Longshore Currents Generated by Obliquely Incident Sea Waves, 1

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By using known results on the radiation stress associated with gravity waves, the total lateral thrust exerted by incoming waves on the beach and in the nearshore zone is rigorously shown to equal $(E_0/4) \sin 2\theta_0$ per unit distance parallel to the coastline, where E_0 denotes the energy density of the waves in deep water and θ_0 denotes the waves' angle of incidence. The local stress exerted on the surf zone in steady conditions is shown to be given by $(D/c) \sin \theta$ per unit area, where D is the local rate of energy dissipation and c is the phase velocity. These relations are independent of the manner of the energy dissipation, but, because breaker height is related to local depth in shallow water, it is argued that ordinarily most of the dissipation is due to wave breaking, not to bottom friction. Under these conditions the local mean longshore stress in the surf zone will be given by $(5/4)\rho u_{max}^3 s \sin \theta$, where ρ is the density, u_{max} is the maximum orbital velocity in the waves, s is the local beach slope, and θ is the angle of incidence. It is further shown that, if the friction coefficient C on the bottom is assumed constant and if horizontal mixing is neglected, the mean longshore component of velocity is given by $(5\pi/8)(s/C)\, u_{max} \sin \theta$. This value is proportional to the longshore component of the orbital velocity. When the horizontal mixing is taken into account, the longshore currents observed in field observations and laboratory experiments are consistent with a friction coefficient of about 0.010.

1. INTRODUCTION

It is well known [*Wiegel*, 1963; *Inman and Bagnold*, 1963] that when sea waves or swell approach a straight coastline at an oblique angle (Figure 1) a mean current tends to be set up parallel to the coastline. Such longshore currents and the associated longshore transport of sand or other sedimentary material are of prime importance for both the coastal engineer and the submarine geologist.

Many hypotheses, of a very rough kind, have been advanced to account for this phenomenon. However, a recent review of the subject by *Galvin* [1967] arrives at the justifiable conclusion that, 'A proven prediction of longshore current velocity is not available, and reliable data on longshore currents are lacking over a significant range of possible flows.'

It has often been suggested [e.g., *Putnam et al.*, [1949] that the magnitude of the longshore current is related in some way to the energy or the momentum of the incoming waves. Of these two approaches, that employing momentum is the more promising since momentum is conserved, whereas energy can be dissipated by breaking and other processes not immediately associated with sediment transport.

It goes without saying that any momentum theory must be correctly formulated. The estimate of the momentum made by *Putnam et al.* [1949] has been already criticized on theoretical grounds by *Galvin* [1967]. Moreover, *Inman and Quinn* [1952] showed that, in order to make the theory fit the observations, the friction coefficient C would have to be assumed to vary over a wide range of 3½ orders of magnitude. A version of the theory of Putnam et al. modified by *Galvin and Eagleson* [1965] also requires a large variation in C.

The aim of this paper is to introduce a more satisfactory estimate of the momentum of the incoming waves, which is based on the concept of the radiation stress as developed by *Longuet-Higgins and Stewart* [1960, 1961, 1962, 1963, 1964]. This estimate of the excess transfer of momentum due to the waves has already proved remarkably successful in the prediction of several wave phenomena, particularly the setup, or change in mean level of the sea sur-

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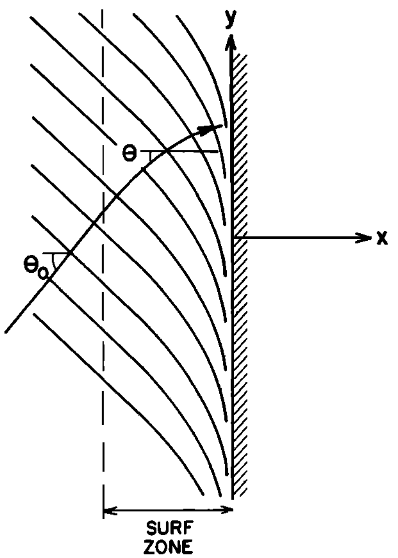

Fig. 1. Definition diagram for waves approaching a straight shoreline at an oblique angle.

face in the breaker zone [*Longuet-Higgins and Stewart*, 1963, 1964; *Bowen*, 1967].

In the present paper it is pointed out, first, that there exists a simple and precise relationship between the total longshore thrust exerted by the incoming waves on the one hand and their direction and amplitude in deep water on the other (see equation 10). This result can be derived either from the concept of the radiation stress mentioned earlier, or by a direct evaluation of the momentum flux due to the waves.

Next it is shown that the local longshore stress due to the waves is very simply related to the local rate of dissipation of wave energy, regardless of whether the dissipation is due to wave breaking or to bottom friction. Hence, using the known relation of breaker height to local depth in the surf zone, one can estimate accurately the local longshore stress due to the waves (section 4).

When the local longshore wave stress is known, it is possible to write an equation of motion for the longshore current that involves in general both the bottom friction and the horizontal mixing by turbulent eddies. If the horizontal mixing is negligible, the momentum balance gives an exceedingly simple expression for the longshore current $\langle v \rangle$. The addition of horizontal mixing generally reduces the current, although not drastically.

A comparison with the available data (section 7) shows that even without the assumption of mixing there is already an order- of-magnitude agreement between the observed and the theoretical current if one takes an a priori estimate of the friction coefficient (about 0.010) based on experiments with flow in rough pipes [*Prandtl*, 1952]. The comparison indicates also that horizontal mixing is significant, though not dominant, in most circumstances.

Note added in processing. Since this paper was prepared, a somewhat similar approach to the theory of longshore currents has been published by *Bowen* [1969]. Besides containing new results, the present treatment differs both in the derivation of equation 34 (since Bowen takes θ to be constant during differentiation) and in the assumed form of the bottom friction. For further comparisons see the companion paper.

2. Waves Approaching Coastline

Imagine a straight coastline, as in Figure 1, in which the local still water depth h is some function of the coordinate x normal to the shoreline and is independent of the distance y along the shore. The shoreline itself is at $x = 0$. A train of two-dimensional waves of amplitude a is advancing from deep water toward the coast, the local direction of propagation being inclined at an angle of incidence θ to the normal, as shown.

Both θ and a will vary with the distance $|x|$ from the shoreline. If σ denotes the frequency of the waves and k denotes the local wave number, Snell's law, which expresses the constancy of the wave number in the direction parallel to the shoreline, can be written as

$$k \sin \theta = \text{constant} \tag{1}$$

or equivalently

$$(\sin \theta)/c = \text{constant} \tag{2}$$

where $c = \sigma/k$ denotes the local phase velocity. If the bottom slope is gradual, so that the proportional change in depth over one wavelength is small, it is reasonable to assume that σ and k are related to the local depth $h(x)$ by the Stokes relation for waves of small amplitude:

$$\sigma^2 = gk \tanh kh \tag{3}$$

The phase velocity c is then given by

$$c = \sigma/k = [(g/k) \tanh kh]^{1/2} \tag{4}$$

and the group velocity, or velocity of energy propagation, is given by

$$c_g = \frac{d\sigma}{dk} = \frac{\sigma}{2k}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{5}$$

With the local energy density per unit horizontal area being given by

$$E = \tfrac{1}{2}\rho g a^2 \tag{6}$$

correct to second order, the flux of energy toward the coast, per unit distance parallel to the shoreline, is given by

$$F_x = Ec_g \cos \theta \tag{7}$$

If the waves are losing no energy by breaking, bottom friction, or otherwise, we have

$$F_x = \text{constant} \tag{8}$$

independently of x, from which one can deduce the law of variation of the wave amplitude a with distance offshore [*Burnside*, 1915; *Longuet-Higgins*, 1956]. Inside the breaker zone, however, some energy will be lost, and hence a will diminish toward the shoreline and become zero at or near $x = 0$. If D denotes the rate of dissipation of wave energy, either by breaking or friction, we have identically

$$\partial F_x/\partial x = -D \tag{9}$$

3. Radiation Stresses

So much is well accepted. We propose now to calculate the force exerted on the nearshore region by the incoming waves, by using the notion of the radiation stress, as introduced by *Longuet-Higgins and Stewart* [1960, 1961, 1962, 1963, 1964].

It can be shown [*Longuet-Higgins and Stewart*, 1960] that the presence of a wave train of amplitude a in water of depth h increases the flux of momentum parallel to the direction of propagation across any plane normal to that direction by an amount

$$S_{11} = E\left(\frac{1}{2} + \frac{2kh}{\sinh 2kh}\right) \tag{10}$$

Similarly the flux of momentum normal to the direction of wave propagation across a plane parallel to the direction of propagation is increased by an amount

$$S_{22} = E(kh/\sinh 2kh) \tag{11}$$

where $E = \frac{1}{2}\rho g a^2$. In general the momentum flux tensor, referred to coordinates (ξ_1, ξ_2) parallel and perpendicular to the direction of wave propagation, is given by

$$S_{ij} = \begin{bmatrix} E\left(\frac{1}{2} + \frac{2kh}{\sinh 2kh}\right) & 0 \\ 0 & E\dfrac{kh}{\sinh 2kh} \end{bmatrix} \tag{12}$$

the off-diagonal elements being zero.

Now let us calculate the flux of y momentum parallel to the shoreline across a plane $x =$ constant, parallel to the shoreline. Since the axes (x, y) are inclined at an angle θ to the principal axes (ξ_1, ξ_2) of the waves, we have

$$\begin{aligned} S_{xy} &= \sum_{i,j} S_{ij} \frac{\partial x}{\partial \xi_i} \frac{\partial y}{\partial \xi_j} \\ &= S_{11} \sin \theta \cos \theta + S_{22} \cos \theta(-\sin \theta) \\ &= E\left(\frac{1}{2} + \frac{kh}{\sinh 2kh}\right) \cos \theta \sin \theta \\ &= E(c_g/c) \cos \theta \sin \theta \end{aligned} \tag{13}$$

By (7) this relation can be written as

$$S_{xy} = F_x(\sin \theta)/c \tag{14}$$

or, if we make use of Snell's law in the form of (2), we then have

$$S_{xy} = F_x(\sin \theta_0)/c_0 \tag{15}$$

where θ_0 and c_0 refer to the (constant) values of θ and c in deep water.

This very simple and exact relation states that the flux of y momentum across the plane $x =$ constant is proportional, by a fixed, known constant, to the energy flux across the same plane.

Because of the simplicity and fundamental importance of relation 15 we give here an alternative proof.

The flux of y momentum across any vertical plane $x =$ constant is simply equal to ρuv, where u and v are the components of velocity in the x and y directions. On integrating this with respect to the vertical coordinate z we find

$$S_{xy} = \left\langle \int_{-h}^{\zeta} \rho uv \, dz \right\rangle \tag{16}$$

The angle brackets denote the mean value with respect to time. Now for waves traveling at an angle θ to the x axis we have

$$u = u_1 \cos \theta \qquad v = u_1 \sin \theta \tag{17}$$

where u_1 denotes the horizontal component of the orbital velocity in the direction of wave propagation. Also in (16) the upper limit of integration can be replaced by the mean value $z = 0$, since the difference $\int_0^{\zeta} \rho uv \, dz$ is only of the third order at most in the wave amplitude. (The mean value is actually of fourth order.) We have then, correct to second order,

$$S_{xy} = \int_{-h}^{0} \rho \langle u_1^2 \rangle \, dz \cos \theta \sin \theta \tag{18}$$

Now the flux of energy in the direction of wave propagation is given by

$$F = \left\langle \int_{-h}^{\zeta} [p + \tfrac{1}{2}\rho(u_1^2 + v_1^2)]u_1 \right\rangle dz \tag{19}$$

So to the same order of approximation

$$F = \int_{-h}^{0} \langle pu_1 \rangle \, dz \tag{20}$$

From the linearized equation of horizontal momentum, however, we have

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial \xi_1} = \frac{1}{\rho c}\frac{\partial p}{\partial t} \tag{21}$$

since in progressive wave motion $\partial/\partial t \sim c \, \partial/\partial \xi_1$. Then on integration with respect to time we have

$$u_1 = (p/\rho c) + \text{constant} \tag{22}$$

On substituting in (18) and noting that for irrotational waves $\langle u_1 \rangle = 0$ correct to second order we obtain from (18) and (20)

$$S_{xy} = (1/c)F \cos \theta \sin \theta \tag{23}$$

The energy flux F_x being equal to $F \cos \theta$, we obtain (14) and hence (15) as before.

From (15) we can at once calculate the total longshore thrust of the waves, as follows.

Outside the breaker line (or the line at which energy losses become significant) we have

$$F_x = \text{constant} = E_0(\tfrac{1}{2}c_0) \cos \theta_0 \tag{24}$$

c_0 being the phase velocity in deep water, where the group velocity $c_g = \tfrac{1}{2}c_0$, and E_0 being the energy density in deep water. Therefore from (15)

$$(S_{xy})_\infty = \tfrac{1}{2}E_0 \cos \theta_0 \sin \theta_0 \tag{25}$$

On the other hand, at the shoreline $x = \delta > 0$ (just beyond the reach of the waves) we have

$$F_x = 0 \qquad S_{xy} = 0 \qquad x = \delta \tag{26}$$

Therefore, by considering the balance of momentum of the water between the breaker line and the shoreline, we see that the total external force G_y parallel to the shoreline acting on the water and sediment inside the breaker zone is given by

$$(S_{xy})_\infty + G_y = 0 \tag{27}$$

In the absence of wind or other surface stresses the only external force must come from bottom friction. Hence the total lateral littoral force exerted by the waves on the bottom is given by $H_y = -G_y$, that is to say

$$H_y = \tfrac{1}{4}E_0 \sin 2\theta_0 \tag{28}$$

It is interesting that the force is a maximum, for a given wave amplitude at infinity, when $\sin 2\theta_0 = 1$ or $\theta_0 = 45°$.

4. Local Wave Stress

Inside the breaker zone F_x gradually diminishes toward the shoreline. A consideration of the momentum balance between two planes $x = x_1$ and $x_1 + dx$ parallel to the shoreline and separated by a distance dx shows at once that the net stress τ_y per unit area exerted by the waves on the water in the surf zone is given by

$$\tau_y = -\partial S_{xy}/\partial x \tag{29}$$

and by (15) this equation becomes

$$\tau_y = -\frac{\partial F_x}{\partial x}\left(\frac{\sin \theta_0}{c_0}\right) = D\left(\frac{\sin \theta_0}{c_0}\right) \tag{30}$$

where D denotes the local rate of energy dissipation. In other words, the local stress exerted by the waves is directly proportional to the local rate of dissipation of wave energy. Outside the breaker zone the mean bottom stress vanishes.

In some situations the loss of wave energy can be attributed to bottom friction (due mainly to the orbital velocity of the waves). However, the observation by *Munk* [1949] that in the surf zone the breaker height is proportional to the mean depth suggests that under normal circumstances most of the loss of wave energy is due to wave breaking, not to bottom friction.

It is found that the rule

$$a = \alpha h \qquad (31)$$

where α is a constant between 0.3 and 0.6 is in agreement both with direct observations (see Table 1 below) and with laboratory measurements of wave setup [*Longuet-Higgins and Stewart*, 1963, 1964; *Bowen*, 1967] the approximate linear shallow-water theory is used. On the basis of this theory we have from section 2, when $kh \ll 1$,

$$c = (gh)^{1/2} = c_g \qquad (32)$$

If it is assumed that in the breaker zone θ is small enough that $\cos\theta$ can be approximated

TABLE 1. Observed and Theoretical Values of α

Investigator	s	α	$\langle\alpha\rangle$
Observed Values			
Putnam et al.[1949]	0.066	0.37	
	0 098	0.36	
	0.100	0.33	
	0.139	0.32	0.35
	0.143	0.37	
	0.144	0.32	
	0 241	0 35	
	0.260	0.36	
Iverson [1952]	0.020	0.41	
	0.033	0 38	0.44
	0.050	0.42	
	0 100	0.52	
Larras [1952]	0 010	0.34	
	0.020	0.37	0.39
	0.091	0.43	
Ippen and Kulin [1955]	0 023	0.60	0.60
Eagleson [1956]	0.067	0.56	0.56
Galvin and Eagleson [1965]	0.104	0.59	0.59
Bowen [1968]	0.082	0 45–0 62	0.56
Values Determined from Solitary Wave Theory			
McCowan [1894]	0.000	0.39	
Davies [1952]	0.000	0.41	
Long [1956]	0.000	0.406	

by unity, we have from (7), (31), and (32)

$$F_x = \tfrac{1}{2}\rho g a^2 c_g = \tfrac{1}{2}\alpha^2 \rho g^{3/2} h^{5/2} \qquad (33)$$

and so from (30)

$$\tau_y = -\tfrac{5}{4}\alpha^2 \rho (gh)^{3/2} \frac{dh}{dx}\frac{\sin\theta}{c}$$

$$= \tfrac{5}{4}\alpha^2 \rho g h (s \sin\theta) \qquad (34)$$

where $s = -dh/dx$ denotes the local bottom slope.

Some values of α as determined by various authors are shown in Table 1. Though the later determinations of α tend to be higher than the earlier ones, no determination departs by more than 50% from the theoretical value of 0.41 calculated by Davies and Long for the solitary wave.

Using (31) and the linear shallow-water theory, we can also express (34) in terms of the maximum horizontal orbital velocity given by

$$u_{\max} = (a\sigma)/(kh) = \alpha(\sigma/k) = \alpha(gh)^{1/2} \qquad (35)$$

Then we have simply

$$\tau_y = \tfrac{5}{4}\rho u_{\max}^2 (s \sin\theta) \qquad (36)$$

where s denotes the bottom slope and θ denotes the local angle of incidence.

We note that in this simple relation there are no adjustable parameters.

Beyond the breaker line, i.e. where the energy dissipation is negligibly small, D vanishes, and so by (31)

$$\tau_y = 0 \qquad (37)$$

5. Bottom Friction

The tangential stress **B** excited by the water on the bottom will be assumed to be given adequately by a relation of the form

$$\mathbf{B} = C\rho\,|\mathbf{u}|\,\mathbf{u} \qquad (38)$$

where **u** is the instantaneous velocity vector near the bottom and C is a constant coefficient.

If there were no longshore velocity, and if the amplitude of the motion were small and the bottom impermeable, the horizontal orbital velocity would be expected to be to-and-fro in the same straight line, making an angle θ with the normal to the shoreline (see Figure 2*a*).

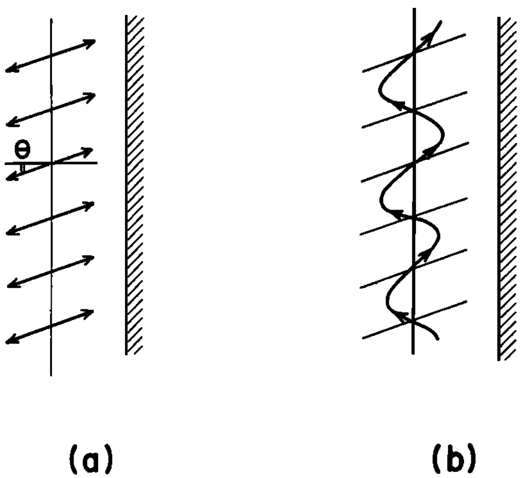

Fig. 2. Schematic representation of particle orbits (*a*) with zero mean littoral velocity and (*b*) with positive littoral velocity $\langle v \rangle$.

The frictional stress **B** given by

$$\mathbf{B} = C\rho\ |\mathbf{u}_{orb}|\ \mathbf{u}_{orb} \tag{39}$$

would then vanish in the mean (according to linear theory).

Now suppose that a small component of velocity $\langle v \rangle$ in the longshore direction is added to the orbital velocity (Figure 2*b*). When θ is small, this component of velocity is almost perpendicular to the orbital velocity. Therefore the magnitude of the velocity $\mathbf{u} = \mathbf{u}_{orb} + (0, \langle v \rangle)$ is unchanged, to first order, but the direction of the bottom stress is changed by a small angle $\langle v \rangle / |\mathbf{u}_{orb}|$ approximately. Hence there is an additional stress in the y direction given by

$$B_y = C\rho\ |\mathbf{u}_{orb}|^2\ (\langle v \rangle / |\mathbf{u}_{orb}|) = C\rho\ |\mathbf{u}_{orb}|\ \langle v \rangle \tag{40}$$

Physically, when the orbital velocity is onshore, the direction of the bottom stress is inclined more toward the positive y direction (if $\langle v \rangle$ is positive); when the orbital velocity is offshore, the bottom stress, now almost in the opposite direction, is again more toward the positive y direction. Taking mean values in (40), we have the relation

$$\langle B_y \rangle = C\rho \langle |\mathbf{u}_{orb}| \rangle \langle v \rangle \tag{41}$$

Assuming $\mathbf{u}_{orb}$ to be sinusoidal, we have

$$\langle |\mathbf{u}_{orb}| \rangle = (2/\pi) u_{max} \tag{42}$$

and hence

$$\langle B_y \rangle = (2/\pi) C\rho u_{max}\ \langle v \rangle \tag{43}$$

As a guide to the appropriate value of the friction coefficient we consider first the values for a rough horizontal plate in uniform flow, as given for example by *Prandtl* [1952] and based on Nikuradse's experiments with roughened pipes. For convenience we reproduce *Prandtl*'s [1952, p. 195] diagram as Figure 3 below. The friction coefficient appears to depend on just two parameters. The first is the Reynolds number

$$Re = Ul/\nu \tag{44}$$

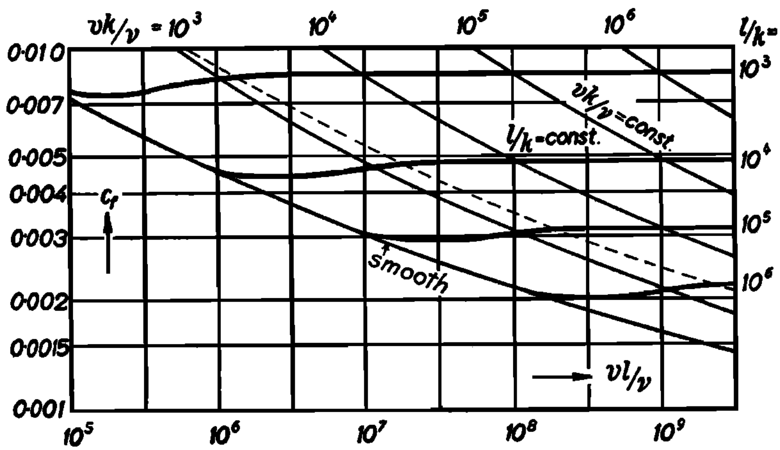

Fig. 3. Values of the friction coefficient C for flow over rough plates, as deduced from the experiments of Nikuradse [from *Prandtl*, 1952]. (Figure reprinted by permission of Haffner Co.)

where U denotes the horizontal velocity, l denotes the length of the plate, and ν is the kinematic viscosity. The second parameter is the ratio (l/K), where K denotes a typical scale for a roughness element. Here we can take as an appropriate value of U the horizontal component of the orbital velocity, u_{max}, and for l the horizontal excursion of a water particle from its mean position, that is $l = u_{max}/\sigma$. Thus we have

$$Re = u_{max}^2/\nu\sigma = \alpha^2 gh/\nu\sigma \tag{45}$$

As typical values for field data we can take

$$\begin{aligned} \alpha &= 0.4 \qquad g = 10\text{m/sec}^2 \\ h &= 1 \text{ meter} \qquad \sigma = 1 \text{ rad/sec} \end{aligned} \tag{46}$$

corresponding to 6-sec waves 0.8 meter high. With the approximate value $\nu = 1.3 \times 10^{-6}$ m^2/sec and with a sand grain diameter of 1 mm we obtain

$$Re = 1.3 \times 10^6 \qquad l/K = 1.3 \times 10^3 \tag{47}$$

and so from Figure 3 $C_f \doteqdot 0.007$. On the other hand, for laboratory data more typical values are

$$h = 0.1\text{meter} \qquad \sigma = 5\text{rad/sec} \tag{48}$$

With the same values of α, g, and ν, this leads to

$$Re = 2.5 \times 10^4 \qquad l/K = 20 \tag{49}$$

if the roughness scale K is the same. In that case Figure 3 suggests that C_f is somewhat larger, about 0.010.

Bretschneider [1954] has found that the observed damping of swell which is propagated over a smooth, level, impermeable sea bed is consistent with a friction coefficient lying between 0.034 and 0.097. These values appear to agree well with Prandtl's values. On the other hand, Bretschneider also found that the spectral limitation of wave growth under the action of wind suggested higher values of C, between 0.01 and 0.02. These coefficients may include other significant effects such as bottom percolation. R. E. Mayer (personal communication) has found, however, that the theory of run-up of surf on beaches [*Shen and Meyer*, 1963; *Freeman and Le Méhauté*, 1964] can be made to agree fairly well with the model experiments of *Miller* [1968] over a hard sloping concrete bottom by assuming that C lies between 0.01 and 0.02. These values cannot be the result of bottom percolation, but might be attributable in part to turbulence arising from the breaking of the waves as they run up the slope.

Taken together, the above data suggest that it is not unreasonable to expect a friction coefficient C of the order of 0.01.

6. Equations for Longshore Current

To estimate the longshore current $\langle v \rangle$, let us assume first that the mean current is steady and two-dimensional, being independent of the time t and of the longshore coordinate y. Then the equation of motion in the longshore direction can be written as

$$0 = \tau_y + \frac{\partial}{\partial x}\left(N \frac{\partial \langle v \rangle}{\partial x}\right) - \langle B_y \rangle \tag{50}$$

where in the surf zone τ_y and $\langle B_y \rangle$ are given by (37) and (43), respectively. The second term represents the exchange of momentum due to horizontal turbulent eddies, with eddy coefficient N.

In this equation the magnitude of N is unknown. Suppose first that the exchange of momentum by turbulence is negligible in comparison with that due to the waves; then in general the second term on the right of (50) can be neglected in comparison with the first. There remains a balance between the first and third terms:

$$\langle B_y \rangle = \tau_y \tag{51}$$

Substituting from equations (36) and (43), we have in the breaker zone

$$\frac{2}{\pi} C\rho u_{max} \langle v \rangle = \tfrac{5}{4}\rho u_{max}^2 (s \sin \theta) \tag{52}$$

and hence

$$\langle v \rangle = (5\pi/8C) u_{max} (s \sin \theta) \tag{53}$$

This very simple relation implies that for constant values of C and s the longshore current is simply proportional to $u_{max} \sin \theta$, or to the longshore component of the orbital velocity.

The proportionality of $\langle v \rangle$ and u_{max} has been inferred on quite different grounds by P. Komar (personal communication, 1969).

Using (36) the relation between u_{max} and the local phase velocity c, we can also write (52) in the form

$$\langle v \rangle = \frac{5\pi}{8}\frac{\alpha}{C} ghs\left(\frac{\sin \theta}{c}\right) \qquad (53')$$

where $c = (gh)^{1/2}$. Now by Snell's law the last factor is a constant. Equation (53′) then states that in a given wave situation, if both C and s are constant, the longshore velocity $\langle v \rangle$ is simply proportional to the local depth h.

If we assume that the shallow-water theory is valid as far out as the breaker line where the depth h is equal to h_B, the mean longshore current, in the absence of horizontal mixing, can be written as

$$\langle v \rangle = \left(\frac{h}{h_B}\right) \times \begin{cases} v_0 & h < h_B \\ 0 & h > h_B \end{cases} \qquad (54)$$

where

$$v_0 = \frac{5\pi}{8}\frac{\alpha}{C} (gh_B)^{1/2}(s \sin \theta_B) \qquad (55)$$

This relation is shown in Figure 4 by the dashed line (corresponding to $\gamma = 0$). The total longshore flux in the surf zone is given by

$$Q = \int_{h=h_B}^{h=0} h\langle v \rangle \, dx$$

$$= \int_0^{h_B} h(h/h_B)v_0 \, dh/s$$

$$= \tfrac{1}{3}h_B^2 v_0/s = \tfrac{1}{3}h_B \, |x_B| \, v_0 \qquad (56)$$

We have so far neglected the horizontal mixing entirely. In this idealized model there is a sharp discontinuity in the velocity profile at the breaker line. The presence of any horizontal mixing, as well as any variability in wave height and position of breaker line, will tend to smooth out the discontinuity at the breaker line and produce a smoother velocity profile; this shifts the maximum velocity closer to shore, as in Figure 4.

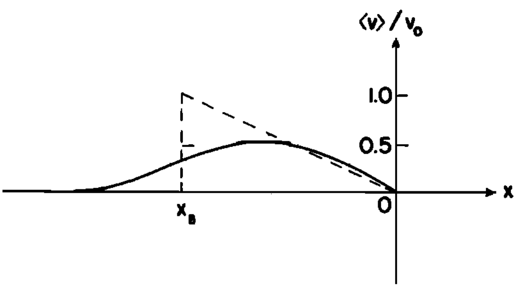

Fig. 4. Schematic representation of the longshore velocity profile as a function of distance offshore. Broken line denotes values without horizontal mixing; full line, with horizontal mixing.

TABLE 2. Theoretical Values of β

$\gamma = L/\|x_B\|$	β
0.00	0.500
0.25	0.386
0.50	0.290
0.75	0.218
1.00	0.167

A very rough estimate of the effect of mixing on the velocity v_B at the breaker line can be obtained by taking the average of the momentum $h\langle v \rangle$ over a distance L on either side of the breaker line, where L represents a mixing length. When L is small in comparison with the width $|x_B|$ of the surf zone, the velocity v_B is equal to the mean value of the velocities on the two sides of the discontinuity in Figure 4. Hence $v_B = \tfrac{1}{2} v_0$. More generally if we take

$$L = \gamma \, |x_B| \qquad 0 < \gamma < 1 \qquad (57)$$

we obtain for constant bottom gradient

$$v_B = \beta v_0 \qquad (58)$$

where

$$\beta = \frac{1 - \gamma}{2} + \frac{\gamma^2}{6} \qquad (59)$$

As y increases from 0 to 1, β decreases from $\tfrac{1}{2}$ to $\tfrac{1}{6}$. Then we can write

$$v_B = \frac{5\pi}{8}\frac{\alpha\beta}{C} (gh_B)^{1/2}(s \sin \theta_B) \qquad (60)$$

where θ_B denotes the angle of incidence at the breaker line and β is a constant between 0.5 and about 0.167. The dependence of β on γ is shown in Table 2.

7. Comparison with Observation

Because of the dependence of the longshore velocity on the distance from the shoreline it is particularly important to define precisely the position of the point of observation relative to the shoreline and breaker line.

TABLE 3. Laboratory Data by Brebner and Kamphuis from *Galvin and Nelson* [1965, p. 12]

H_B, feet	θ_B, deg	s	v_B, ft/sec	$(gH_B)^{1/2}/v_B$	$s \sin \theta_B$	r
0.092	7.0	0.10	0.44	3.92	0.012	0.048
0.097	7.5	0.10	0.47	3.77	0.013	0.049
0.110	9.0	0.10	0.67	2.81	0.016	0.045
0.118	10.0	0.10	0.82	2.38	0.017	0.041
0.118	7.5	0.10	0.49	3.97	0.013	0.052
0.138	8.0	0.10	0.67	3.15	0.014	0.044
0.153	10.0	0.10	0.83	2.68	0.017	0.047
0.159	12.0	0.10	0.99	2.29	0.021	0.047
0.157	9.0	0.10	0.63	3.57	0.016	0.056
0.159	9.5	0.10	0.80	2.83	0.016	0.047
0.200	12.0	0.10	0.96	2.65	0.021	0.055
0.203	13.0	0.10	1.07	2.39	0.022	0.054
0.177	9.0	0.10	0.63	3.79	0.016	0.059
0.220	11.0	0.10	0.88	3.02	0.019	0.058
0.228	12.5	0.10	1.04	2.60	0.022	0.056
0.231	14.0	0.10	1.16	2.35	0.024	0.057
0.092	10.0	0.10	0.60	2.87	0.017	0.050
0.112	11.0	0.10	0.81	2.35	0.019	0.045
0.110	13.0	0.10	0.84	2.24	0.022	0.050
0.118	15.0	0.10	0.91	2.14	0.026	0.055
0.118	11.0	0.10	0.83	2.35	0.019	0.145
0.133	12.5	0.10	0.97	2.14	0.022	0.046
0.153	15.0	0.10	1.04	2.14	0.026	0.055
0.159	17.0	0.10	1.14	1.99	0.029	0.058
0.170	13.0	0.10	0.94	2.49	0.022	0.056
0.158	14.0	0.10	1.12	2.01	0.024	0.049
0.200	17.0	0.10	1.25	2.03	0.029	0.059
0.194	18.0	0.10	1.32	1.89	0.031	0.058
0.184	13.0	0.10	1.07	2.28	0.022	0.051
0.204	16.0	0.10	1.25	2.05	0.027	0.056
0.231	18.0	0.10	1.29	2.12	0.031	0.065
0.234	21.0	0.10	1.32	2.08	0.036	0.074
0.085	12.0	0.10	0.70	2.36	0.021	0.049
0.097	14.0	0.10	0.83	2.13	0.024	0.052
0.110	17.0	0.10	0.88	2.14	0.029	0.062
0.112	18.0	0.10	1.05	1.81	0.031	0.056
0.118	14.0	0.10	0.91	2.14	0.024	0.052
0.133	16.0	0.10	0.96	2.16	0.027	0.059
0.141	18.0	0.10	1.10	1.94	0.031	0.060
0.147	21.0	0.10	1.22	1.79	0.036	0.064
0.151	17.0	0.10	1.08	2.04	0.029	0.060
0.153	18.0	0.10	1.18	1.88	0.031	0.059
0.176	22.0	0.10	1.36	2.75	0.037	0.066
0.187	24.0	0.10	1.53	1.60	0.041	0.065
0.177	17.0	0.10	1.21	1.97	0.029	0.057

The profiles of velocity versus offshore distance measured by *Galvin and Eagleson* [1965] show a maximum velocity about halfway between the mean shoreline (not the still water level) and the breaker line, as one would expect from section 6 if horizontal mixing were important. In the above instance, however, the flow was being accelerated downstream from a side wall, so that the compensating inflow would also contribute to the redistribution of longshore momentum and could have an effect similar to the presence of a large horizontal eddy viscosity.

A useful summary of the available field and laboratory data has been compiled by *Galvin and Nelson* [1967]; these data have been critically discussed by *Galvin* [1967]. It seems that the most commonly observed parameters of the wave field are the breaker height

$$H_B = 2\alpha h_B \tag{61}$$

and the angle of incidence θ_B at the breaker line, though in some instances these quantities must be deduced from the wave height and angle of incidence as measured in deep water. *Galvin and Eagleson* [1965] have shown that there is considerable uncertainty in the measurement of H_B and θ_B (especially θ_B) even under laboratory conditions.

Now substituting for h_B in (60) we have for the longshore velocity v_B at the breaker line

$$v_B = \frac{5\pi}{8\sqrt{2}} \frac{\sqrt{\alpha}\,\beta}{C} (gH_B)^{1/2}(s \sin \theta_B) \tag{62}$$

In other words, if we write

$$\frac{(gH_B)^{1/2}}{v_B} (s \sin \theta_B) = r \tag{63}$$

a dimensionless ratio, we have

$$C = 1.39\sqrt{\alpha}\,\beta r \tag{64}$$

With little uncertainty we can take α to be the mean value of the entries in the last column of Table 1, namely $\alpha = 0.42$; then (64) further simplifies to

$$C = 0.90\beta r \tag{65}$$

For each entry in the data compiled by *Galvin and Nelson* [1967] we have computed the quantity r as given by (63). The results of these computations for a typical page of laboratory data [*Brebner and Kamphuis*, 1963] are shown in Table 3 and for the field data of *Inman and Quinn* [1952] in Table 4. Despite the great range in the values of the breaker height H_B it will be seen that the computed value of r remains remarkably consistent. There is somewhat more scatter in the field data than

TABLE 4. Field Data by Inman and Quinn from *Galvin and Nelson* [1965, p. 17]

H_B, feet	θ_B, deg	s	v_B, ft/sec	$(gH_B)^{1/2}/v_B$	$s \sin \theta_B$	r
2.8	6.5	0.027	0.38	25.0	0.0030	0.076
3.1*	1.5*	0.027	0.04	25.0	0.0007	0.018*
3.7	4.0	0.027	0.22	49.6	0.0019	0.093
3.6*	0. *	0.027	0.04	269.0	0.0000	0.000*
4.9	5.0	0.027	0.84	14.9	0.0024	0.035
3.8	5.0	0.027	0.21	52.6	0.0024	0.124
3.4*	0. *	0.027	0.55	19.0	0.0000	0.000*
2.6*	0. *	0.035	0.04	22.9	0.0000	0.000*
3.0*	1.0*	0.035	0.01	98.2	0.0006	0.600*
2.7*	0. *	0.035	0.15	62.1	0.0000	0.000*
3.5*	0. *	0.035	0.09	117.9	0.0000	0.000*
4.9*	0. *	0.035	0.21	59.8	0.0000	0.000*
2.9*	0. *	0.035	0.50	19.3	0.0000	0.000*
4.6*	0. *	0.035	0.88	13.8	0.0000	0.000*
3.7*	0. *	0.028	0.20	54.5	0.0000	0.000*
5.1	6.0	0.027	0.29	44.1	0.0028	0.124
4.7	7.0	0.027	0.53	23.2	0.0033	0.076
4.5	4.0	0.027	0.70	17.2	0.0019	0.032
4.8	4.0	0.027	1.19	10.4	0.0019	0.020
4.2	4.5	0.027	0.40	29.1	0.0021	0.062
2.0	4.0	0.027	0.36	22.3	0.0019	0.042
1.7	7.0	0.027	0.23	31.2	0.0033	0.103
2.9	5.0	0.027	0.56	17.2	0.0023	0.041
1.6	5.0	0.027	0.11	65.2	0.0023	0.153
6.2	5.0	0.014	0.54	26.1	0.0012	0.032
3.1	7.0	0.014	0.62	16.1	0.0017	0.028
4.5	3.0	0.014	0.49	24.6	0.0007	0.018
3.5	4.0	0.014	0.17	62.4	0.0010	0.061
2.7	3.5	0.014	0.13	71.7	0.0009	0.061
4.7	7.0	0.014	1.37	9.0	0.0017	0.015
2.6*	2.0*	0.014	0.04	228.6	0.0005	0.116*
2.0	4.0	0.014	0.11	72.9	0.0010	0.071
1.8	2.5	0.014	0.06	126.8	0.0006	0.077

* Values for which θ_B is reckoned to be 2° or less.

TABLE 5. Summary of Observations: Mean Values

Investigators	Type of Beach	⟨s⟩	$\langle H_B \rangle$	$\langle \theta_B \rangle$	N	⟨r⟩
Putnam et al. [1949]*	Bonded sand	0.133	0.28	14.4	14	0.121
	Metal or smooth cement	0.172	0.23	36.8	14	0.134
	Gravel, 1/4 inch in diam.	0.123	0.22	22.0	9	0.322
Saville [1950]*	Concrete or 0.3 mm sand	0.100	0.14	6.5	7	0.087
Brebner and Kamphuis [1963]		0.100	0.15	13.9	45	0.054
	Roughened concrete	0.100	0.14	21.2	48	0.068
		0.100	0.16	14.6	48	0.035
Galvin and Eagleson [1965]	Smooth concrete	0.109	0.16	11.8	18	0.044
Putnam et al. [1949]	Oceanside	0.021	6.42	11.1	18	0.020
Inman and Quinn [1951]	Torrey Pines and Pacific Beach	0.022	3.58	4.9	21	0.064
Galvin and Savage [1966]	Nags Head	0.027	3.75	15.4	4	0.035

* Data rejected by *Galvin* [1967].

in the laboratory data, as is to be expected, especially considering the difficulty in measuring the angle of incidence θ_B. If we omit from consideration all observations (marked with an asterisk) for which θ_B is reckoned to be 2° or less, the mean value of the entries in the last column is $\langle r \rangle = 0.054$ for the laboratory measurements and $\langle r \rangle = 0.064$ for the field data.

A summary of such mean values is given in Table 5, for all the data compiled by *Galvin and Nelson* [1967] with the exception of the field observations of *Moore and Scholl* [1961], which contained a large proportion of zero or negative values of v_B and were thought to be influenced by disturbances other than wave action. In the laboratory measurements of *Saville* [1950] and of *Galvin and Eagleson* [1965] the entries corresponding to angles θ_B less than 6° have also been discarded on the grounds of unreliability.

On quite different grounds *Galvin* [1967] has rejected all the early laboratory measurements of *Putnam et al.* [1949] since they were found not to be reproducible under almost the same conditions either by *Brebner and Kamphuis* [1963] or by *Galvin and Eagleson* [1965]. It is possible that Putnam et al. employed a different definition of breaker height than Brebner and Kamphuis or Galvin and Eagleson. Galvin also suggests that less weight should be attached to the observations of *Saville* [1950], since he did not actually measure H_B and θ_B; these entries in the table are estimated from H_0 and θ_0.

Retaining then only the most reliable measurements in Table 5 (namely those not rejected by *Galvin* [1967]), we find for the field observations $\langle r \rangle = 0.040$ and for the laboratory data $\langle r \rangle = 0.050$.

According to (65), these values of r correspond to mean values of the friction coefficient C given by

$$C = 0.036\beta \qquad C = 0.045\beta \qquad (66)$$

where β, as we have seen, is between 0.50 and 0.167, depending on the horizontal mixing.

Assuming a friction coefficient C of about 0.010, we see that both field observations and laboratory data are consistent with a mean value of β equal to about 0.2. This suggests that horizontal mixing played some part, though not a dominant one, in the distribution of the longshore current.

A more precise estimate of the effects of horizontal mixing are given in the accompanying paper.

8. Conclusions

By the use of the concept of radiation stress and the small-amplitude theory of water waves, we have shown that the total longshore thrust exerted by the waves on the water and sea bed inside the surf zone is very simply related to the energy density and direction of propagation of the waves in deep water (equation 28). This relation is quite different from that given by previous authors, and it would be interesting to test it directly by experiment.

The local wave stress τ_y is also simply related to the local rate of energy dissipation, and, it would be interesting to test this relation also.

The comparisons so far made between theory and observation suggest that the rational prediction of longshore currents may be practically possible. There is no need, as some authors have suggested, to fall back on empirical correlations.

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