5.4	0.191	1.000	1.01	1.62	0.22	2.08	0.25	−0.04
3	5.1	0.167	1.125	1.06	1.53	0.28	1.84	0.35	−0.40
4	3.3	0.143	1.250	1.07	1.33	0.34	1.12	0.48	−0.86
5	2.3	0.121	1.375	1.15	1.24	0.42	0.74	0.69	−1.72
6	3.7	0.105	1.500	1.04	1.17	0.50	1.18	0.35	−0.40
7	2.6	0.050	1.250	0.62	0.62	0.34	0.59	0.40	−0.58
8	3.1	0.098	1.250	0.77	0.87	0.34	0.83	0.35	−0.40
9	3.8	0.124	1.250	1.03	1.21	0.34	1.21	0.41	−0.62
10	3.7	0.130	1.250	1.03	1.07	0.34	1.12	0.46	−0.80
11	4.0	0.156	1.250	1.08	1.44	0.34	1.40	0.46	−0.80
Series III									
2	14.1	0.191	1.000	0.95	1.52	0.22	5.01	0.39	−0.55
3	12.1	0.167	1.125	1.05	1.51	0.28	4.30	0.42	−0.65
4	10.1	0.143	1.250	1.05	1.44	0.34	3.48	0.56	−1.17
5	9.2	0.121	1.375	1.05	1.13	0.42	2.80	0.61	−1.37
6	6.9	0.105	1.500	1.06	1.04	0.50	2.04	0.70	−1.77
8	6.6	0.098	1.250	0.75	0.85	0.34	1.76	0.66	−1.58
9	8.7	0.124	1.250	0.89	1.11	0.34	2.64	0.60	−1.33
11	11.2	0.156	1.250	1.10	1.55	0.34	4.01	0.49	−0.90
Series IV									
2	28.0	0.191	1.000	0.84	1.40	0.22	9.29	0.37	−0.47
3	21.9	0.167	1.125	0.88	1.15	0.28	6.74	0.44	−0.72
4	18.6	0.143	1.250	1.02	1.22	0.34	5.87	0.53	−1.05
5	15.8	0.121	1.375	1.17	1.32	0.42	5.04	0.55	−1.13
6	8.6	0.105	1.500	0.90	0.91	0.50	2.37	0.55	−1.13
7	13.4	0.050	1.250	0.59	0.69	0.34	3.18	0.39	−0.55
8	14.3	0.098	1.250	0.79	0.83	0.34	3.71	0.48	−0.86
9	19.7	0.124	1.250	0.86	1.19	0.34	6.13	0.39	−0.55
10	19.7	0.130	1.250	0.97	1.27	0.34	6.34	0.47	−0.83
11	22.6	0.156	1.250	1.12	1.29	0.34	7.26	0.46	−0.80
12	6.0	0.062	1.500	0.54	0.57	0.50	1.31	0.57	−1.21
13	20.2	0.110	1.000	0.72	0.88	0.22	5.37	0.46	−0.80

whole length of the beach. (There was some uncertainty in the measurement of θ_B [see *Galvin and Eagleson*, 1965] and hence in the appropriate values of v_0 and V_{max}.)

The results of the comparison are shown in Figure 4. Each plot in the diagram is identified by the number of the corresponding run in Table 2. It will be seen that most of the points lie between two of the theoretical curves derived in section 2, namely the curves corresponding to $P = 0.4$ and $P = 0.1$. Individual profiles (such as that numbered 11 in Figure 4*a* tend to follow quite closely the predicted profile for some particular value of P; the maximum velocity always lies not far from the dotted curve, which represents the locus of maximums in Figure 1.

The values of V_{max} corresponding to each of the profiles in Figure 4 are shown in Table 2. Also shown are the corresponding values of P derived from Figure 2. It can be seen that P varies from 0.40 to about 0.01.

How does P vary with other parameters: the wave height and period, the angle of incidence, and the beach slope? Before a definite answer can be given, further experiments covering a wider range of conditions are necessary. There is some evidence from Table 2 that P is an increasing function of the wave frequency ($2\pi/T$) and also of the breaker height in deep water (H_D). The value of P can also increase with the angle of incidence θ_B.

It is striking that few profiles correspond to values of P greater than the critical value of 0.4 (see section 3). At this value of P the gra-

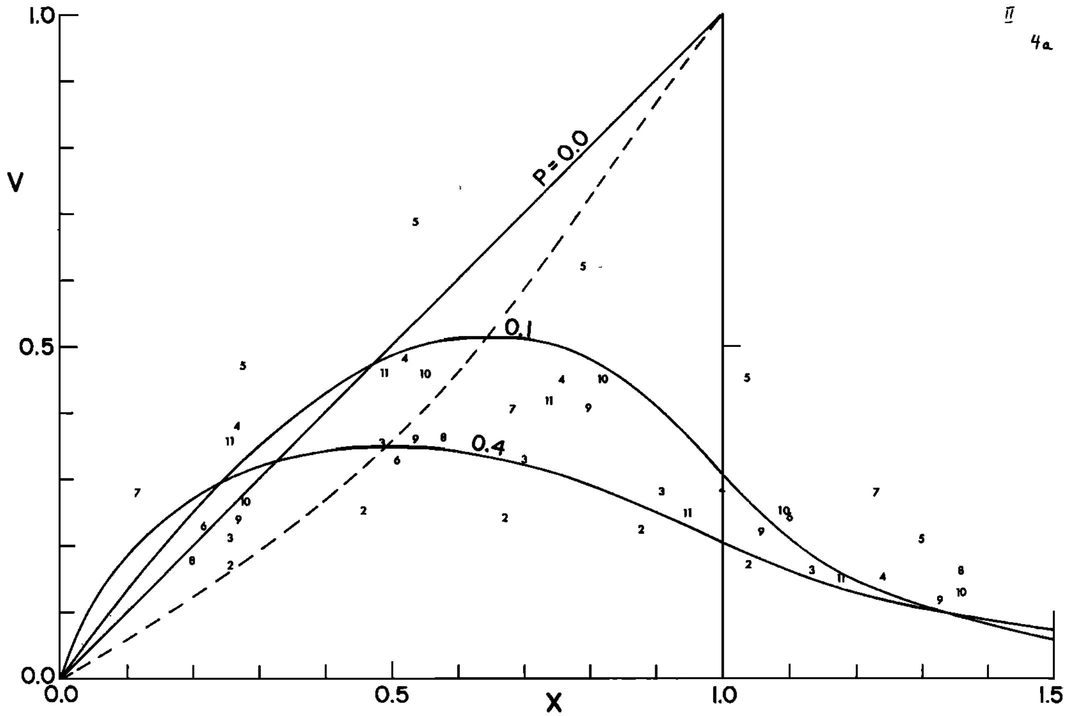

Fig. 4*a*. Comparison of the longshore velocities measured by *Galvin and Eagleson* [1965; series II] with the theoretical profiles derived in section 2 of the present paper. The plotted numbers correspond to the number of each run (Table 2).

dient of the current at $X = 0$ becomes infinite. One is tempted to conjecture that this is in fact the greatest possible value of P, and that the presence of the shoreline controls the horizontal mixing so that the value of 0.40 is not exceeded.

In support of this conjecture we can note that the corresponding value of V_B is about 0.20, or one-fifth, whereas a rough argument in section 6 of paper 1 showed that V_B should never be less than about one-sixth. As shown in paper 1, both laboratory measurements and field observations seem to point to 0.2 as being a common value for V_B.

Appendix 1. An Allowance for Wave Setup

It is well known [*Saville*, 1961] that waves approaching a beach cause a change in the mean water level, or wave setup, in the shelving zone. The effect is caused by the onshore component of radiation stress [*Longuet-Higgins and Stewart*, 1962, 1963]. Outside the breaker zone, the wave setup η is slightly negative, but inside the surf zone, the region with which we are concerned here, $d\eta/dx$ and hence η become appreciably positive. Thus the local depth $h(x)$ should, in practice, be replaced by the effective depth $(h + \eta)$, and the bottom slope s should be replaced by

$$s^* = -\frac{d}{dx}(h + \eta) \qquad \text{(A1)}$$

Now it has been shown both theoretically and experimentally by *Bowen et al.* [1968] that within the surf zone

$$\frac{d\eta}{dx} = -\frac{1}{1 + (2/3\alpha^2)}\frac{dh}{dx} \qquad \text{(A2)}$$

(see equation 12 and Figure 5 of their paper, with $y = 2\alpha$). Thus, if α is constant, the gradient of the mean surface level is simply proportional to the gradient of the beach. From (A2) it follows that

$$\frac{dh}{dx} + \frac{d\eta}{dx} = \frac{1}{1 + 3\alpha^2/2}\frac{dh}{dx} \qquad \text{(A3)}$$

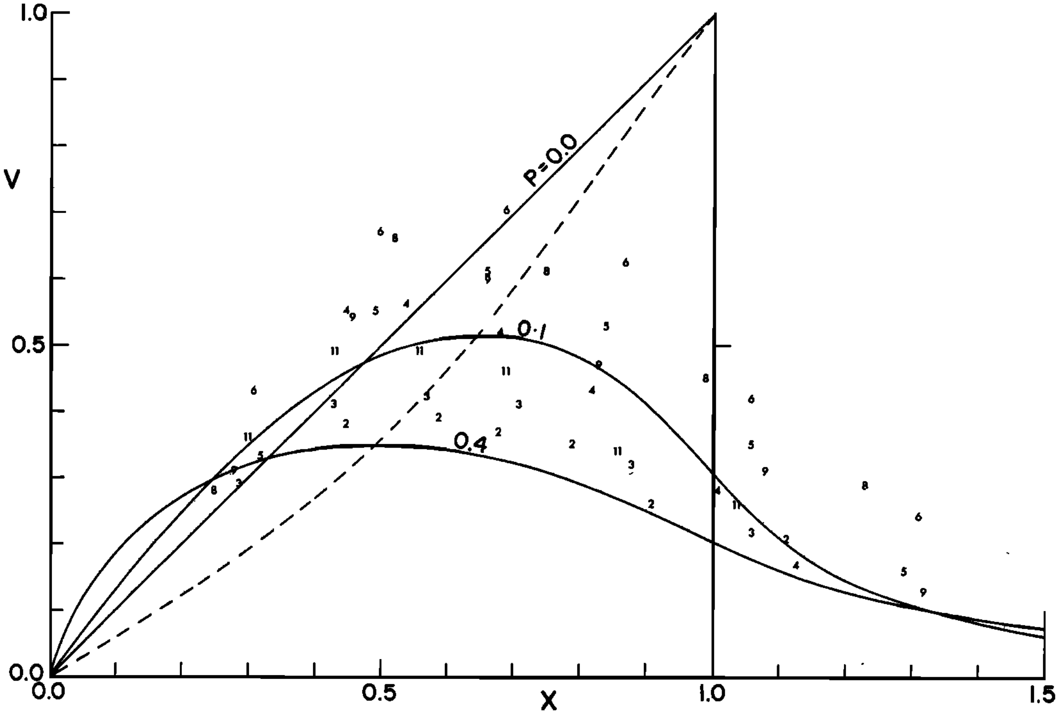

Fig. 4b. Comparison of the longshore velocities measured by *Galvin and Eagleson* [1965; series III] with the theoretical profiles derived in section 2 of the present paper. The plotted numbers correspond to the number of each run (Table 2).

and hence

$$s^* = \frac{s}{1 + 3\alpha^2/2} \tag{A4}$$

Equation 10 can therefore be modified to read

$$v_0 = \frac{5\pi}{8}\frac{\alpha^* s}{C}(gh_B)^{1/2} \sin \theta_B \tag{A5}$$

where

$$\alpha^* = \frac{\alpha s^*}{s} = \frac{\alpha}{1 + 3\alpha^2/2} \tag{A6}$$

A graph of α^* as a function of α is shown in Figure 5. It should be noted that over much of the range of α the value of α^* differs little from its minimum value:

$$\alpha_{max}^* = (6)^{-1/2} = 0.4082 \tag{A7}$$

In fact there is little error if we take $\alpha = 0.5$, about in the middle of the observed range. Correspondingly we have $\alpha^* = 0.36$. Hence the appropriate value of v_0 might be some 12% less than the value given by (10) with $\alpha = 0.41$.

Appendix 2. The Stokes Velocity

In any fluctuating field of motion there is a systematic difference between the mean velocity measured by a freely floating object (the Lagrangian mean velocity) and the mean velocity recorded by a current meter at a fixed point (the Eulerian mean velocity). The difference between the Lagrangian and the Eulerian mean velocities has been called the Stokes velocity [*Longuet-Higgins*, 1969] after *G. G. Stokes* [1847], who discovered it for surface waves on water of uniform depth. A general expression for the Stokes velocity $\mathbf{U}_s$ is given by

$$\mathbf{U}_s = \left\langle \int \mathbf{u}_1 \, dt \cdot \nabla \mathbf{u}_1 \right\rangle \tag{B1}$$

where $\mathbf{u}_1$ denotes the first-order particle motion, assumed periodic with mean zero, and the angle brackets denote the mean value with respect to time. Thus $\mathbf{U}_s$ depends on the space gradient of the orbital velocity $\mathbf{u}_1$.

In shallow-water wave theory, the orbital velocity is independent of the vertical coordinate,

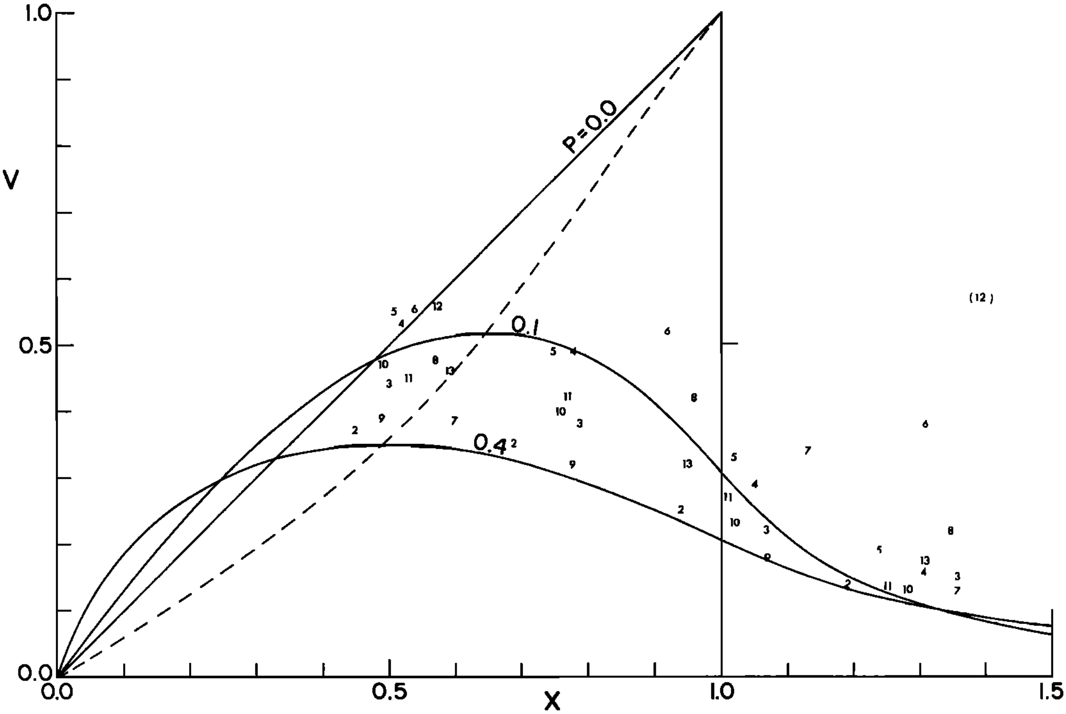

Fig. 4c. Comparison of the longshore velocities measured by *Galvin and Eagleson* [1965; series IV] with the theoretical profiles derived in section 2 of the present paper. The plotted numbers correspond to the number of each run (Table 2).

and so likewise is $\mathbf{U}_s$. Nevertheless, the horizontal gradients of $\mathbf{u}_1$ still give rise to a non-zero Stokes velocity. If u_1 and v_1 denote the x and y components of $\mathbf{u}_1$, the longshore component of $\mathbf{U}_s$ is given by

$$V_s = \left\langle \int u_1 \, dt \, \frac{\partial v_1}{\partial x} \right\rangle + \left\langle \int v_1 \, dt \, \frac{\partial v_1}{\partial y} \right\rangle \quad \text{(B2)}$$

Consistent with our previous use of the linearized shallow-water theory, we have

$$\begin{aligned} u_1 &= \alpha(gh)^{1/2} \cos\theta \cos(lx + my - \sigma t) \\ v_1 &= \alpha(gh)^{1/2} \sin\theta \cos(lx + my - \sigma t) \end{aligned} \quad \text{(B3)}$$

where

$$l = k \cos\theta \qquad m = k \sin\theta \quad \text{(B4)}$$

k being the absolute wave number. By Snell's law, the longshore wave number m is a constant, but the onshore wave number l is related to the frequency σ and the local depth h by the relation

$$\sigma^2 = k^2 gh = (l^2 + m^2)gh \quad \text{(B5)}$$

On substituting from (B3) into (B2) and noting that $\partial m/\partial x$, $\partial m/\partial y$, $\partial l/\partial y$, $\partial h/\partial y$, and $\partial\theta/\partial y$ all vanish we obtain

$$V_s = \tfrac{1}{2}\alpha^2(gh/\sigma)[(l + x\,\partial l/\partial x) + m] \cdot \cos\theta \sin\theta \quad \text{(B6)}$$

If we ignore $\partial l/\partial x$ in comparison with l/x, the above expression reduces to

$$V_s = \tfrac{1}{2}\alpha^2(gh/c)[\cos\theta + \sin\theta] \cos\theta \sin\theta \quad \text{(B7)}$$

where $c = \sigma/k$ denotes the phase velocity. When θ^2 is negligible, (B7) becomes simply

$$V_s = \tfrac{1}{2}\alpha^2 gh(\sin\theta/c) \quad \text{(B8)}$$

The last factor is constant, by Snell's law. Hence the Stokes velocity increases linearly with the

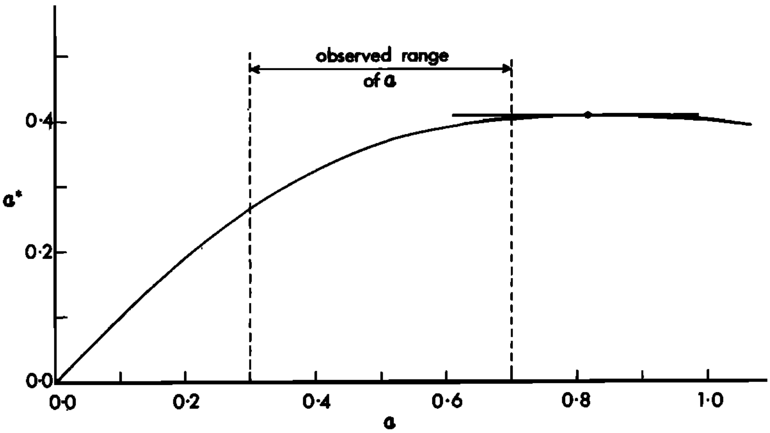

Fig. 5. The virtual coefficient α^* as a function of α.

depth h and hence linearly with distance from the shoreline.

To show that $\partial l/\partial x$ is negligible in comparison with l/x, we have from (B5)

$$\begin{aligned} l^2 &= (\sigma^2/gh) - m^2 \\ 2l\,\partial l/\partial x &= -(\sigma^2/gh^2)\,\partial h/\partial x \end{aligned} \qquad \text{(B9)}$$

Therefore, if $h = sx$

$$2l\,\partial l/\partial x = (l^2 + m^2)/x \qquad \text{(B10)}$$

Thus $\partial l/\partial x \ll l/x$, provided that

$$l^2 + m^2 \ll 2l^2 \qquad \text{(B11)}$$

that is to say $m^2 \ll l^2$ or $\tan^2\theta \ll 1$. Since θ^2 has been neglected, this condition is already satisfied by our previous assumptions.

Let us now compare the magnitude of the Stokes velocity, as given by (B8), with the scale velocity v_0 defined by (10). At the breaker line the ratio of the two is given by

$$\frac{V_s}{v_0} = \frac{4}{5\pi}\frac{\alpha C}{s} \qquad \text{(B12)}$$

With the values $s = 0.11$, $C = 0.010$, and $\alpha = 0.41$ we have simply $V_s/v_0 = 0.01$, which is negligible.

On the other hand, with very gently sloping beaches, where s is much smaller than the value in Galvin and Eagleson's experiments, the Stokes velocity may very well have to be taken into account.

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