The generation of capillary waves by steep gravity waves

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A theory is given for the non-linear transfer of energy from gravity waves on water to capillary waves. When a progressive gravity wave approaches its maximum steepness it develops a sharp crest, at which the surface tension must be locally important. This gives rise to a travelling disturbance which produces a train of capillary waves ahead of the crest, i.e. on the forward face of the gravity wave. The capillary waves, once formed, then take further energy from the gravity wave through the radiation stresses, at the same time losing energy by viscosity.

The steepness of the capillary waves is calculated and found to be in substantial agreement with some observations by C. S. Cox. An approximate expression for the ripple steepness near the crest of the gravity wave is

$$(2\pi/3)\, e^{-g/6T'K^2},$$

where T' is the surface tension constant and K is the curvature at the wave crest. The ripple steepness also varies with distance from the wave crest.

Under favourable circumstances the dissipation of energy by the capillary waves can be many times the dissipation in the gravity waves. The capillary waves may therefore play a significant role in the generation of waves by wind, in that they tend to delay the onset of breaking.

1. Introduction

On the forward face of gravity waves in water there is sometimes observed a train of short capillary waves, carried forward more or less steadily with the gravity waves. The capillaries are especially noticeable when the gravity waves have wavelengths of 5–50 cm and are near to their maximum steepness.

Attention was drawn to these capillaries many years ago by Scott Russell (1844). More recently Munk (1955) suggested that they might be due to some kind of disturbance (of unknown origin) located near the crest of the gravity waves. On one form of Munk's hypothesis the wavelength of the capillaries is such that their phase velocity is equal to the combined phase velocity and orbital velocity of the gravity waves. The reason for their occurrence on the *forward* face is that capillary waves generated by a travelling source tend to occur forward of the source (Lamb, 1932, § 270).

The nature of the disturbance at the crest has however remained an unsolved problem. In some interesting experiments in a model wave tank, Cox (1958) has found that in free gravity waves, generated by a plunger in the absence of wind, the capillary waves were still present. A wind blowing in the same direction did indeed augment the ripples, but the ripples were still present even when the mean wind-speed was zero (see figure 1*a*).

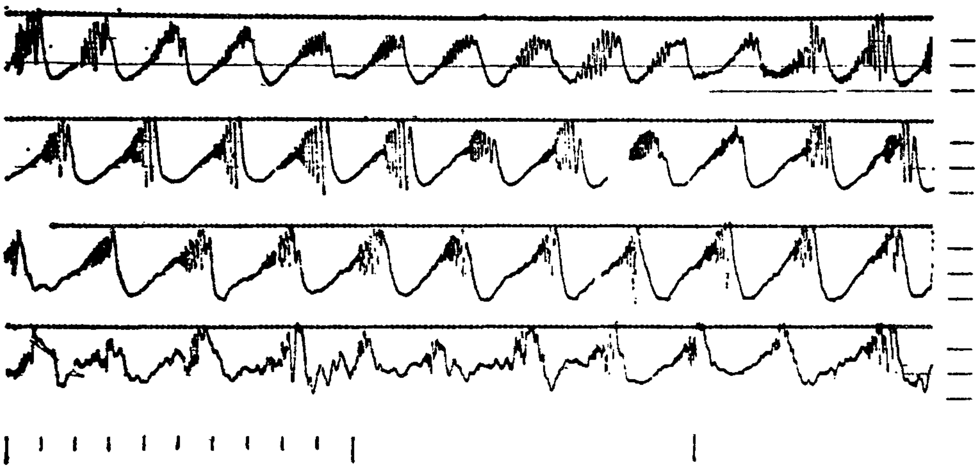

FIGURE 1. (After Cox 1958, figure 9.) Slopes of plunger-generated waves: (*a*) without wind; (*b*) with 2·8 m/sec wind; (*c*) with 9·0 m/sec wind; (*d*) with 11·6 m/sec wind. Horizontal scale: time between vertical lines is 0·1 sec or 1 sec. Vertical scale: difference in slope between horizontal lines is 0·5.

The purpose of the present paper is to suggest a mechanism for the generation of these capillaries. The idea is as follows.

It is observed that when gravity waves in deep water are on the point of breaking they develop sharp crests. In a classical paper (1847) Stokes showed that if indeed a sharp crest is formed and if the motion is progressive and irrotational then the angle at the crest is 120°. It seems probable that just short of the formation of the highest wave the curvature of the surface becomes quite large, being infinite at a sharp crest, if gravity alone is taken into account. But where the curvature is great, the surface tension becomes locally quite important. Its effect is to produce an increase in normal stress near the crest. The suggestion is that it is this travelling stress which is responsible for generating the capillary waves. The present paper contains essentially a calculation of the amplitude and wavelength of the capillaries on this hypothesis.

The present treatment of the effect of capillarity differs somewhat from the conventional one. Usually (e.g. Lamb 1932, § 266) surface-tension terms are treated on a similar footing to the gravitational terms; expansion is made in powers of a small parameter corresponding roughly to the maximum steepness of the wave. In the first approximation the effect of capillarity is to modify the wavelength of the waves. Higher approximations in powers of the steepness have been investigated, for example by Wilton (1915) and Sekerz-Zenkovitch (1956).† Each step in the approximation introduces, in general, one extra harmonic in a Fourier series. However, with steep waves the method fails for

† For related work see also Pierson & Fife (1961).

two related reasons. First, the small parameter used in the expansion becomes unsuitably large (of the order of 0·5). Secondly, the curvature of the surface, and hence the surface tension, becomes very unequally distributed over the surface. Thus, instead of affecting the wave uniformly, the surface tension produces simply a local disturbance near the crests. Such a local effect is not well represented by a Fourier expansion in harmonics of the fundamental wavelength. This limitation is typical of many non-linear problems when the amplitude of the perturbation parameter becomes unsuitably large.

Moreover, viscosity acts to damp the short waves produced by the local disturbance; since they are effectively destroyed within one wavelength of the gravity waves their wavelength is not necessarily a harmonic of the length of the gravity waves.

The present analysis, therefore, introduces surface tension terms in an altogether different way. In the first approximation the waves are treated as pure gravity waves, of nearly the maximum steepness, and then surface tension is introduced as a perturbation on this basic flow. The perturbation is still treated as small, and this limits the validity of the analysis to a certain range of wavelengths and steepnesses—roughly those over which the phenomenon is observed.

The amplitude of the capillary waves calculated in this way is compared with Cox's observations and is found to be substantially in agreement.

The effective transfer of energy from the gravity waves to the capillaries results in an appreciable damping of the gravity waves, which may exceed considerably the direct damping of the waves by viscosity. This fact has implications for theories of wave generation by wind.

2. Formulation of the problem

Suppose that two-dimensional, irrotational waves in a perfect fluid are travelling horizontally with velocity $-c$, in the direction of x increasing negatively. Let the motion be reduced to a steady state by superposing on it a uniform positive velocity c. Define a velocity potential ϕ and a stream function ψ by the relations

$$\left.\begin{aligned} d\phi &= u\,dx + v\,dy, \\ d\psi &= -v\,dx + u\,dy, \end{aligned}\right\} \tag{2.1}$$

where u and v are the components of velocity; the y-axis is taken to be vertically upwards. Writing

$$x+iy = z, \quad \phi + i\psi = \chi,$$

we have

$$u - iv = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{d\chi}{dz}.$$

Let q and θ denote the magnitude and direction of the velocity, i.e.

$$(u, v) = (q\cos\theta,\ q\sin\theta);$$

then if τ is defined by

$$q = c\,e^{\tau},$$

we have also

$$u - iv = q\,e^{-i\theta} = c\,e^{\tau - i\theta} = c\,e^{\zeta}, \tag{2.2}$$

so that

$$\zeta = \tau - i\theta$$

is a regular function of $(u-iv)$ and hence of χ.

We take ϕ and ψ as co-ordinates, and attempt to find z and ζ in terms of them.

It will be noticed that the curvature κ along any streamline $\psi = \text{const.}$ is given by

$$\kappa = \frac{\partial\theta}{\partial s} = \frac{\partial\theta}{\partial\phi}\frac{\partial\phi}{\partial s} = q\frac{\partial\theta}{\partial\phi},$$

where s denotes arc length. And since $\partial\theta/\partial\phi = \partial\tau/\partial\psi$ we have

$$\kappa = c\,e^{\tau}\frac{\partial\tau}{\partial\psi} = c\frac{\partial}{\partial\psi}(e^{\tau}) = \frac{\partial q}{\partial\psi}. \tag{2.3}$$

Boundary conditions

It is assumed that the depth of water is effectively infinite, so that as $y \to -\infty$, so $(u-iv) \to c$ and $\zeta \to 0$; i.e.

$$\zeta \to 0 \quad \text{when} \quad y \to -\infty. \tag{2.4}$$

The free surface, being a streamline, may be chosen as $\psi = 0$, and we also take $\phi = 0$ at the crest of the wave. From Bernoulli's equation we have then

$$\frac{p}{\rho} + \tfrac{1}{2}q^2 + gy = \text{const.} \quad \text{when} \quad \psi = 0. \tag{2.5}$$

At the free surface the pressure p is given by

$$p = \text{const.} - T\kappa,$$

where T is the surface tension. Thus writing $T/\rho = T'$ and substituting in (2.5) we have

$$\tfrac{1}{2}q^2 + gy - T'\frac{\partial q}{\partial\psi} = \text{const.} \quad \text{when} \quad \psi = 0, \tag{2.6}$$

that is to say

$$\tfrac{1}{2}c^2 e^{2\tau} + gy - cT'\frac{\partial}{\partial\psi}(e^{\tau}) = \text{const.} \tag{2.7}$$

or

$$\tfrac{1}{2}e^{2\tau} + \frac{g}{c^2}y - \frac{T'}{c}\frac{\partial}{\partial\psi}(e^{\tau}) = \text{const.} \tag{2.8}$$

The left-hand side may be differentiated with respect to ϕ. Since

$$\frac{\partial y}{\partial\phi} = \frac{1}{q^2}\frac{\partial\phi}{\partial y} = \frac{1}{q^2}.q\sin\theta = \mathscr{R}\frac{1}{ic}e^{-(\tau-i\theta)}, \tag{2.9}$$

we have then

$$\frac{\partial}{\partial\phi}\left[\tfrac{1}{2}e^{2\tau} - \frac{T'}{c}\frac{\partial}{\partial\psi}(e^{\tau})\right] + \mathscr{R}\frac{g}{ic^3}e^{-(\tau-i\theta)} = 0 \quad (\psi = 0). \tag{2.10}$$

3. Method of approximation

In the zero-order approximation we neglect the surface tension entirely and assume that the flow corresponds to a pure gravity wave of finite amplitude. Let all quantities referring to this basic flow be denoted by a suffix 0. Thus from (2.6)

$$\tfrac{1}{2}q_0^2 + gy_0 = \text{const.} \quad (\psi = 0), \tag{3.1}$$

and from (2.10)

$$\frac{\partial}{\partial\phi}(\tfrac{1}{2}e^{2\tau_0}) + \mathscr{R}\frac{g}{ic^3}e^{-(\tau_0-i\theta_0)} = 0 \quad (\psi = 0).$$

Dividing by $e^{2\tau_0}$ we have then

$$\frac{\partial \tau_0}{\partial \phi}+\frac{g}{c^3}e^{-3\tau_0}\sin\theta_0 = 0 \quad (\psi = 0), \tag{3.2}$$

the surface condition obtained by Levi-Civita (1925) for pure gravity waves. Also

$$\tau_0 - i\theta_0 \to 0 \quad \text{as} \quad \psi \to -\infty. \tag{3.3}$$

Now let us write $\quad \zeta = \zeta_0+\zeta_1, \quad q = q_0+q_1, \quad$ etc.,

in the equations of § 2, where ζ_0, q_0, etc., represent the basic gravity wave and ζ_1, q_1, etc., represent the perturbation due to surface tension. Squares and higher powers of the perturbation terms will be neglected.

The boundary condition (2.6) becomes

$$(\tfrac{1}{2}q_0^2+q_0q_1)+g(y_0+y_1)-T'\left(\frac{\partial q_0}{\partial \psi}+\frac{\partial q_1}{\partial \psi}\right) = \text{const.},$$

and subtracting from this the boundary condition (3.1) for the basic flow we have

$$q_0q_1+gy_1-T'\frac{\partial q_1}{\partial \psi} = T'\frac{\partial q_0}{\partial \psi}+\text{const.} \tag{3.4}$$

It will be noticed that the term $T'\,\partial q_0/\partial\psi$, which was neglected in the zero-order approximation, becomes a forcing function for the perturbation. Similarly, from the differentiated form of the boundary condition, equation (2.10) we have

$$\frac{\partial}{\partial \phi}\left[e^{2\tau_0}\tau_1-\frac{T'}{c}\frac{\partial}{\partial \psi}(e^{\tau_0}\tau_1)\right]-\mathscr{R}\left[\frac{g}{ic^3}e^{-(\tau_0-i\theta_0)}(\tau_1-i\theta_1)\right] = \frac{T'}{c}\frac{\partial^2}{\partial\phi\,\partial\psi}(e^{\tau_0}) \quad (\psi = 0). \tag{3.5}$$

Also

$$\tau_1 - i\theta_1 \to 0 \quad \text{as} \quad \psi \to -\infty. \tag{3.6}$$

If the perturbations are to be small relative to the basic flow it is necessary that the surface tension term $T'\,\partial q_0/\partial\psi$ be small compared to other terms in the boundary conditions (3.1). So in particular we must have

$$T'\,\partial q_0/\partial\psi \ll \tfrac{1}{2}q_0^2,$$

that is

$$T'\frac{\partial}{\partial\psi}(c\,e^{\tau_0}) \ll \tfrac{1}{2}c^2\,e^{2\tau_0},$$

or

$$\frac{T'}{c}\frac{\partial \tau_0}{\partial \psi} \ll \tfrac{1}{2}e^{\tau_0}. \tag{3.7}$$

4. The zero-order approximation

The expansion adopted by Levi-Civita (1925) for solving equation (3.2) is suitable only for waves of small steepness, when τ_0 and $\theta_0 \ll 1$. For waves of finite steepness the approach suggested by Davies (1951) appears more suitable. Davies noted that if in the second term $\sin\theta_0$ is replaced by $\frac{1}{3}l\sin 3\theta_0$ where l is a constant one has

$$\frac{\partial \tau_0}{\partial \phi}+\frac{lg}{3c^3}e^{-3\tau_0}\sin 3\theta_0 = 0 \quad (\psi = 0), \tag{4.1}$$

i.e.

$$\mathscr{R}\left[\frac{\partial \zeta_0}{\partial \chi}+\frac{lg}{3ic^3}e^{-3\zeta_0}\right] = 0 \quad (\psi = 0),$$

of which an exact solution symmetrical about $\phi = 0$ is

$$e^{3\zeta_0} = 1 - A\, e^{-im\chi}, \tag{4.2}$$

where A is a real constant lying between 0 and 1 and

$$m = lg/c^3. \tag{4.3}$$

Thus
$$u_0 - iv_0 = c\, e^{\zeta_0} = c(1 - A\, e^{-im\chi})^{\frac{1}{3}}. \tag{4.4}$$

This approximate expression is also the first term in a series found by Havelock (1918).

The highest wave

In the limiting case when $A = 1$ equation (4.4) becomes

$$u_0 - iv_0 = c(1 - e^{-im\chi})^{\frac{1}{3}}. \tag{4.5}$$

In the neighbourhood of the wave crest, when $m\chi$ is small, the right-hand side reduces to $c(im\chi)^{\frac{1}{3}}$. Thus we have

$$\frac{d\chi_0}{dz} = u_0 - iv_0 \doteqdot c(im\chi)^{\frac{1}{3}},$$

whence
$$cz \doteqdot \int_0^{\chi} \frac{d\chi}{(im\chi)^{\frac{1}{3}}} = \frac{3}{2}\frac{\chi^{\frac{2}{3}}}{(im)^{\frac{1}{3}}},$$

and
$$\chi = (im)^{\frac{1}{2}}\,(\tfrac{2}{3}cz)^{\frac{3}{2}}. \tag{4.6}$$

This is equivalent to Stokes's 120° angle solution, satisfying the exact boundary condition

$$\frac{1}{2}\left|\frac{d\chi}{dz}\right|^2 + gy = 0 \quad (\arg z = -\tfrac{1}{2}\pi \pm \tfrac{1}{3}\pi),$$

provided that we take

$$m = \frac{3}{2}\frac{g}{c^3}, \tag{4.7}$$

i.e. $l = \frac{3}{2}$ in equation (4.3). So the approximation is exact in this limiting case.

The almost-highest wave

Since θ_0 generally lies between $\pm\frac{1}{6}\pi$ one would expect that the appropriate value of l should lie somewhere between 1 and $\frac{3}{2}$. Certainly the value 1 is appropriate for very low waves, when θ_0 is uniformly small. On the other hand since we are dealing with steep waves where $\theta_0 = \pm\frac{1}{6}\pi$ at the two extremes of the flow it seems preferable to take $l = \frac{3}{2}$. The relative error introduced by this assumption may be expected to be of order

$$(\tfrac{3}{2} - 1)/(\tfrac{3}{2} + 1) = 20\,\%.$$

Both Havelock (1918) and Davies (1951) obtained solutions in the form of infinite series which satisfy the exact boundary condition (3.2). Thus Davies shows that

$$e^{3\zeta_0} = 1 - A\, e^{-im\chi} + \tfrac{1}{54}A^3\, e^{-3im\chi} - A^4(\tfrac{1}{27}e^{-2im\chi} - \tfrac{1}{81}e^{-4im\chi}) - A^5(\tfrac{1}{108}e^{-3im\chi} - \tfrac{5}{648}e^{-5im\chi}) + \ldots,$$

where
$$m = \frac{g}{c^3}(1 - \tfrac{1}{9}A^2 - \tfrac{5}{162}A^4 - \ldots)^{-1},$$

and A is a real parameter lying between 0 and about 0·992. However, though the coefficients in this series do appear to be diminishing fairly rapidly it is evident that no finite number of terms can give an adequate representation of the flow in the neighbourhood of the crest of a steep wave (except possibly in the limiting case), since θ_0 is a rapidly varying function there, which cannot be well described by a finite number of harmonics. Indeed, one may expect a 'Gibbs' phenomenon' of the slope θ_0 which would render the partial sum of such a series misleading in that neighbourhood. This is not to say that the series does not give a fair representation of the wave surface over the remaining part of the wave.

Similar comments apply to Havelock's (1918) expansion which, though different from Davies's, still has the form of a Fourier series.

Accordingly, in approximating ζ_0 we shall adopt equation (4.4) with $m = 3g/2c^3$, recognizing that there may be errors of order 20 % in the final solution. Let us write

$$A = 1-\delta, \tag{4.8}$$

where δ is a small quantity. From (4.4) we have

$$e^{\tau_0 - i\theta_0} \doteqdot [1-(1-\delta)\,e^{-im\chi}]^{\frac{1}{3}}. \tag{4.9}$$

In the neighbourhood of the crest, where $m\chi$ is comparable to δ, we have

$$e^{\tau_0 - i\theta_0} \doteqdot (\delta + im\chi)^{\frac{1}{3}},$$

and so

$$\left.\begin{aligned} e^{\tau_0} &\doteqdot [(\delta - m\psi)^2 + (m\phi)^2]^{\frac{1}{6}}, \\ \theta_0 &\doteqdot -\tfrac{1}{3}\tan^{-1}\left(\frac{m\phi}{\delta - m\psi}\right). \end{aligned}\right\} \tag{4.10}$$

At the free surface $\psi = 0$ we have

$$\left.\begin{aligned} e^{\tau_0} &\doteqdot (\delta^2 + m^2\phi^2)^{\frac{1}{6}}, \\ \theta_0 &\doteqdot -\tfrac{1}{3}\tan^{-1}(m\phi/\delta). \end{aligned}\right\} \tag{4.11}$$

Also

$$\left.\begin{aligned} \frac{\partial \tau_0}{\partial \psi} &\doteqdot -\frac{m\delta}{3(\delta^2 + m^2\phi^2)}, \\ \kappa_0 = c\,e^{\tau_0}\frac{\partial \tau_0}{\partial \psi} &\doteqdot -\frac{cm\delta}{3(\delta^2 + m^2\phi^2)^{\frac{5}{6}}}. \end{aligned}\right\} \tag{4.12}$$

At the crest of the wave, where $\phi = 0$, we have

$$\left.\begin{aligned} q_0 = c\,e^{\tau_0} &\doteqdot c\delta^{\frac{1}{3}}, \\ \kappa_0 \doteqdot -\frac{cm}{3\delta^{\frac{2}{3}}} &= -\frac{g}{2c^2\delta^{\frac{2}{3}}}, \end{aligned}\right\} \tag{4.13}$$

since $m = 3g/2c^3$. Hence the vertical acceleration of the fluid at the crest is given by

$$\kappa_0 q_0^2 \doteqdot -\tfrac{1}{2}g, \tag{4.14}$$

which is the limiting value for the Stokes 120° angle (see Appendix).

When $m\chi$ is no longer small, that is to say at distances from the crest which are comparable to a wavelength, we find from (4.4) when $\psi = 0$

$$\left.\begin{aligned} e^{\tau_0} &\doteqdot (1-2A\cos m\phi + A^2)^{\frac{1}{6}}, \\ \theta_0 &\doteqdot -\tfrac{1}{3}\tan^{-1}\left(\frac{A\sin m\phi}{1-A\cos m\phi}\right), \\ \frac{\partial \tau_0}{\partial \psi} &\doteqdot -\frac{1}{3}\frac{mA(\cos m\phi - A)}{(1-2A\cos m\phi + A^2)}. \end{aligned}\right\} \tag{4.15}$$

Putting $A = 1-\delta$ we find in particular

$$e^{\tau_0} \doteqdot [\delta^2 + 4(1-\delta)\sin^2\tfrac{1}{2}m\phi]^{\frac{1}{6}}. \tag{4.16}$$

5. The perturbation solution

We now seek a solution to the perturbation equations (3.4) to (3.6). (In equations (3.4) and (3.5) it is assumed that q_0, y_0 and τ_0 represent the exact gravity wave solution; only after τ_1 and θ_1 have been found do we make use of the approximate expressions found in § 4.)

Since we expect a solution in the form of a capillary wave, we shall in the first place neglect the gravitational terms in (3.4) and (3.5) in comparison to the surface tension term. The effect of this assumption will be discussed later. Equation (3.4) then becomes simply

$$q_0 q_1 - T'\frac{\partial q_1}{\partial \psi} = T'\frac{\partial q_0}{\partial \psi} \tag{5.1}$$

(a constant term on the right-hand side is omitted as having no effect on the motion). Since $q_0 = c\,e^{\tau_0}$ and $q_1 = c\,e^{\tau_0}\tau_1$ this equation becomes

$$e^{2\tau_0}\tau_1 - \frac{T'}{c}\frac{\partial}{\partial \psi}(e^{\tau_0}\tau_1) = \frac{T'}{c}\frac{\partial}{\partial \psi}(e^{\tau_0}), \tag{5.2}$$

or

$$\left(e^{\tau_0} - \frac{T'}{c}\frac{\partial \tau_0}{\partial \psi}\right)\tau_1 - \frac{T'}{c}\frac{\partial \tau_1}{\partial \psi} = \frac{T'}{c}\frac{\partial \tau_0}{\partial \psi}. \tag{5.3}$$

It is convenient to write this as

$$P(\phi)\,\tau_1 - \frac{T'}{c}\frac{\partial \tau_1}{\partial \psi} = \frac{T'}{c}Q(\phi) \qquad (\psi = 0), \tag{5.4}$$

where

$$\left.\begin{aligned} P(\phi) &= e^{\tau_0} - \frac{T'}{c}\frac{\partial \tau_0}{\partial \psi}, \\ Q(\phi) &= \frac{\partial \tau_0}{\partial \psi}. \end{aligned}\right\} \tag{5.5}$$

The function τ_1 has also to satisfy Laplace's equation and the condition

$$\tau_1 \to 0 \quad \text{as} \quad \psi \to -\infty. \tag{5.6}$$

Now let us make the conformal transformation

$$\alpha + i\beta = \int_0^{\phi + i\psi} P(\chi)\,d\chi, \tag{5.7}$$

where $P(\chi)$ is a function of the complex variable $\chi = \phi + i\psi$ which is equal to $P(\phi)$ on the real axis $\psi = 0$. Then on $\psi = 0$ we have $\beta = 0$ for all ϕ and so also $\partial\beta/\partial\phi = 0$. Hence

$$\begin{aligned} \frac{\partial\tau_1}{\partial\psi} &= \frac{\partial\tau_1}{\partial\alpha}\frac{\partial\alpha}{\partial\psi} + \frac{\partial\tau_1}{\partial\beta}\frac{\partial\beta}{\partial\psi}, \\ &= -\frac{\partial\tau_1}{\partial\alpha}\frac{\partial\beta}{\partial\phi} + \frac{\partial\tau_1}{\partial\beta}\frac{\partial\alpha}{\partial\phi}, \\ &= P(\phi)\frac{\partial\tau_1}{\partial\beta}, \end{aligned} \tag{5.8}$$

when $\psi = 0$. The condition (5.4) thus reduces to

$$P(\phi)\left(\tau_1 - \frac{T'}{c}\frac{\partial\tau_1}{\partial\beta}\right) = \frac{T'}{c}Q(\phi) \qquad (\psi = 0), \tag{5.9}$$

that is

$$\frac{c}{T'}\tau_1 - \frac{\partial\tau_1}{\partial\beta} = R(\alpha) \qquad (\beta = 0), \tag{5.10}$$

where

$$R(\alpha) = Q(\phi)/P(\phi). \tag{5.11}$$

Also τ_1 is to satisfy Laplace's equation in the co-ordinates α, β and

$$\tau_1 \to 0 \quad \text{as} \quad \beta \to -\infty, \tag{5.12}$$

provided that $\beta \to -\infty$ as $\psi \to -\infty$.

Now $R(\alpha)$ is an even function of ϕ and so of α. Let us define

$$r(t) = \int_{-\infty}^{\infty} R(\alpha)\, e^{i\alpha t}\, d\alpha; \tag{5.13}$$

then $r(t)$ is a real, even function of t and

$$\begin{aligned} R(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} r(t)\, e^{-i\alpha t}\, dt \\ &= \frac{1}{\pi}\int_0^{\infty} r(t)\, e^{-i\alpha t}\, dt. \end{aligned} \tag{5.14}$$

On substituting this expression in (5.13) we see that

$$\tau_1 = -\mathscr{R}\frac{1}{\pi}\int_0^{\infty} \frac{r(t)\, e^{-it(\alpha+i\beta)}}{t - c/T'}\, dt \tag{5.15}$$

is a solution of the equations. So, since $\tau_1 - i\theta_1$ is a regular function of $\alpha + i\beta$ we have

$$\tau_1 - i\theta_1 = -\frac{1}{\pi}\int_0^{\infty} \frac{r(t)\, e^{-it(\alpha+i\beta)}}{t - c/T'}\, dt, \tag{5.16}$$

where $r(t)$ is given by (5.13). Since $R(\alpha)$ is a real, even function of α we have also

$$r(t) = 2\int_0^{\infty} R(\alpha)\cos\alpha t\, d\alpha. \tag{5.17}$$

In (5.16) the contour of integration is to be taken to pass above the singularity at $t = c/T'$.

We are interested especially in the behaviour of this solution for large values of $|\xi|$ and so of $|\alpha|$. If the function $r(t)$ is suitably bounded at infinity, the chief contribution to the integrand will arise from the residue at $t = c/T'$, and will be given by

$$\tau_1 - i\theta_1 \sim \begin{cases} 2ir(c/T')\, e^{-i(c/T')(\alpha+i\beta)}, & \alpha < 0, \\ 0, & \alpha > 0. \end{cases} \tag{5.18}$$

In particular when $\alpha < 0$ we have

$$\tau_1 - i\theta_1 \sim -ib\, e^{-i(c/T')(\alpha+i\beta)}, \tag{5.19}$$

where

$$\begin{aligned} b &= -2r(c/T') \\ &= -4\int_0^\infty R(\alpha)\cos(\alpha c/T')\,d\alpha \\ &= -4\int_0^\infty \frac{Q(\phi)}{P(\phi)}\cos(\alpha c/T')\frac{\partial\alpha}{\partial\phi}\,d\phi. \end{aligned}$$

Since $\partial\alpha/\partial\phi = P(\phi)$ when $\beta = 0$, and $Q(\phi) = \partial\tau_0/\partial\psi$ this last expression becomes

$$b = -4\int_0^\infty \frac{\partial\tau_0}{\partial\psi}\cos(\alpha c/T')\,d\phi, \tag{5.20}$$

where

$$\alpha = \int_0^\phi P(\phi)\,d\phi. \tag{5.21}$$

We note that in the expression for $P(\phi)$ in (5.5) the ratio of the second term to the first is small, by (3.7); so that to this approximation we have

$$P(\phi) \doteqdot e^{\tau_0}, \tag{5.22}$$

and hence

$$\alpha \doteqdot \int_0^\phi e^{\tau_0}\,d\phi. \tag{5.23}$$

6. Discussion

The solution (5.19) represents a wave upstream of the crest, i.e. on the forward face of the gravity wave. It is stationary with respect to the moving co-ordinate system; hence the phase velocity with respect to the surrounding medium is equal to $-q_0$.

Now from (5.19) the wave-number k_c is given by

$$k_c = \frac{\partial}{\partial s}(\alpha c/T') \doteqdot q_0\frac{\partial}{\partial\phi}(\alpha c/T'). \tag{6.1}$$

But

$$\partial\alpha/\partial\phi = P(\phi) \doteqdot e^{\tau_0}$$

as we have just seen. Hence

$$k_c \doteqdot q_0(c/T')\,e^{\tau_0} = q_0^2/T'$$

or

$$q_0 = (T'k_c)^{\frac{1}{2}}. \tag{6.2}$$

But $\pm(T'k_c)^{\frac{1}{2}}$ is the classical expression for the velocity of free capillary waves, of small amplitude (Lamb 1932). So it appears that to this approximation the waves at some distance from the crest are free capillary waves.

Capillary waves in deep water have a group velocity equal to $\frac{3}{2}$ times their phase velocity. The fact that the group velocity exceeds the phase velocity, while the phase velocity is minus the velocity of the opposing stream, explains why the waves are found up-stream of the source of energy. (See, for example, Lamb 1932, § 269.)

The variation of the wave *amplitude* may be derived by considering its relation to the local energy density E. If a_c denotes the amplitude and k_c the wave-number of a capillary wave on a locally uniform stream, the amplitude of the surface slope θ' is

$$|\theta'| = a_c k_c, \tag{6.3}$$

and the potential energy per unit distance, which equals the work done in stretching the surface against surface tension is $\frac{1}{4}T\,|\theta'|^2$. The total energy density is twice this, or

$$E = \tfrac{1}{2}T\,|\theta'|^2. \tag{6.4}$$

To interpret the variation of E, and hence $|\theta'|$, with distance s along the surface, consider the balance of capillary wave energy. By analogy with known results for gravity waves on slightly non-uniform currents (Longuet-Higgins & Stewart 1961), and other types of wave motion, one expects that the capillary wave energy density will satisfy an equation of the form

$$\frac{\partial}{\partial s}[E(c_g+q_0)]+S_x\frac{\partial q_0}{\partial s} = 0, \tag{6.5}$$

where c_g denotes the group-velocity of the capillary waves and S_x is an interaction coefficient between the waves and current, called the *radiation stress.* Essentially S_x represents a mean transfer of momentum across a fixed plane normal to the undisturbed surface. In the case of capillary waves it is found† that

$$S_x = \tfrac{3}{2}T\,|\theta'|^2 = \tfrac{3}{2}E.$$

Substituting $c_g = -\frac{3}{2}q_0$ and $S_x = \frac{3}{2}E$ in equation (6.5) we have

$$\frac{\partial}{\partial s}[-\tfrac{1}{2}Eq_0]+\tfrac{3}{2}E\frac{\partial q_0}{\partial s} = 0. \tag{6.6}$$

Hence

$$-\frac{1}{2}\frac{\partial E}{\partial s}q_0+E\frac{\partial q_0}{\partial s} = 0,$$

and so

$$\frac{1}{E}\frac{\partial E}{\partial s} = \frac{2}{q_0}\frac{\partial q_0}{\partial s}. \tag{6.7}$$

Therefore

$$E \propto q_0^2, \tag{6.8}$$

and since $E \propto |\theta'|^2$ it follows that

$$|\theta'| \propto q_0. \tag{6.9}$$

In other words there is a gradual variation of the steepness of free capillary waves along the surface; the wave steepness is proportional to the magnitude of the underlying current.

In equation (5.19), on the other hand, the steepness of the capillary waves is equal to b, a constant. It will now be shown that this discrepancy is due to our previous neglect of the gravitational terms in the free surface condition (3.4).

† R. W. Stewart, private communication.

In order to eliminate y_1 from the free surface condition we must use the differentiated form, equation (3.5). It will suffice to consider only *free* waves in which the right-hand side of (3.5) is replaced by zero. Thus, let us take

$$\frac{\partial}{\partial\phi}\left[e^{2\tau_0}\tau_1-\frac{T'}{c}\frac{\partial}{\partial\psi}(e^{\tau_0}\tau_1)\right]-\mathscr{R}\left[\frac{g}{ic^3}e^{-(\tau_0-i\theta_0)}(\tau_1-i\theta_1)\right]=0 \quad (\psi=0). \quad (6.10)$$

By the same transformation of co-ordinates as in (5.7) this becomes

$$\frac{\partial}{\partial\phi}\left[e^{\tau_0}P(\phi)\left(\tau_1-\frac{T'}{c}\frac{\partial\tau_1}{\partial\beta}\right)\right]-\mathscr{R}\left[\frac{g}{ic^3}e^{-(\tau_0-i\theta_0)}(\tau_1-i\theta_1)\right]=0. \quad (6.11)$$

Now let us write
$$\tau_1-i\theta_1=S\,e^{-i(c/T')(\alpha+i\beta)}, \quad (6.12)$$

where S is, by hypothesis, a complex amplitude which varies slowly compared to the phase in the exponent. Then we have when $\psi=\beta=0$

$$\begin{aligned}P(\phi)\left(\tau_1-\frac{T'}{c}\frac{\partial\tau_1}{\partial\beta}\right)&=-\mathscr{R}P(\phi)\frac{T'}{c}\frac{\partial S}{\partial\beta}e^{-i(c/T')\alpha}\\&=-\mathscr{R}\frac{T'}{c}\frac{\partial S}{\partial\psi}e^{-i(c/T')\alpha}\\&=\mathscr{R}\frac{T'}{ic}\frac{\partial S}{\partial\phi}e^{-i(c/T')\alpha}.\end{aligned}$$

Substituting in (6.11) gives

$$\mathscr{R}\left\{\frac{\partial}{\partial\phi}\left[e^{\tau_0}\frac{T'}{ic}\frac{\partial S}{\partial\phi}e^{-i(c/T')\alpha}\right]-\frac{g}{ic^3}e^{-(\tau_0-i\theta_0)}S\,e^{-i(c/T')\alpha}\right\}=0.$$

Since the exponent varies rapidly compared to S we may carry out the differentiation with respect to ϕ in the exponent only, giving

$$\mathscr{R}\left\{\left[e^{\tau_0}P(\phi)\frac{\partial S}{\partial\phi}+\frac{g}{ic^3}e^{-(\tau_0-i\theta_0)}S\right]e^{-i(c/T')\alpha}\right\}\doteqdot 0.$$

To the same approximation, the factor multiplying the exponential must vanish. Hence

$$\frac{1}{S}\frac{\partial S}{\partial\phi}\doteqdot\frac{ig\,e^{-2\tau_0}}{c^3P(\phi)}e^{i\theta_0}. \quad (6.13)$$

Taking the real part of each side and replacing $P(\phi)$ by e^{τ_0} we have

$$\mathscr{R}\frac{\partial}{\partial\phi}(\log S)\doteqdot-\frac{g}{c^3}e^{-3\tau_0}\sin\theta_0=\frac{\partial\tau_0}{\partial\phi},$$

the last step following from equation (3.2). Integration now gives

$$\mathscr{R}\log S\doteqdot\tau_0+\text{const.}$$

Hence
$$|S|\propto e^{\tau_0}. \quad (6.14)$$

From (5.12) we have now, when $\beta=0$,

$$|\theta_1|\propto e^{\tau_0}\propto q_0, \quad (6.15)$$

so that in this, higher, approximation the wave steepness is indeed proportional to the stream velocity.

Consider now the effect of neglecting the gravitational term in the non-homogeneous perturbation, equation (3.4) or (3.5). In equation (3.4), for example, the forcing term is the surface tension pressure $-T'\kappa_0$. We saw in § 4 that near to the crest of the gravity wave this term varies as $(\delta^2+m^2\phi^2)^{-\frac{5}{8}}$. On the other hand, the velocity q_0 of the underlying stream varies as $(\delta^2+m^2\phi^2)^{\frac{1}{8}}$ or

$$q_0 \propto (T'\kappa_0)^{-\frac{1}{5}}. \tag{6.16}$$

Thus over the important range of variation of the perturbing pressure the velocity q_0 varies comparatively little. Since the capillary wave amplitude is, for free waves, proportional to q_0, we expect that the effect on the wave amplitude at the edge of the forcing zone is affected comparatively little by the neglect of the gravitational terms. Thus the capillary wave amplitude near the crest is still given approximately by (5.19) and (5.20). Only at some distance from the crest does the variation in current q_0 become important.

This reasoning suggests that under the condition stated, and in the absence of viscosity, the steepness of the capillary waves is given approximately by

$$|\theta_1| = \frac{bq_0}{(q_0)_{\text{crest}}}, \tag{6.17}$$

where b is given by (5.20) and $(q_0)_{\text{crest}}$ denotes the velocity at the crest of the wave.

7. The effect of viscosity

So far the viscosity of the fluid has been neglected. One may expect that its direct effect is mainly on the short capillary waves, where the velocity gradient and hence the loss of energy, is comparatively great.

The rate of energy dissipation by viscosity in a capillary wave in deep water (see, for example, Lamb 1932, § 347) is given by

$$\partial E/\partial t = 4\nu k_c^2 E = 4\nu(q_0^2/T')^2 E. \tag{7.1}$$

Including this term in the energy equation (6.5) we have

$$\frac{\partial}{\partial s}[E(c_g+q_0)]+S_x\frac{\partial q_0}{\partial s}+\frac{4\nu}{T'^2}q_0^4E = 0.$$

Substituting for c_g and S_x the same values as before we obtain

$$-\frac{1}{2}\frac{\partial E}{\partial s}q_0+E\frac{\partial q_0}{\partial s}+\frac{4\nu}{T'^2}q_0^4E = 0,$$

and so

$$\frac{1}{E}\frac{\partial E}{\partial s} = \frac{2}{q_0}\left[\frac{\partial q_0}{\partial s}+\frac{4\nu}{T'^2}q_0^4\right]. \tag{7.2}$$

Since $\partial/\partial s = q_0\partial/\partial\phi$ we have

$$\frac{1}{E}\frac{\partial E}{\partial \phi} = \frac{2}{q_0}\frac{\partial q_0}{\partial \phi}+\frac{8\nu}{T'^2}q_0^2. \tag{7.3}$$

On the forward face of the wave $\partial q_0/\partial\phi$ is negative and hence for small values of q_0, E will tend to increase with distance away from the crest. But as q_0 increases

E may tend ultimately to decrease away from the crest. There will be a maximum wave steepness where

$$\frac{\partial q_0}{\partial \phi} = -\frac{4\nu}{T'^2} q_0^3. \tag{7.4}$$

Equation (7.3) may be integrated to give

$$\log E = 2 \log q_0 + \frac{8\nu}{T'^2} \int_0^{\phi} q_0^2 d\phi + \text{const.},$$

and so

$$E \propto q_0^2 \exp\left[\frac{8\nu}{T'^2} \int_0^{\phi} q_0^2 d\phi\right]. \tag{7.5}$$

Hence

$$|\theta_1| \propto q_0 \exp\left[\frac{4\nu}{T'^2} \int_0^{\phi} q_0^2 d\phi\right]. \tag{7.6}$$

Adjusting the constant of proportionality so that $|\theta_1| = b$ at the crest we have finally

$$|\theta_1| = \frac{b q_0}{(q_0)_{\text{crest}}} \exp\left[\frac{4\nu}{T'^2} \int_0^{\phi} q_0^2 d\phi\right]. \tag{7.7}$$

8. Explicit formulae for $|\theta_1|$

Up to this point the explicit expressions for the zero-order approximation which were obtained in § 4 have not been used. In (4.11) and (4.12) let us introduce the non-dimensional co-ordinates

$$(\xi, \eta) = \frac{m}{\delta} (\phi, \psi), \tag{8.1}$$

so that near to the wave crest

$$\left.\begin{aligned} e^{\tau_0} &= \delta^{\frac{1}{3}} (1+\xi^2)^{\frac{1}{6}}, \\ \frac{\partial \tau_0}{\partial \psi} &= -\frac{1}{3} \frac{m}{\delta(1+\xi^2)}. \end{aligned}\right\} \tag{8.2}$$

Then from (5.23) we have

$$\alpha = \frac{\delta^{\frac{4}{3}}}{m} \alpha', \tag{8.3}$$

where

$$\alpha' = \int_0^{\xi} (1+\xi^2)^{\frac{1}{3}} d\xi, \tag{8.4}$$

and from (5.20)†

$$b = \frac{4}{3} \int_0^{\infty} \frac{\cos \lambda \alpha'}{1+\xi^2} d\xi, \tag{8.5}$$

where λ is the non-dimensional parameter

$$\lambda = \frac{c \delta^{\frac{4}{3}}}{m T'}. \tag{8.6}$$

Since $m = 3g/2c^3$ this parameter can also be expressed as

$$\lambda = \frac{2c^4 \delta^{\frac{4}{3}}}{3gT'}. \tag{8.7}$$

† It can be shown that the conditions for $P(\phi)$ and $Q(\phi)$ in § 5 are satisfied.

If we denote by K the absolute value of the curvature at the crest,

$$K = \frac{mc}{3\delta^{\frac{2}{3}}} = \frac{g}{2c^2\delta^{\frac{4}{3}}}, \tag{8.8}$$

then we have also

$$\lambda = \frac{g}{6K^2T'}. \tag{8.9}$$

Equation (8.5) expresses the initial steepness b as a function of λ only. The integral was computed, and is shown in figure 2. For small values of λ, since $(1+\xi^2)^{\frac{1}{3}}$ does not differ greatly from unity, (8.4) shows that $\alpha' \doteqdot \xi$ and so we expect

$$b \doteqdot \frac{4}{3}\int_0^\infty \frac{\cos\lambda\xi}{1+\xi^2}\,d\xi = \frac{2\pi}{3}e^{-\lambda}. \tag{8.10}$$

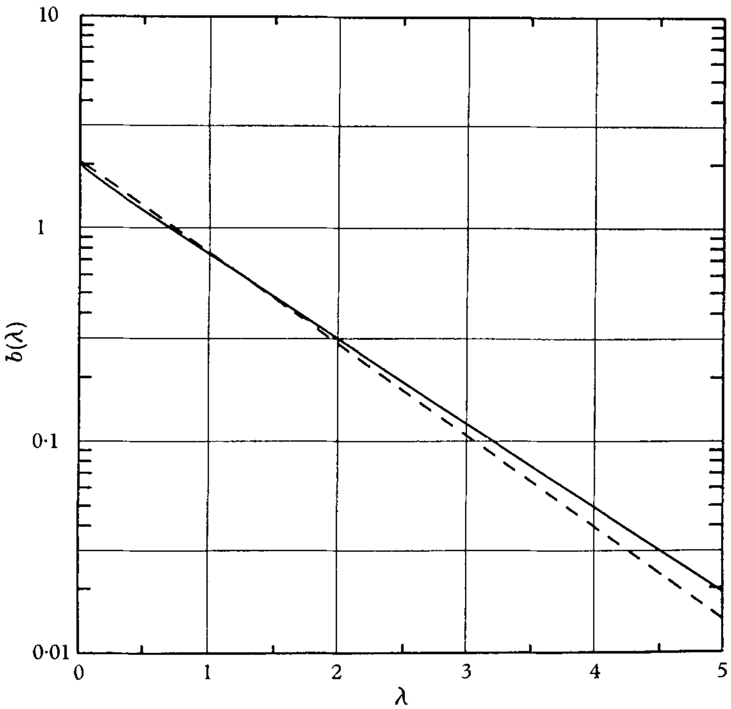

FIGURE 2. The solid curve represents $b(\lambda)$, giving the steepness of the capillary waves near to the crest of the gravity wave, as a function of $\lambda = g/6T'K^2$. The broken line represents the approximate value $(2\pi/3)\,e^{-\lambda}$.

The function $(2\pi/3)\,e^{-\lambda}$ is also shown in figure 2, compared with the computed value of b. It is convenient also to write

$$b = \frac{2\pi}{3}e^{-\lambda} \times B(\lambda), \tag{8.11}$$

where $B(\lambda)$ is a function of order unity. Some computed values of $B(\lambda)$ are given in table 1.

It will be seen how critically b depends upon λ, which in turn is inversely proportional to K^2.

The *variation* in steepness of the capillary waves is found by substitution in (7.7). At least near the crest of the wave we have

$$|\theta_1| = b(1+\xi^2)^{\frac{1}{6}} \exp\left[\epsilon \int_0^{\xi} (1+\xi^2)^{\frac{1}{3}}\, d\xi\right], \tag{8.12}$$

where ϵ is a non-dimensional parameter depending on the viscosity:

$$\epsilon = \frac{4\nu}{T'^2}\frac{c^2\delta^{\frac{5}{3}}}{m} = \frac{4\lambda\nu c\delta^{\frac{1}{3}}}{T'} = \frac{4\lambda\nu}{T'}\left(\frac{g}{2K}\right)^{\frac{1}{2}}.$$

λ	$B(\lambda)$	λ	$B(\lambda)$
0·0	1·0000	4·5	1·3053
0·2	0·9543	5·0	1·3630
0·4	0·9538	5·5	1·4236
0·6	0·9606	6·0	1·4871
0·8	0·9707	6·5	1·5537
1·0	0·9827	7·0	1·6234
1·5	1·0180	7·5	1·6964
2·0	1·0582	8·0	1·7726
2·5	1·1019	8·5	1·8519
3·0	1·1486	9·0	1·9341
3·5	1·1981	9·5	2·0195
4·0	1·2503	10·0	2·1086

TABLE 1. Values of $B(\lambda) = \frac{2}{\pi} e^{\lambda} \int^{\infty} \frac{\cos \lambda\alpha'}{1+\xi^2}\, d\xi$

Eliminating K from this expression by means of (8.9) we have

$$\epsilon = 4(\tfrac{3}{2})^{\frac{1}{4}} \lambda^{\frac{5}{4}} (\nu g^{\frac{1}{4}}/T'^{\frac{3}{4}}). \tag{8.13}$$

Taking $\nu = 0{\cdot}0178$, $g = 981$, $T' = 74$ c.g.s. units we have

$$\nu g^{\frac{1}{4}}/T'^{\frac{3}{4}} = 3{\cdot}95 \times 10^{-3}, \tag{8.14}$$

a dimensionless quantity. So

$$\epsilon = 1{\cdot}75 \times 10^{-2} \lambda^{\frac{5}{4}}. \tag{8.15}$$

It is convenient to express $|\theta_1|$ as a function of the number n of ripple cycles measured from the crest of the gravity wave:

$$n = -\frac{1}{2\pi}(\alpha c/T') = -\frac{\lambda}{2\pi}\int_0^{\xi} (1+\xi^2)^{\frac{1}{6}}\, d\xi. \tag{8.16}$$

In figure 3, $|\theta_1|$ is plotted against n, for various values of λ, using equations (8.5), (8.12) and (8.16).

If the capillary waves extend over an appreciable fraction of a wavelength, it will be necessary to use the more general equation (5.16) for determining the variation in steepness of the capillaries. In terms of the non-dimensional coordinates (ξ, η) we have

$$e^{\tau_0} = \delta^{\frac{1}{3}}\left[1 + (1-\delta)\left(\frac{\sin \frac{1}{2}\delta\xi}{\frac{1}{2}\delta}\right)^2\right]^{\frac{1}{6}}. \tag{8.17}$$

So on substitution in (7.7) we have

$$|\theta_1| = b\left[1+(1-\delta)\left(\frac{\sin\frac{1}{2}\delta\xi}{\frac{1}{2}\delta}\right)^2\right]^{\frac{1}{6}} \exp\left[\epsilon\int_0^{\xi}\left\{1+(1-\delta)\left(\frac{\sin\frac{1}{2}\delta\xi}{\frac{1}{2}\delta}\right)^2\right\}^{\frac{1}{3}} d\xi\right]. \quad (8.18)$$

Also n is given in terms of ξ by

$$n = -\frac{\lambda}{2\pi}\int_0^{\xi}\left\{1+(1-\delta)\left(\frac{\sin\frac{1}{2}\delta\xi}{\frac{1}{2}\delta}\right)^2\right\}^{\frac{1}{6}} d\xi. \quad (8.19)$$

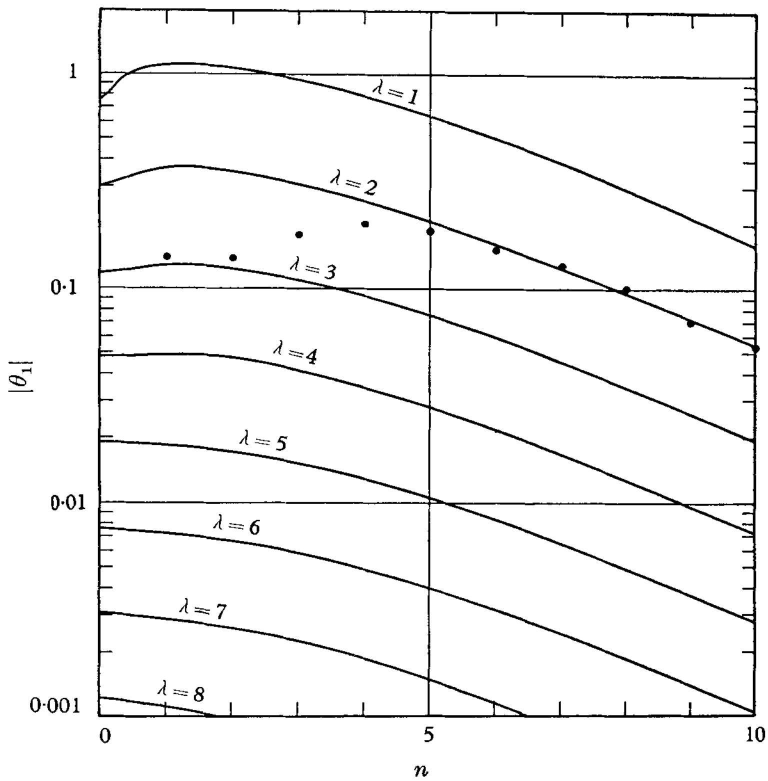

FIGURE 3. Theoretical values of the capillary wave steepness $|\theta_1|$ as a function of the number n of wavelengths from the crest of the gravity wave.

The assumptions made during the course of the analysis imply certain limitations on the values of the physical parameters:

First, the capillary waves must be short compared to the basic gravity wave. This is equivalent to saying that in the classical dispersion relation $\sigma^2 = gk + T'k^3$ the surface tension term is relatively small, or

$$Tk^3 \ll gk.$$

Hence

$$k \ll (g/T')^{\frac{1}{2}}. \quad (8.20)$$

This implies that the length of the gravity waves must be large compared with $2\pi(T'/g)^{\frac{1}{2}}$, which is about 1·7 cm.

Since the capillary waves have been treated as perturbations on the main flow the capillary wave steepness $|\theta_1|$ must be reasonably small, say $< 0{\cdot}3$. From figure 2 we see then that λ must exceed about 2.

A similar restriction follows from the assumption that the surface tension term in the boundary condition for the basic gravity wave is relatively small; for if $T'K \ll \frac{1}{2}q_0^2$ while $Kq_0^2 = \frac{1}{2}g$ at the crest (see § 3) then $T'K^2 \ll \frac{1}{4}g$, or

$$\lambda \gg \tfrac{2}{3}. \tag{8.21}$$

In order that the disturbance at the crest shall produce ripples upstream it is necessary (Lamb 1932, § 270) that the velocity q_0 of the stream shall exceed the minimum phase velocity of gravity-capillary waves. This velocity is equal to $(2g'T')^{\frac{1}{4}}$, where g' is the apparent value of gravity, i.e. $g' = \frac{1}{2}g$. Hence we have $q_0 > (gT')^{\frac{1}{4}}$, or $q_0^4 > gT'$. But $q_0^2 = g/2K$ and so this leads to

$$\lambda > \tfrac{2}{3} \tag{8.22}$$

which is already satisfied.

In order that the capillary waves shall be damped out in less than one wavelength of the gravity wave we must have from (7.7)

$$\exp\left[\frac{4\nu}{T'^2}\int_0^{cL} q_0^2\,d\phi\right] \ll 1.$$

The integral is of the same order as

$$\int_0^{cL} c^2\,d\phi = -Lc^3 = -L\left(\frac{gL}{2\pi}\right)^{\frac{3}{2}},$$

where L is the wavelength. If the exponent is to be greater than 2 in absolute magnitude we must have

$$\frac{4\nu}{T'^2}\frac{g^{\frac{3}{2}}L^{\frac{5}{2}}}{(2\pi)^{\frac{3}{2}}} > 2,$$

or

$$L^{\frac{5}{2}} > (2\pi^3)^{\frac{1}{2}}\frac{T'^2}{\nu g^{\frac{3}{2}}}.$$

In c.g.s. units this becomes $L^{\frac{5}{2}} > 81\,\mathrm{cm}^{\frac{5}{2}}$ or

$$L > 5{\cdot}8\,\mathrm{cm}. \tag{8.23}$$

This condition is consistent with (8.20).

There appears to be no theoretical upper limit to the wavelength L. However with waves longer than a few metres it becomes unlikely that the sharp curvatures required by the condition (8.2) will be attained for any appreciable time.

9. Comparison with observation

In the experiments of Cox (1958) corresponding to figure 1 the frequency of the gravity waves was 6·6 c/s and the wavelength was 4·7 cm so that the phase velocity c was given by $c = 6{\cdot}6 \times 4{\cdot}7 = 30{\cdot}9$ cm/sec. The value of the curvature K at the crest of the gravity wave can be estimated from the data in more than one way. For example one might attempt to apply the theory of gravity waves of finite amplitude, assuming a maximum surface slope about 0·35. However, this procedure would not give a very reliable value for K since, as we have seen, even the finite amplitude theory of Davies (1951) is liable to be in error near the crest. For this reason we prefer to derive K directly from figure 1 *a*. The trace in figure 1 *a* represents the wave slope; the surface curvature is given by the gradient of the slope profile where the profile passes through zero. In figure 1 *a* the vertical and horizontal scales are such that a gradient of unity corresponds to a rate of change of slope of $6{\cdot}80\,\text{sec}^{-1}$; the wave speed being about $30{\cdot}9$ cm/sec this is equivalent to a curvature of $6{\cdot}80/30{\cdot}9 = 0{\cdot}22\,\text{cm}^{-1}$.

The individual waves in figure 1 are by no means identical. The variability is probably due mainly to the reflexion of waves at the far end of the tank, which is difficult to eliminate experimentally. Even a small amount reflexion will have a pronounced effect on the slope of waves which approach their maximum steepness. In the circumstances it seems best to take mean values over the 13 wave cycles in figure 1 *a*. Allowing, as far as possible, for the existence of a ripple near the crest we find that the tangent to the profile at the point of zero slope (corresponding to the wave crest) makes an angle of $75{\cdot}8^\circ$ with the time axis. The corresponding curvature is therefore $K = 0{\cdot}22 \times \tan 75{\cdot}8^\circ = 0{\cdot}87\,\text{cm}^{-1}$. Hence we have

$$\lambda = g/6T'K^2 = 2{\cdot}92.$$

Since the condition (8.24) is only marginally satisfied, we can expect no more than rough agreement between theory and observation.

In figure 3 we have plotted the observed values of the ripple slopes as measured† from the profiles in figure 1 *a*. Each plotted point represents a mean value over the 13 wave cycles. The plotted points lie not far from the curve corresponding to $\lambda = 2$, and generally fall between $\lambda = 2$ and $\lambda = 3$.

We conclude that the ripples can be attributed at least in part to the mechanism that has been described.

10. Energy dissipation by capillary waves

Consider a gravity wave of amplitude a and wave-number k. The dissipation of energy by viscosity in such a wave is equal to

$$4\nu k^2 E = 2\rho g \nu (ak)^2 \tag{10.1}$$

per unit time and horizontal distance. Suppose that at the crest of each wave a capillary wave of amplitude a_c and wave-number k_c is generated, and that the whole of this capillary wave energy is dissipated before the capillary wave reaches the next crest. Then in one wavelength the dissipation of energy‡ by the capillaries is equal to $\frac{1}{2}T'(a_c k_c)^2 \times \frac{3}{2}c_c$ and per unit horizontal distance the dissipation is

$$\frac{k}{2\pi} \times \tfrac{1}{2}T(a_c k_c)^2 \times \tfrac{3}{2}c_c = \frac{3}{8\pi}Tk(a_c k_c)^2 c_c. \tag{10.2}$$

If now the slope $a_c k_c$ of the capillaries is assumed to be of the same order of magnitude as the gradient ak in the gravity waves, and if the phase velocity c_c of the ripples is comparable to the phase velocity σ/k of the gravity waves, then the ratio of (10.2) to (10.1) is of order

$$r = \frac{3}{16\pi}\frac{T'\sigma}{g\nu} = 0{\cdot}35\sigma \tag{10.3}$$

† The author is indebted to Dr Cox for kindly supplying a copy of the original record, from which these measurements were taken.

‡ In this calculation, energy supplied by the radiation stress is neglected.

in c.g.s. units. In gravity waves of period 0·2 sec, or length 6 cm, we have $\sigma = 2\pi/0{\cdot}2 = 31{\cdot}4$, so that r is of order 10. In other words, the capillary waves are ten times more effective in damping the gravity waves than is the direct action of viscosity. As the wave period increases so r diminishes in inverse proportion. r is reduced to unity only when the wave period is about 2 sec (wavelength 6 m).

It has been assumed that the steepness of the capillaries is comparable to that of the gravity waves; in general r is proportional also to $(a_c k_c)^2/(ak)^2$. Since it appears that the amplitude of the capillaries depends very critically on the maximum curvature of the gravity waves, one expects that the ratio will increase very rapidly just before breaking occurs. That is to say the damping by the capillaries comes into action just before the gravity waves break.

11. Conclusions

When gravity waves, of length greater than a few centimetres, approach their maximum amplitude, the effect on them of tension is localized near the wave crests, not distributed over the whole wave. Instead of a modification to the wavelength, the effect is to produce a train of ripples on the forward face of the wave, having a phase velocity such that the ripples appear stationary to an observer moving with the wave. Energy is fed into the ripples not only by the surface tension at the crest but also by interaction with the gravity wave on its forward face, through the radiation stress. Ripple energy is simultaneously being drained away by viscosity. On the rear face of the gravity wave both the radiation stress and the viscosity tend to reduce the ripple energy.

The ripple steepness is given, to within an order of magnitude, by equation (7.7) at least for regular waves. It has been shown that

$$b \doteqdot \frac{2\pi}{3} \exp\left(-g/6T'K^2\right),$$

where K is the curvature at the crest. The ripple energy thus depends very critically upon K.

The present analysis applies in the first place only to regular waves. On an irregular wave train, such as would be encountered on a surface subject to wind action, the formation of wave crests is spasmodic; the nature of the ripples associated with a sharp crest then depends on the length of time for which the sharp crest is in existence. If the crest exists for more than a few ripple periods then the present analysis may apply. Hence ripple formation by sharp wave crests may well inhibit the breaking of wind-generated waves at a certain stage of their growth.

On a water surface subject to a wind velocity of more than about 5 m/sec capillary waves are observed (Roll 1951) which are probably due to shear instability, as described, for example, by Miles (1962). In addition, some ripples may be generated locally by the complex airflow in the neighbourhood of any sharp-pointed crest. These mechanisms are quite distinct from the one discussed above. Nevertheless it may be noted that capillary waves of whatever origin will still tend to draw energy from the gravity waves on which they ride, by means of the radiation stresses, and to lose energy through viscosity. The gain or loss of energy to the capillaries will depend on their position relative to the crests of the gravity waves.

The present calculations have been based on a theory for gravity waves of finite amplitude which, in the critical region near to the crest of the wave, is at best approximate. A more exact theory for gravity waves of nearly maximum amplitude would not only be useful in the present application but also of some interest in itself.

I am indebted to Mr J. A. Grant for carrying out the computations for figures 2 and 3 at the Mathematical Laboratory, Cambridge, and also to Dr J. C. P. Miller for his interest in the problem.

Appendix: The Stokes 120° angle

It has been shown both theoretically and experimentally by Taylor (1953) that in a standing gravity wave of maximum amplitude the vertical acceleration at the sharp crest is equal to $-g$. It is not so well known that in a progressive wave of limiting height the acceleration near the crest is equal to $\frac{1}{2}g$ directed away from the crest; at the crest itself the acceleration is indeterminate.

In Stokes's limiting angle (Stokes 1847) the velocity potential ϕ is given by

$$\phi + i\psi = C(x+iy)^{\frac{3}{2}} = Cr^{\frac{3}{2}}\, e^{3i\gamma/2}, \tag{A 1}$$

where

$$C = \tfrac{2}{3}(-ig)^{\frac{1}{2}}, \tag{A 2}$$

and $r\cos\gamma = x$, $r\sin\gamma = y$. It is easily seen that the lines $\gamma = -\frac{1}{2}\pi \pm \frac{1}{3}\pi$ are streamlines. Also since

$$u - iv = \tfrac{3}{2}C(x+iy)^{\frac{1}{2}} = \tfrac{3}{2}Cr^{\frac{1}{2}}\, e^{i\gamma/2}, \tag{A 3}$$

we have

$$(u^2+v^2) = \tfrac{9}{4}\,|C|^2\, r = gr.$$

Thus the Bernoulli condition $\frac{1}{2}(u^2+v^2) = -gy$ is satisfied on the free surface.

To obtain the acceleration, differentiate (A 3). This gives

$$\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} = \tfrac{3}{4}C(x+iy)^{-\frac{1}{2}} = \tfrac{3}{4}Cr^{-\frac{1}{2}}\, e^{-i\gamma/2}. \tag{A 4}$$

Multiplying (A 3) by the complex conjugate of (A 4) we find

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + i\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \tfrac{9}{8}\,|C|^2\, e^{i\gamma}. \tag{A 5}$$

The expression on the left represents the vector acceleration $\mathbf{a}$, say. Substituting for C on the right-hand side, we have simply

$$\mathbf{a} = \tfrac{1}{2}g\, e^{i\gamma}. \tag{A 6}$$

In other words the acceleration has a magnitude $\frac{1}{2}g$ and is directed everywhere outwards from the vertex.

On the free surface the acceleration is the same as that of a particle sliding freely down a plane inclined at $\frac{1}{6}\pi$ to the horizontal, that is, it is $\frac{1}{2}g$ directed down the plane.

Particles vertically beneath the wave crest, in the plane of symmetry, have an acceleration equal to $\frac{1}{2}g$ directed vertically downwards.

At the crest itself the acceleration is indeterminate; the acceleration in the neighbourhood of the crest depends upon the direction from which the crest is approached.

The difference between the vertical accelerations in the two cases of the standing and the progressive wave of limiting height shows that the value of the vertical acceleration is not a sufficient criterion for the breaking of gravity waves under all conditions.

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