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Mass transport in the boundary layer at a free oscillating surface

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In a previous paper (1953*b*) it was shown theoretically that just below the boundary layer at the surface of a free wave the mass-transport gradient should be exactly twice that given by Stokes's irrotational theory. The present paper describes careful experiments which confirm the higher value of the gradient.

The results have an implication for any oscillatory boundary layer at the free surface of a fluid; such a boundary layer must generate a second-order mean vorticity which diffuses inwards into the interior of the fluid.

1. Introduction

Although it was Stokes (1847) who first theoretically predicted the existence of a mean forwards velocity of the particles (mass transport) in a water wave, only since the experimental work of Bagnold (1947) has it been realized that the mass-transport velocities may be very different quantitatively from those given by Stokes's irrotational theory. For example, Bagnold observed a strong forwards 'jet' close to the bottom, a phenomenon unaccountable on the hypothesis of irrotational motion. The whole distribution of the mass transport is in fact strongly influenced by viscous boundary layers both at the bottom and at the free surface, as was shown by the present author (1953*b*; this paper will be referred to as (I)).

The boundary-layer theory of (I) yielded two surprising results; first, that just above the boundary layer at the bottom the forward mass-transport velocity is independent of the viscosity and has the value

$$U = \frac{5}{4}\frac{a^2\sigma k}{\sinh^2 kh}, \tag{1.1}$$

where a denotes the amplitude of the waves at the surface, $2\pi/\sigma$ the wave period, $2\pi/k$ the wavelength and h the mean depth. This value is quite different from that obtained on the non-viscous theory of Stokes (1847).*

Secondly, just below the boundary layer at the *free surface* the vertical gradient of the mass-transport velocity is given by

$$\frac{\partial U}{\partial z} = -4a^2\sigma k^2 \coth kh, \tag{1.2}$$

* Stokes's theory is given partially in Lamb's *Hydrodynamics* (1932, Ch. 9); the mass-transport velocity is there derived only in the case of deep water ($kh \gg 1$).

z being measured vertically downwards. This is also independent of the viscosity and moreover is exactly twice the value given by Stokes.

The value (1.1) has been well verified by the recent experimental studies of Vincent & Ruellan (1957), Russell & Osorio (1958) and Allen & Gibson (1959), though with a considerable scatter of observations in the last case.*

On the other hand no careful verification has yet been attempted of the gradient (1.2), which may be no less important in determining the distribution of mass transport throughout the fluid. Partly, no doubt, this is due to the greater experimental difficulty in making measurements close to a moving surface, and partly also to the weak stability of the motion near the surface, which can be easily disturbed by external influences, as will be explained in § 4.

Our purpose is to describe some experiments designed to measure the velocity gradient near the free surface. These do in fact confirm equation (1·2) as against Stokes's prediction.

The opportunity is taken to correct a theoretical calculation of Harrison (1908) on the same subject. Harrison's corrected calculation leads also to equation (1.2) but over a more restricted range of the amplitude a.

The validity of (1.2) implies that a second-order vorticity is generated by the oscillatory boundary layer and is diffused inwards into the fluid, as described in § 5 below. Moreover, a similar phenomenon must occur in any fluid motion where there is a free oscillating boundary, even though the mean velocity is zero to the first order. The value of the vorticity ω generated by any free surface in this way is given by equation (6.12).

2. Theory of the boundary layer

A general theory for an oscillating free boundary has been given in § 8 of (I). Here we shall present a simplified version, relying however on (I) for some of the results quoted.

On account of the thinness of the boundary layer in relation to the usual amplitude of the waves it is desirable to take co-ordinates (s, n) measured along and normal to the free surface itself (and therefore moving with the fluid). The co-ordinate n is supposed to be directed normally inwards into the fluid. In the notation of (I) the components of velocity tangential and normal to the surface are written q_s, q_n and these are related to the stream function ψ by

$$(q_s, q_n) = \left(\frac{\partial \psi}{\partial n}, -\frac{1}{\eta}\frac{\partial \psi}{\partial s}\right), \tag{2.1}$$

where $\eta = 1 - n\kappa$ and κ denotes the curvature of the surface. The vorticity is equal to $\nabla^2\psi$ where

$$\nabla^2 \equiv \frac{1}{\eta}\left[\frac{\partial}{\partial s}\left(\frac{1}{\eta}\frac{\partial}{\partial s}\right) + \frac{\partial}{\partial n}\left(\eta\frac{\partial}{\partial n}\right)\right]. \tag{2.2}$$

* Vincent & Ruellan, as well as Allen & Gibson, actually compared their observations with the predicted *maximum* velocity in the boundary layer, which is about 10 % greater than (1·1).

An extension of the result (1·1) to the case of turbulent flow is given by the author in an appendix to the paper of Russell & Osorio (1958).

It is supposed that q_s, q_n and ψ can be expanded asymptotically in the series

$$\left.\begin{aligned} q_s &= \epsilon q_{s1}+\epsilon^2 q_{s2}+\ldots,\\ q_n &= \epsilon q_{n1}+\epsilon^2 q_{n2}+\ldots,\\ \psi &= \epsilon\psi_1+\epsilon^2\psi_2+\ldots,\end{aligned}\right\} \tag{2.3}$$

where ϵ is a small parameter which may tend to zero and where the mean values of q_{s1}, q_{n1} are identically zero. The mass-transport velocity is then shown in (I) § 6 to be of order ϵ^2; its stream function is denoted by $\epsilon^2\Psi$.

The case when the stresses vanish at the surface $n = 0$ is included in the discussion in (I) § 8. By applying a 'boundary-layer' approximation it is shown that ψ_1 satisfies the differential equation

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial n^2}\right)\nabla^2\psi_1 = 0. \tag{2.4}$$

Assuming that the tangential stress at the surface vanishes we have the boundary condition

$$\nabla^2\psi_1 = -2\frac{\partial q_{n1}}{\partial s} \quad (n = 0) \tag{2.5}$$

(derived from (220) of (I) by setting $p_{ns}^{(0)} = 0$ and $\kappa_0 = 0$). When the motion is simple-harmonic with angular frequency σ the appropriate solution of (2.4) and (2.5) is

$$\nabla^2\psi_1 = -2\frac{\partial q_{n1}^{(0)}}{\partial s}e^{-\alpha n}, \tag{2.6}$$

where $q_{n1}^{(0)}$ denotes the value of q_{n1} at $n = 0$ and

$$\alpha = \frac{1+i}{\delta}, \qquad \delta = (2\nu/\sigma)^{\frac{1}{2}}. \tag{2.7}$$

It is understood that in (2.6) the real part of the right-hand side is to be taken. So the first-order vorticity $\nabla^2\psi_1$ vanishes exponentially inwards, in a distance of order δ.

The differential equation satisfied by Ψ is given by (211) of (I) (in which V_{s1} is to be set equal to zero). We have

$$\frac{\partial^4\Psi}{\partial n^4} = 4\overline{\int\frac{\partial^2}{\partial n^2}\nabla^2\psi_1\,dt\,\frac{\partial q_{s1}^{(0)}}{\partial s}}, \tag{2.8}$$

where a bar denotes mean values with respect to the time. On integrating from outside the boundary layer, where $\partial^3\Psi/\partial n^3$ is assumed to be relatively small, we have

$$\frac{\partial^3\Psi}{\partial n^3} = 4\overline{\int\frac{\partial}{\partial n}\nabla^2\psi_1\,dt\,\frac{\partial q_{s1}^{(0)}}{\partial s}}. \tag{2.9}$$

At the free surface $n = 0$, the boundary condition for Ψ is that

$$\frac{\partial^2\Psi}{\partial n^2} = 0 \quad (n = 0) \tag{2.10}$$

(see equation (218) of (I)), and so from (2.9) and (2.10)

$$\frac{\partial^2\Psi}{\partial n^2} = 4\overline{\int(\nabla^2\psi_1)_0^n\,dt\,\frac{\partial q_{s1}^{(0)}}{\partial s}}. \tag{2.11}$$

In the case of simple-harmonic motion, when $\nabla^2\psi_1$ is given by (2.6) we have

$$\frac{\partial^2\Psi}{\partial n^2} = \frac{4}{i\sigma}\frac{\partial q_{n1}^{(0)}}{\partial s}\frac{\partial q_{s1}^{(0)*}}{\partial s}(1-e^{-\alpha n}), \tag{2.12}$$

where a star denotes the complex conjugate quantity and the real part of the product is to be taken. Hence, just beyond the boundary layer ($n \gg \delta$) we have

$$\frac{\partial^2\Psi}{\partial n^2} = \frac{4}{i\sigma}\frac{\partial q_{n1}^{(0)}}{\partial s}\frac{\partial q_{s1}^{(0)*}}{\partial s} \quad (n \gg \delta), \tag{2.13}$$

a value independent of the viscosity.

Now in the case of progressive gravity waves in water of uniform depth the orbital velocities at the surface are given by

$$\left.\begin{aligned} \epsilon q_{s1}^{(0)} &= a\sigma \coth kh\, e^{i(\sigma t-ks)}, \\ \epsilon q_{n1}^{(0)} &= ia\sigma\, e^{i(\sigma t-ks)}, \end{aligned}\right\} \tag{2.14}$$

approximately, so that (2.12) and (2.13) become

$$\epsilon^2\frac{\partial^2\Psi}{\partial n^2} = 4a^2\sigma k^2 \coth kh(1-e^{-n/\delta}\cos n/\delta) \tag{2.15}$$

and

$$\epsilon^2\frac{\partial^2\Psi}{\partial n^2} = -4a^2\sigma k^2\coth kh \quad (u \gg \delta). \tag{2.16}$$

Since $\epsilon^2\partial\Psi/\partial n$ is equal to the mean tangential component of mass transport ((I), § 6) and since n is approximately vertical, equation (2.16) is equivalent to (1.2).

As already remarked, this result represents exactly twice the gradient given by the irrotational theory of Stokes (1847).

3. Digression on a result of Harrison

It was pointed out to me by Professor P. S. Eagleson that a conclusion apparently different from the above was obtained by Harrison (1908). Harrison's method consisted of a direct expansion of the equations of motion and boundary conditions in terms of rectangular co-ordinates (x, z). This method has the disadvantage of being valid only for waves whose amplitude a is small compared with the thickness δ of the boundary layer. Nevertheless, since the range of validity of equation (2.16) certainly includes such small amplitudes, one would expect Harrison's result to agree with equation (2.16) over the restricted range.

Another formal difference between Harrison's solution and ours is that we have assumed the motion to be periodic in time, whereas Harrison allows for a slight exponential decrement proportional to $\nu k^2 t$. It can, however, be shown that such a slight decrement, or one proportional to $(\nu/\sigma)^{\frac{1}{2}}k^2x$, does not affect the boundary-layer theory outlined in § 2.

Translated into our notation, Harrison's expression for the elevation of the free surface in waves on deep water is

$$\begin{aligned} -z &= a\,e^{-2\nu k^2 t}\cos(kx-\sigma t) \\ &\quad + a^2k\,e^{-4\nu k^2 t}[\tfrac{1}{2}\cos 2(kx-\sigma t) - k(\nu^2/4gk)^{\frac{1}{4}}\sin 2(kx-\sigma t)] \end{aligned} \tag{3.1}$$

and for the mass-transport velocity (p. 115)

$$\begin{aligned} U = {} & a^2\sigma k\, e^{-4\nu k^2 t} \\ & \times [e^{-2kz} + \tfrac{1}{2}k\delta\{4(\cos z/\delta - \sin z/\delta)\, e^{-(k+\delta^{-1})z} - \sin 2kz\} \\ & \qquad + \nu k^2/\sigma\{4 \sin z/\delta\, e^{-(k+\delta^{-1})z} - 3e^{-2z/\delta}\}] \end{aligned} \tag{3.2}$$

terms of higher order in ν being neglected. (Harrison's β, λ, μ, p, y are equivalent to our a, δ^{-1}, δ^{-1}, $-\sigma$, $-z$.) On differentiation with respect to z this gives us for the terms of highest order

$$\frac{\partial U}{\partial z} = -2a^2\sigma k^2(1 - 2\, e^{-z/\delta} \cos z/\delta) \tag{3.3}$$

which is not in agreement with (2.15). (The slight exponential decrement $e^{-4\nu k^2 t}$ is ignored.)

An examination of Harrison's analysis reveals the source of the discrepancy. For his second boundary condition on p. 111 he has assumed that the stress component p_{xy} vanishes at the free surface. This is incorrect, for in fact it is not p_{xy} but p_{ns} which must vanish, and the two differ by the second-order quantity

$$4\mu \frac{\partial \eta}{\partial x} \frac{\partial^2 \psi}{\partial x\, \partial y}, \tag{3.4}$$

where $z = -\eta$ denotes the equation of the free surface. Hence to the left-hand side of his equation (12) should be added a term

$$2pk^2\beta^2\, e^{2\alpha t}. \tag{3.5}$$

This would add to his expression for U on p. 115 a term

$$-pk\beta^2 \sin 2ky\, e^{2\alpha t}, \quad \sim -a^2\sigma k\, e^{-4\nu k^2 t} \sin 2kz, \tag{3.6}$$

whence the gradient of U, for values of z comparable with δ, would be

$$\frac{\partial U}{\partial z} = -2a^2\sigma k^2(2 - 2e^{-z/\delta} \cos z/\delta) \tag{3.7}$$

in agreement with (2.15).

Doubtless the reason why Harrison overlooked his algebraical slip was that it happens by chance to bring the value for $(\partial U/\partial z)_{z \gg \delta}$ into exact agreement with the irrotational theory of Stokes, as might at first sight have been expected.

4. Observations

The actual measurement of velocity gradients near the free surface is a somewhat delicate matter owing not only to the movement of the surface itself but to the external influences by which it is easily affected. The presence of grease or other impurity in the form of a thin surface film may completely alter the gradient by restricting the free tangential motion of the particles and producing a forwards jet, as in the boundary layer at the bottom. Turbulent currents in the air may have a disturbing influence; and very slight temperature changes can produce strong velocity gradients associated with density currents; for if the length of the tank or wave system is restricted, the forwards mass transport at one level must be compensated by a backwards flow at other levels; hence any temperature stratification will tend to intensify the horizontal shearing.

Further, the presence of obstacles in the water, even the vertical walls of a measuring tank, will alter the distribution of mass transport in the neighbourhood. The observations must therefore be made far enough from such obstacles for their effect to be negligible.

Lastly, it is extremely important to ensure that the wave motion is purely sinusoidal. In relatively deep water, for example ($\coth kh \doteqdot 1$) equation (1.2) shows that $\partial U/\partial z$ is proportional to $a^2\sigma k^2$, that is to σ^5 since $\sigma^2 \doteqdot gk$. If any second harmonic, of amplitide a', is present in the wave motion its effect on the velocity gradient will be in the ratio

$$32a'^2/a^2. \tag{4.1}$$

Even if a'/a is as little as $1/10$ this will be sufficient to increase the observed velocity gradient by 32 %. In shallow water waves, however, the relative effect is correspondingly less.

For the purpose of the experiments the wind-wave tank at the Hydraulics Research Station, Wallingford, was kindly made available. A full description of the apparatus is given in the paper by Russell & Osorio (1958). The tank is 185 ft. long, 4 ft. wide and has a maximum depth of 22 in. The wave generator is of a paddle type with a fixed or movable hinge at the bottom. The wave absorber consists of a shingle beach, at a slope of about 1 in 10. All the present observations were made from the centre window of the tank, which is about 90 ft. from the wave generator. The tank is covered, and is fitted with a fan capable of drawing air over the surface in the direction of the beach, at a mean speed of about 25 ft./s.

In preliminary trials of the wavemaker it was found that on switching off the motor, and after the main group of waves had passed the point of observation half-way down the tank, there persisted a train of waves of twice the frequency but of smaller amplitude, until their group-velocity in turn has carried them past the point of observation. The existence of this second harmonic was attributed to the form of linkage used in the wave generator which had in fact been designed so as to increase the velocity of the forward stroke compared with that of the backward stroke. The linkage was therefore modified with the result that the amplitude of the second harmonic at the point of observation was reduced to about 4 % of that of the first harmonic. The error corresponding to (4.1) was thus reduced to about 5 %.

Method of observation

A grid of lines was drawn (see figure 1) at an angle $\tan^{-1} 2$ to the vertical, and was attached to a metal sheet on the far side of the tank, in the plane of motion. A drop of black ink (Waterman's ink, diluted 3 : 1) was let fall from a syringe at a height of about 18 in. above the surface, between the viewer and the grid of lines. The drop penetrated about 1 cm below the surface, leaving a nearly vertical streak. Owing to the velocity gradient the streak gradually became inclined (figure 1), and the time τ taken for its mean inclination to become parallel to the grid of lines was measured with a stop-watch. The velocity gradient was then taken to be

$$\frac{\partial U}{\partial z} = \frac{2}{\tau}. \tag{4.2}$$

The height $2a$ of the waves was measured against a grid on the near window of the tank, and the wave period was measured over 20 cycles with a stop-watch.

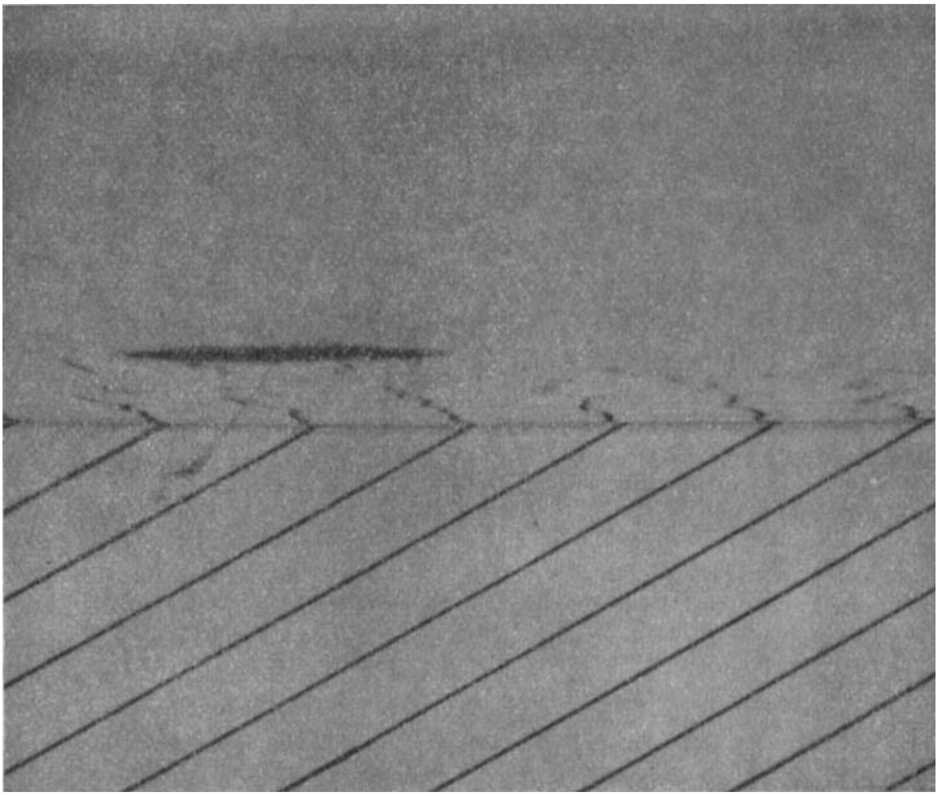

FIGURE 1. A photograph taken through the window of the tank from a point just below the surface of the wave. This shows the streak left by a drop of dilute ink, inclined after the passage of a few waves. (Some ink is left on the surface.) The lines in the background were drawn at an angle $\tan^{-1} 2$ to the vertical. The vertical separation of the lines was $\frac{1}{2}$ in.

Precautions

It was found that a thin film of oil or grease was nearly always present on the surface. In order to remove this, the fan was switched on for about 15 min, so that the surface film was driven up to the far end of the tank. Without further precautions the film would quickly have returned and covered the surface of the tank (at a rate of several cm/s). Accordingly, when the wind ceased a plastic curtain was immediately inserted in the tank at about 15 ft. from the beach. This prevented the return of the surface film, while allowing the transmission of waves from the generator to the beach.

The vigorous action of the wind also had the effect of thoroughly stirring the water, so that suspended particles were observed to be carried quickly from near the bottom to the surface and back again. In this way any temperature gradient present in the tank was temporarily destroyed.

After switching off the wind, a period of 30–45 min was allowed for the tank to become quiet again, until a drop of ink or dye inserted in the water showed that the velocity gradients were negligible. The wave generator was then started and allowed to run for 5 min before observations were begun. This period gave time for the vorticity generated in the boundary layer to penetrate at least the upper 5 mm below the free surface (see § 5 below). Only those observations were

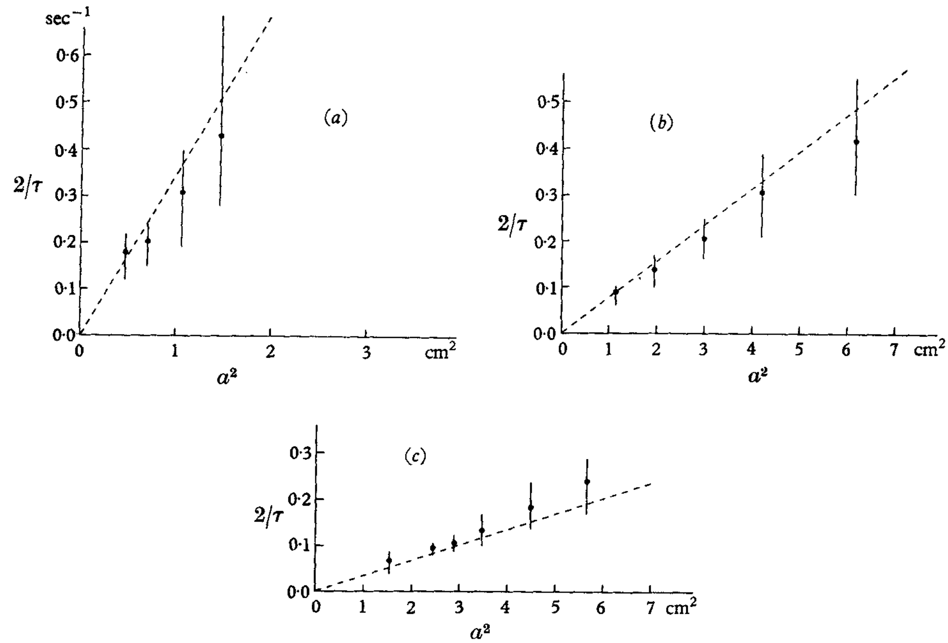

FIGURE 2. Results of observations at three different wave periods: (a) $kh = 2{\cdot}81$, (b) $kh = 1{\cdot}54$, (c) $kh = 1{\cdot}06$.

accepted in which the ink trace was initially vertical and remained in a practically straight line from 2 to 10 mm below the surface. At each run, ten acceptable observations were recorded: these were completed between 5 and 15 min after switching on the wave generator.

The choice of parameters for the experiments was limited: first, by the design of the wavemaker, which could not run safely at periods much less than 0·7 sec; secondly, by the method of observation, which is satisfactory only if the time of observation τ contains a sufficient number of wave cycles (about 10), for otherwise it is hard for the observer to judge accurately the moment when the mean inclination of the ink trace becomes parallel with the grid of lines; thirdly, by the presence of very weak gradients due to turbulent air movements and to residual effects of the stirring process described above: these last set the lower limit to the observed gradients.

Finally, three periods were chosen: $T = 0{\cdot}65$, $0{\cdot}925$ and $1{\cdot}20$ sec corresponding to $kh = 2{\cdot}81$, $1{\cdot}54$ and $1{\cdot}06$, respectively, and a number of runs were made at different amplitudes a, while the wave period was kept constant. The mean depth h was kept at 29·7 cm throughout the experiments.

The observed results are shown in figure 2, plotted in each case against the square of the wave amplitude a^2. At each wave period, according to equation (1.2) the plotted points would be expected to lie on a straight line through the origin, as has been indicated in the figure. Each plotted point represents the mean of just ten consecutive observations, and the vertical lines through the points represent the total range of the same ten observations.

The most obvious feature of the results is the very wide scatter of observations, but this is in fact not much greater than would be expected considering the method of observation. It is perhaps puzzling that in figures 2(*a*) and (*b*) the mean values tend to lie slightly below the theoretical value while in figure 2(*c*) they lie somewhat above it; but this could be brought about by quite a small error in the measurement of wave period.

Two fairly definite conclusions may be drawn: (1) over the ranges of T, a and kh covered by the experiments the velocity gradient does tend to increase proportionally to a^2 approximately; and (2) the constant of proportionality is not far from that given by equation (1.2), and is certainly closer to (1.2) than to half this value.

5. Consequences for the interior of the fluid

Some implications of these results for the motion in the main body of the fluid may be briefly mentioned here.

Before the waves are started the vorticity is everywhere zero, and since vorticity cannot be generated within the fluid it follows that immediately after starting the waves the motion *beyond* the boundary layer is irrotational and will be given by Stokes's theory approximately. The difference between (1.2) and Stoke's value corresponds to a vorticity

$$\omega = -2a^2\sigma k^2 \coth kh, \tag{5.1}$$

which can be regarded as having been generated in the boundary layer itself, that is, within a distance of order $(\nu/\sigma)^{\frac{1}{2}}$ from the free surface. This vorticity will then begin to diffuse into the interior of the fluid.

Now it was shown in (I) that in a quasi-steady (that is, perfectly periodic) state of motion the vorticity ω in the interior of the fluid ($n \gg (\nu/\sigma)^{\frac{1}{2}}$) satisfies the equation

$$\mathbf{U}\,.\,\nabla\omega - \nu\nabla^2\omega = 0, \tag{5.2}$$

where $\mathbf{U}$ denotes the mass-transport velocity.* The first term in (5.2) represents the transport of vorticity by convection with the mass-transport velocity, and the second term represents the transport of vorticity by viscous conduction. This result might also be expected on physical grounds (cf. Lamb 1932, § 328); the motion being two-dimensional, there is no stretching of the vortex lines.

Before the vorticity in the interior is fully established the motion is not

* ω is of second order; the first-order vorticity vanishes in the interior; see (I), equation (50).

exactly periodic beyond the first order; but similar arguments suggest that the differential equation governing the distribution of vorticity in the interior is

$$\frac{\partial \omega}{\partial t} + \mathbf{U}.\nabla\omega - \nu\nabla^2\omega = 0. \tag{5.3}$$

In a regular progressive wave the mass-transport velocity is almost horizontal, apart from effects at the ends of tank. Any decrease in amplitude with distance also produces small vertical mass transports, but since the third term in (5.3) initially predominates near the surface it can be seen that most of the transport of vorticity in the layers nearest the free surface will take place by viscous 'conduction'.

Thus the equation governing the initial distribution of the vorticity just beyond the boundary layer is

$$\frac{\partial \omega}{\partial t} = \nu\nabla^2\omega \doteqdot \nu\frac{\partial^2\omega}{\partial z^2}. \tag{5.4}$$

The solution of this equation with initial conditions

$$\omega = \begin{cases} 0, & z = 0, \quad t < 0, \\ \omega_0, & z = 0, \quad t > 0, \end{cases} \tag{5.5}$$

is well known:

$$\omega(z,t) = \frac{2\omega_0}{\sqrt{\pi}}\int_{z/2(\nu t)^{\frac{1}{2}}}^{\infty} e^{-\theta^2}\,d\theta, \tag{5.6}$$

which may also be written

$$1 - \frac{\omega}{\omega_0} = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\theta^2}\,d\theta \equiv f(Z), \tag{5.7}$$

where

$$Z = \frac{z}{2(\nu t)^{\frac{1}{2}}}. \tag{5.8}$$

For a fixed value of z, as $t \to \infty$ so $Z \to 0$ and $f(Z) \to 0$; hence $\omega \to \omega_0$. The following table gives some typical values of $f(Z)$

$2Z$	$f(Z)$
0·0018	0·001
0·0035	0·002
0·0089	0·005
0·0177	0·01
0·0355	0·02
0·095	0·05
0·178	0·01

Thus to ensure that the vorticity is within 10 % of its value ω_0 at $z = 0$ we must have

$$\frac{z}{(\nu t)^{\frac{1}{2}}} \leqslant 0{\cdot}178. \tag{5.9}$$

Taking $\nu = 0{\cdot}013\,\mathrm{cm^2/s}$ and $t = 5$ min we find $z \leqslant 0{\cdot}47$ cm.

However, the *mean* value of ω between O and z, defined by

$$\frac{\omega_{\text{mean}}}{\omega_0} = \frac{1}{Z}\int_0^z f(Z)\,dZ, \tag{5.10}$$

is much closer to unity, as may be seen from the series expansion

$$\frac{\omega_{\text{mean}}}{\omega_0} = 1-\frac{1}{\sqrt{\pi}}\left(Z-\frac{Z^3}{6}+\frac{Z^5}{30}-\frac{Z^7}{180}+\ldots\right). \tag{5.11}$$

With the same values: $t = 5\,\text{min}$, $z = 0{\cdot}47\,\text{cm}$, we find

$$\frac{\omega_{\text{mean}}}{\omega_0} = 0{\cdot}950, \tag{5.12}$$

so that the mean velocity gradient in the upper 5 mm is within 5 % of its ultimate value. It was this mean value which was measured in the experiments of § 4.

Generally, the width of the upper region influenced by the diffusion of vorticity from the oscillatory boundary layer will be of order $(\nu t)^{\frac{1}{2}}$, within the first few minutes. This region constitutes an outer 'boundary layer' of a different type from the oscillatory layer, the thickness of the latter being of order $(\nu T)^{\frac{1}{2}}$ only.*

At later times the fluid may be influenced by vorticity diffused in a similar way from the boundary layers at the bottom and sides of the tank. However, in most wave tanks it seems likely that the transport of vorticity by convection from the ends of the tank will intervene and ultimately predominate, so that conditions beyond the first few cm of fluid at the surface and bottom may depend somewhat on the type of wave generator or wave absorber which is used. The time required for vorticity to be convected from the ends of the tank through a distance x is clearly of order x/U. In the experiments described above this time was considerably greater than the time taken for the observations.

All the above predictions rely upon the assumption that the motion is laminar and the mass-transport current is predominantly horizontal and parallel to the wave velocity. Even slight winds may completely alter the character of the circulation (except near the bottom, where the motion is controlled by the bottom boundary layer). The possibility of the shearing motion becoming unstable of its own accord when the waves are sufficiently short and steep has also to be borne in mind. Indeed, this mechanism may contribute to the very marked turbulence that is present in all waves under the action of wind.

6. General implications

The analysis of § 2 shows that the effects just described are by no means peculiar to water waves, but will occur whenever there is an oscillatory fluid motion with a free boundary.

Thus from equation (2.11) we have just beyond the boundary, where $\nabla^2\psi_1 \to 0$,

$$\frac{\partial^2\Psi}{\partial n^2} = -4\,\overline{\int (\nabla^2\psi_1)_0\,dt\,\frac{\partial q_{s1}^{(0)}}{\partial s}} \quad (n \gg \delta), \tag{6.1}$$

which combined with (2.5) becomes

$$\frac{\partial^2\Psi}{\partial n^2} = 8\,\overline{\int \frac{\partial q_{n1}^{(0)}}{\partial s}\,dt\,\frac{\partial q_{s1}^{(0)}}{\partial s}} \quad (n \gg \delta). \tag{6.2}$$

* Harrison's solution for the interior (equation (3.2)) contains a term in sin kz. It is hard to see how such a motion, not tending to zero as $z \to \infty$, could be realized in practice.

Since we are now dealing with the part of the fluid beyond the inner boundary layer, we may, to the present order in ϵ, replace the velocity q_{s1}, q_{n1} by $\partial\psi_1/\partial z$, $-\partial\psi_1/\partial x$, where (x, z) are rectangular co-ordinates tangential and normal to the mean boundary. Thus

$$\frac{\partial^2\Psi}{\partial z^2} = -8\,\overline{\int\frac{\partial^2\psi_1}{\partial x^2}dt\,\frac{\partial^2\psi_1}{\partial x\,\partial t}}. \tag{6.3}$$

Since the mean boundary forms a streamline for the mass transport we must have also

$$\frac{\partial\Psi}{\partial x} = 0, \quad \frac{\partial^2\Psi}{\partial x^2} = 0, \tag{6.4}$$

so that the left-hand side of (6.3) can be replaced by $\nabla^2\Psi$. Now the mass-transport velocity is given in terms of ψ_1 and ψ_2 by

$$\Psi = \overline{\psi}_2 + \overline{\int\frac{\partial\psi_1}{\partial z}dt\,\frac{\partial\psi_1}{\partial x}} \tag{6.5}$$

(see (I) § 3), and since by equation (85) of (I)

$$\nabla^2\overline{\int\frac{\partial\psi_1}{\partial z}dt\,\frac{\partial\psi_1}{\partial x}} = -4\,\overline{\int\frac{\partial^2\psi_1}{\partial x^2}dt\,\frac{\partial^2\psi_1}{\partial x\,\partial z}}, \tag{6.6}$$

it follows, on operating on both sides of (6.5) with ∇^2, that

$$-8\,\overline{\int\frac{\partial^2\psi_1}{\partial x^2}dt\,\frac{\partial^2\psi_1}{\partial x\,\partial z}} = \nabla^2\overline{\psi}_2 - 4\,\overline{\int\frac{\partial^2\psi_1}{\partial x^2}dt\,\frac{\partial^2\psi_1}{\partial x\,\partial z}}. \tag{6.7}$$

Hence

$$\nabla^2\overline{\psi}_2 = -4\,\overline{\int\frac{\partial^2\psi_1}{\partial x^2}dt\,\frac{\partial^2\psi_1}{\partial x\,\partial z}}. \tag{6.8}$$

This last expression, multiplied by ϵ^2, represents the vorticity just in the interior of the fluid, for, beyond the boundary layer,

$$\nabla^2\psi_1 \to 0, \quad \nabla^2\psi_2 \to \nabla^2\overline{\psi}_2 \tag{6.9}$$

(see (I) § 4) and therefore

$$\omega = \nabla^2\psi \to \epsilon^2\nabla^2\overline{\psi}_2 + \dots. \tag{6.10}$$

Now write

$$\epsilon\frac{\partial\psi_1}{\partial z} = u, \quad -\epsilon\frac{\partial\psi_1}{\partial x} = w, \tag{6.11}$$

for the components of velocity tangential and normal to the mean position of the boundary (these differ from their corresponding values at the boundary by quantities of order ϵ^2). Our result can thus be written

$$\omega = 4\,\overline{\frac{\partial u}{\partial x}\int\frac{\partial w}{\partial x}dt}. \tag{6.12}$$

In other words, the presence of the free boundary produces a mean vorticity in the interior given by (6.12). The modifications of this result needed to take account of any arbitrary tangential stresses applied at the surface may be deduced from the general formulae of (I) § 8.

It is interesting to note that so far as the distributon of mass transport in the interior is concerned the presence of the boundary is equivalent to a virtual stress

$$4\mu \overline{\frac{\partial u}{\partial x}\int \frac{\partial w}{\partial x}dt}. \tag{6.13}$$

Since $\int w\,dt$ is equal to the surface elevation η, this last expression may be written

$$4\mu \overline{\frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x}} \tag{6.14}$$

which, as we saw in § 3, is equal to the difference between the stress components p_{xz} and p_{ns}. It is p_{ns} that vanishes and p_{xz} generally differs from zero on account of the tilting of the surface through an angle $\partial\eta/\partial x$. Thus we might say that the virtual stress (6.13) was due simply to the corrugation of the free surface; but this would be to neglect the structure of the boundary layer itself, throughout which the tangential stress, like the vorticity, is not uniform.

7. Conclusions

Our concluding picture is as follows. The periodic motion of the fluid produces in the first place boundary layers at both bottom and free surface whose thickness is of the order of $(\nu T)^{\frac{1}{2}}$, where ν denotes kinematic viscosity and T the period of the waves. In practical cases these oscillatory boundary layers have a thickness of only a few millimetres. But within the surface layer there is produced a second-order mean vorticity which, from the moment of starting the waves, begins to diffuse downwards into the fluid. After a time t the region affected by the vorticity is of order $(\nu t)^{\frac{1}{2}}$, provided t is not too great. This latter region may be thought of as a kind of outer boundary layer which finally may fill the whole fluid. Some vorticity may, however, be transported by convection as well as by viscous diffusion. Finally, the motion is no longer irrotational, but contains everywhere a second-order vorticity determined by the oscillatory layers at the boundaries.

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