

The refraction of sea waves in shallow water

By M. S. LONGUET-HIGGINS

National Institute of Oceanography, Wormley

(Received 2 December 1955)

SUMMARY

This paper considers the changes that occur in the character of short-crested sea waves when they are refracted by a shallowing depth of water. Besides a change in mean wavelength and direction there is also a change (usually an increase) in the mean length of the crests. If the waves approach obliquely they become skew, that is, the crests become staggered one behind another.

When a short-crested sea is superposed on a long-crested swell, refraction tends to amplify the longer waves more than the shorter ones. This also produces an increase in the mean length of the crests.

Numerical examples are given.

INTRODUCTION

It is often noticed that sea waves tend to become more regular as they approach a coast, and that the length of the crests perpendicular to the direction of propagation appears to increase. Jeffreys (1924) considered two possible explanations: (1) non-linear effects reduce the short-crested waves by breaking, while the long-crested waves, being of smaller amplitude, do not break until later; (2) the reduction in depth has a greater magnifying effect on the long-crested waves than on the short-crested waves of the same period. On the basis of theoretical analysis, Jeffreys decided in favour of the first alternative.

There are other possible causes, however. The most important of these, now to be considered, is the refraction of the waves on entering shallow water.

It is well known that when waves enter shallow water their velocity and wavelength decrease; also, if they approach obliquely, the direction of the crests changes so as to become nearly parallel to the shore. But in general the crest-length also changes. If the water surface were a simple sine-wave, the waves would be infinitely long-crested and would remain so after refraction. But in fact the sea surface must be thought of as the sum of a whole spectrum of harmonic components having different wavelengths and directions; the 'long-crestedness' depends on the angular deviation of the wave components from the mean direction: the smaller the deviation, the greater is the ratio of mean crest-length to mean wavelength. Refraction thus affects the crest-length in the following two ways.

(1) Wave components of the same wavelength and different direction are 'collimated' so as to become more nearly parallel to the coast and to one another; this tends to diminish the angular deviation of the wave components and to make the waves more long-crested (§ 2 (a)).

(2) Wave components in the same direction but of different wavelength are refracted through different angles and so separated (as in the separation of white light by a prism); this tends to increase the angular deviation of the wave components and to make the waves more short-crested (§ 2 (b)).

The two effects (collimation and separation) are illustrated in § 2 by considering a sea made up of just two harmonic components. The full representation of the sea surface, which requires a continuous spectrum of harmonic components, is discussed in § 3. In general, the mean wavelength, mean crest-length, etc., must be defined statistically (see Longuet-Higgins 1956 a*). The changes in these quantities are deduced in § 4 and § 5, and a numerical example is given in § 6.

One of the more interesting consequences of refraction is that waves which were originally symmetrical about their mean direction become skew, i.e. the principal direction of the wave envelope is in a different direction to that of the crests. The waves then tend to line up in an échelon pattern, each wave crest being displaced (on the average) sideways relative to the preceding crest.

The change in the amplitude of the waves is also studied, and it is shown that the increase in amplitude may itself be considered as a result of refraction.

Finally, in § 7, we consider what happens when two distinct bands of swell, of different wavelength, are present in the spectrum. For example, swell from a distant storm may be present simultaneously with shorter waves due to local winds. In general, the longer waves are amplified more than the shorter waves, and so, since the longer waves are usually also more long-crested, there is an increase in the average crest-length. We calculate the extent of this increase and give a numerical example.

1. REFRACTION OF A SIMPLE WAVE TRAIN

Consider a single, infinitely long-crested train of waves of wavelength $\lambda = 2\pi/w$ and period $\tau = 2\pi/\sigma$ approaching a straight coast at an angle θ to the normal (see figure 1 (a)). If the depth of water varies gradually the wavelength will be related to the local depth h by the equation (see Lamb 1932, ch. 9)

$$w \tanh wh = \frac{\sigma^2}{g} = w', \quad (1.1)$$

where g denotes gravity and a prime (') denotes the value in deep water. The direction θ will depend on the wavelength according to Snell's law:

$$\frac{\sin \theta}{\sin \theta'} = \frac{\lambda}{\lambda'} = \frac{w'}{w}, \quad (1.2)$$

which may also be written

$$w \sin \theta = w' \sin \theta' = \text{constant}, \quad (1.3)$$

that is, the wave-number parallel to the coast is a constant.

* This paper will subsequently be referred to as I.

We shall find it useful to represent the waves by a point P in a wave-number diagram as in figure 1(b): the direction of OP is opposite to the direction of propagation of the waves, and the length of OP is w , the wave-number in the direction of propagation. It may easily be shown that the wave-number in any other direction is the projection of OP in that direction; in particular the coordinates

$$(u, v) = (w \cos \theta, w \sin \theta) \quad (1.4)$$

are the wave-numbers perpendicular and parallel to the coast.

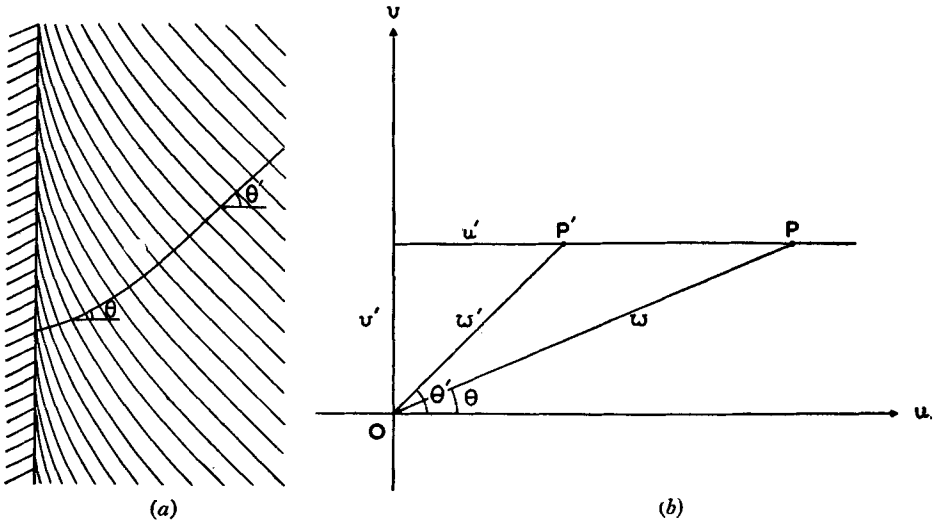


Figure 1. (a) A regular, long-crested train of waves approaching a straight coast ;
(b) its representation in the wave-number diagram.

Let $P' = (u', v')$, represent the wave train in deep water, and $P = (u, v)$ the refracted waves. Then, by (1.1),

$$\frac{OP}{OP'} = \frac{w}{w'} = \coth wh > 1, \quad (1.5)$$

and, by (1.3)

$$v = v', \quad (1.6)$$

so that the effect of refraction is to displace the representative point P away from the origin and parallel to the axis of u .

2. TWO LONG-CRESTED WAVE TRAINS

Two effects of refraction can be simply seen by considering the sum of a pair of long-crested wave trains in the following two cases.

(a) Wave trains having the same wave-number w and slightly different directions θ and $\theta + \delta\theta$

The sum of two such trains is a short-crested pattern whose wave-number is w in a direction perpendicular to the crests and $w\delta\theta$ along the crests (see figure 2(a)). The ratio of the wavelength to the crest-length is $\delta\theta$. Let

the two wave trains be represented by points P_1 and P_2 in the wave-number diagram (figure 2(b)). Then $OP_1 = OP_2 = w$ and $\angle P_1OP_2 = \delta\theta$. Also $P_1P_2 = w\delta\theta$, the wave-number along the crests. Now refraction displaces P_1 and P_2 parallel to the u -axis, and

$$\frac{P_1P_2}{P_1'P_2'} = \frac{\delta v \sec \theta}{\delta v \sec \theta'} = \frac{\cos \theta'}{\cos \theta} < 1, \quad (2.1)$$

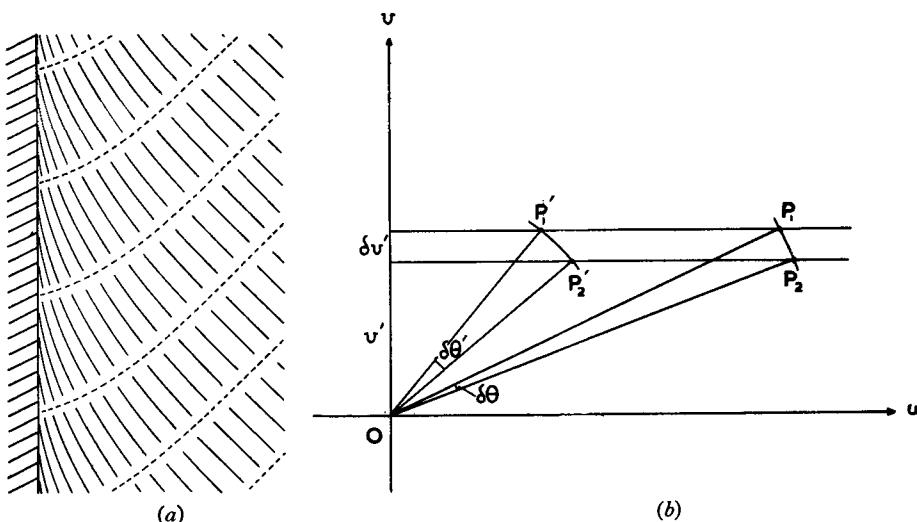


Figure 2. (a) The pattern of crests formed by two wave trains of equal wavelength and slightly different direction ; (b) their representation in the wave-number diagram.

since $\theta < \theta'$. Therefore, refraction increases the length of the crests, though the wavelength is diminished. The angle between the component wave trains, which is equal to P_1P_2/OP_1 , is also diminished.

(b) *Wave trains having the same direction θ but different wave-numbers w and $w + \delta w$*

The sum of two such waves is a pattern which is long-crested but of varying amplitude, the waves coming in 'groups' of length $\pi/\delta w$. In the wave-number diagram the unrefracted waves are represented by two points P_1' and P_2' on the same line through the origin (figure 3(b)) and with $P_1'P_2' = \delta w'$. However, (if P_1 and P_2 represent the refracted waves) P_1P_2 does not in general pass through O ; for in that case we should have $\delta w/w = \delta v/v = \delta w'/w'$ and so $\delta w'/\delta w = w'/w$, whereas in fact we have, on differentiating equation (1.1),

$$\frac{\delta w'}{\delta w} = \tanh wh + wh \operatorname{sech}^2 wh > \tanh wh = \frac{w'}{w}. \quad (2.2)$$

Hence P_1P_2 subtends a positive angle at the origin, and the refracted wave trains have differing directions. Thus the refracted pattern is short-crested, i.e. the crest-length is reduced.

Since the refracted waves are also of different wavelength, the resulting pattern is 'staggered' (see I). A similar effect has been pointed out by Stoneley (1935) in connection with seismic waves refracted at a discontinuity.

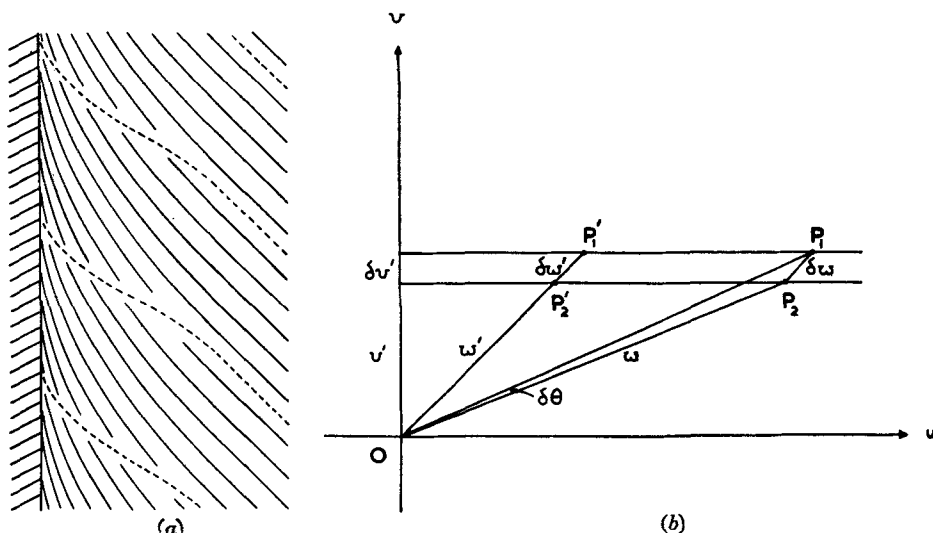


Figure 3. (a) The pattern of crests formed by two wave trains initially in the same direction and of slightly different wavelength ; (b) their representation in the wave-number diagram.

3. GENERAL REPRESENTATION

To describe actual, irregular sea waves we use the representation described in I. It is assumed that the waves in any given locality are statistically uniform, and that they contain wave components of every possible wavelength and direction. Corresponding to each wave number (u, v) we define the energy density $E(u, v)$ such that $E du dv$ is the energy associated with wave-numbers in a small region $du dv$ of the (u, v) -plane. Thus the total energy per unit of the sea surface, for example, is given by

$$\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(u, v) du dv. \quad (3.1)$$

Many of the frequency characteristics of the sea surface may be derived in terms of E (see I). Here we shall make the assumption that the spectrum is narrow, i.e. that the energy E is mainly clustered near the mean wave-number $(\bar{u}, \bar{v})$, defined by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u - \bar{u}) E du dv = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (v - \bar{v}) E du dv = 0. \quad (3.2)$$

The waves are then more or less long-crested, having a mean wave-number $\bar{w}$ in the direction $\bar{\theta}$ perpendicular to the crests, where

$$(\bar{u}, \bar{v}) = (\bar{w} \cos \bar{\theta}, \bar{w} \sin \bar{\theta}) \quad (3.3)$$

(see figure 4). Let coordinates (ξ, η) be taken with origin at the mean point and with ξ and η measured along and perpendicular to $O\bar{P}$, so that, approximately,

$$\left. \begin{aligned} \xi &= w - \bar{w} = \delta w, \\ \eta &= \bar{w}(\theta - \bar{\theta}) = \bar{w} \delta \theta, \end{aligned} \right\} \quad (3.4)$$

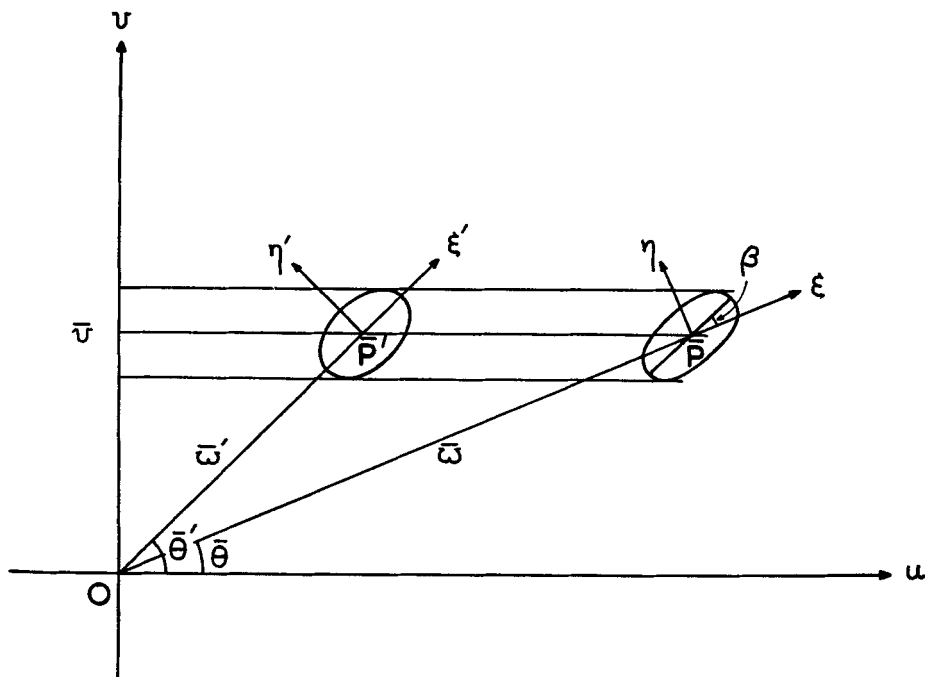


Figure 4. The effect of refraction on a continuous spectrum.

say. We define the moments

$$\left. \begin{aligned} \mu_{20} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi^2 E \, d\xi d\eta, \\ \mu_{11} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi \eta E \, d\xi d\eta, \\ \mu_{02} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta^2 E \, d\xi d\eta, \end{aligned} \right\} \quad (3.5)$$

an interpretation of which is as follows (see I). First,

$$\mu_{20} = \mu \bar{w}^2 \nu^2, \quad (3.6)$$

where ν is the r.m.s. proportional spread in wave-number in a direction perpendicular to the crests; ν is inversely proportional to the average number of waves in a 'group'. Secondly,

$$\mu_{02} = \mu \bar{w}^2 \gamma^2, \quad (3.7)$$

where γ is the r.m.s. angular deviation of the waves from their mean direction; it may be shown that

$$\gamma = \frac{N_{\min}}{N_{\max}}, \quad (3.8)$$

where $N_{\max}$ is the average number of times per unit distance that the surface crosses a line at mean level perpendicular to the crests, and $N_{\min}$ is the corresponding number for a line parallel to the crests. Thus if $\bar{\lambda}$ is the mean wavelength and $\bar{\kappa}$ the mean crest-length,

$$\bar{\lambda} = 2/N_{\max}, \quad \bar{\kappa} = 2/N_{\min}, \quad \gamma = \bar{\lambda}/\bar{\kappa}. \quad (3.9)$$

Thirdly,

$$\mu_{11} = \mu \bar{w}^2 (v^2 - \gamma^2) q, \quad (3.10)$$

where q is a measure of the *skewness* of the waves; it may be shown that

$$q = \frac{1}{2} \tan 2\beta, \quad (3.11)$$

where β is the angle between the mean direction of the waves and the principal direction of their *envelope* (see I).

4. CHANGES DUE TO REFRACTION

If the wave amplitude is sufficiently small, the energy corresponding to each wave-number is refracted independently. Further, it is assumed that the energy reflected from the coast is negligible. Under these conditions, it may be shown (see Longuet-Higgins 1956 b) that if $E'(u, v)$ denotes the energy density before refraction and $E(u, v)$ the energy density after refraction, then E is related simply to E' by

$$E(u, v) = E'(u', v'), \quad (4.1)$$

where (u, v) is related to (u', v') as in § 1. Hence a contour of energy density $E' = \text{constant}$ is transformed into a contour $E = \text{constant}$ (see figure 4), though the area enclosed by the contour may be changed, as also may be its shape.

We shall consider how the parameters μ, ν, γ, q defined above will change on refraction of the waves. Let (ξ', η') be coordinates defined for the unrefracted waves similarly to (ξ, η) for the refracted waves (equations (3.4)). Now, to the first order of small quantities,

$$\delta w = \frac{dw}{dw'} \delta w', \quad (4.2)$$

where from (1.1)

$$\frac{dw}{dw'} = \frac{1}{\tanh wh + wh \operatorname{sech}^2 wh}. \quad (4.3)$$

Also, on differentiating (1.2) logarithmically, we have

$$\frac{\delta \theta}{\tan \theta} - \frac{\delta \theta'}{\tan \theta'} = \delta w - \delta w', \quad (4.4)$$

and so, using (4.2) and (1.2),

$$w \delta \theta = \left(\frac{w}{w'} - \frac{dw}{dw'} \right) \tan \theta \delta w' + \frac{w}{w'} \frac{\tan \theta}{\tan \theta'} w' \delta \theta'. \quad (4.5)$$

Equations (4.2) and (4.5) may be written

$$\left. \begin{aligned} \xi &= \alpha_{11}\xi' + \alpha_{12}\eta', \\ \eta &= \alpha_{21}\xi' + \alpha_{22}\eta', \end{aligned} \right\} \quad (4.6)$$

where

$$\left. \begin{aligned} \alpha_{11} &= \frac{\partial \xi}{\partial \xi'} = \frac{dw}{dw'}, & \alpha_{12} &= \frac{\partial \xi}{\partial \eta'} = 0, \\ \alpha_{21} &= \frac{\partial \eta}{\partial \xi'} = \left(\frac{w}{w'} - \frac{dw}{dw'} \right) \tan \theta, & \alpha_{22} &= \frac{\partial \eta}{\partial \eta'} = \frac{w}{w'} \frac{\tan \theta}{\tan \theta'}. \end{aligned} \right\} \quad (4.7)$$

Therefore,

$$\frac{\partial(\xi, \eta)}{\partial(\xi', \eta')} = \alpha_{11} \alpha_{22}. \quad (4.8)$$

Over the narrow region of integration, $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ may be treated as constants. Hence, on changing the variable in (3.1) and remembering (4.1), we have

$$\mu = \alpha_{11} \alpha_{22} \mu'; \quad (4.9)$$

similarly, from (3.5) we find

$$\left. \begin{aligned} \mu_{20} &= \alpha_{11} \alpha_{22} (\alpha_{11}^2 \mu'_{20}), \\ \mu_{11} &= \alpha_{11} \alpha_{22} (\alpha_{11} \alpha_{21} \mu'_{20} + \alpha_{11} \alpha_{22} \mu'_{11}), \\ \mu_{02} &= \alpha_{11} \alpha_{22} (\alpha_{21}^2 \mu'_{20} + 2\alpha_{21} \alpha_{22} \mu'_{11} + \alpha_{22}^2 \mu'_{02}), \end{aligned} \right\} \quad (4.10)$$

where μ' is the initial value of μ , and so on. Thus ν, q, γ are transformed as follows:

$$\left. \begin{aligned} \nu^2 &= \frac{w'^2}{w^2} \alpha_{11}^2 \nu'^2, \\ (\nu^2 - \gamma^2)q &= \frac{w'^2}{w^2} [\alpha_{11} \alpha_{21} \nu'^2 + \alpha_{11} \alpha_{22} (\nu'^2 - \gamma'^2)q'], \\ \gamma^2 &= \frac{w'^2}{w^2} [\alpha_{21}^2 \nu'^2 + 2\alpha_{21} \alpha_{22} (\nu'^2 - \gamma'^2)q' + \alpha_{22}^2 \gamma'^2], \end{aligned} \right\} \quad (4.11)$$

(the bar in $\bar{w}$ has been dropped). We may assume that q' , the skewness of the unrefracted waves, is zero. Then, on inserting the values of $\alpha_{11}, \alpha_{12}, \alpha_{22}$ in (4.11), we have

$$\left. \begin{aligned} \nu^2 &= r^2 \nu'^2 \\ (\nu^2 - \gamma^2)q &= r(1-r) \tan \theta \nu'^2 \\ \gamma^2 &= (1-r)^2 \tan^2 \theta \nu'^2 + \frac{\tan^2 \theta}{\tan^2 \theta'} \gamma'^2 \end{aligned} \right\} \quad (4.12)$$

where

$$r = \frac{w'}{w} \frac{dw}{dw'} \quad (4.13)$$

and θ is given by Snell's law (1.2).

5. DISCUSSION

The first of equations (4.12) gives

$$\nu = r\nu', \quad (5.1)$$

that is to say, the average number of waves in a group is divided by r on refraction. Now, from (4.13), the factor r may be written

$$r = \frac{\sigma/w}{d\sigma/dw} \div \frac{\sigma'/w'}{d\sigma'/dw'} \quad (5.2)$$

since, if the frequencies are preserved, $\sigma = \sigma'$. But σ/w and $d\sigma/dw$ are the phase-velocity and group-velocity respectively, whose ratio varies from 2 in deep water to 1 in shallow water (see Lamb 1932, ch. 9). Thus in shallow water $r = 1/2$, and hence the number of waves in a group is doubled when the waves reach shallow water. (This effect was noticed by Barber (1950) in the case when the waves approach the shore normally.) However, if the waves were recorded in time at a fixed point, the groups in the record would appear to have the same average number of waves, regardless of the depth of water:

Equations (4.12) show that ν and q depend only on ν' , the initial spread in wave-number, but that γ has a contribution from both ν' and γ' . By setting $\gamma' = 0$ and $\nu' = 0$ in turn, we see that γ may be either greater or less than γ' .

The average length of the crests is given by

$$\bar{\kappa} = \bar{\lambda}/\gamma = 2\pi/\gamma w. \quad (5.3)$$

So from the third of equations (4.12) we have

$$\frac{1}{\bar{\kappa}^2} = \frac{1}{4\pi^2} (1 - r^2) w^2 \nu'^2 \tan^2 \theta + \frac{\cos^2 \theta'}{\cos^2 \theta} \frac{1}{\bar{\kappa}'^2}, \quad (5.4)$$

which shows that the mean crest-length, also, may be either increased or decreased by refraction.

A particularly simple case is when θ' is small, or when the wave crests are nearly parallel to the coast. Then from (1.3),

$$w\theta = w'\theta', \quad (5.5)$$

approximately, and from (4.12) and (5.4),

$$\gamma^2 = \frac{w'^2}{w^2} \gamma'^2, \quad \bar{\kappa} = \bar{\kappa}' \quad (5.6)$$

(assuming $\nu'\theta' \ll \gamma'$). Thus the average crest-length remains unchanged, although the average wavelength and the angular deviation are both diminished.

The wave amplitude

The change in amplitude of the waves may be found as follows. If a denotes the r.m.s. wave amplitude, the mean energy per unit area of the sea surface is $\frac{1}{2}\rho g a^2$. But this is equal to μ , by definition. Therefore, from (4.9),

$$\frac{a^2}{a'^2} = \frac{\mu}{\mu'} = \alpha_{11} \alpha_{22} = \frac{dw}{dw'} \frac{\cos \theta'}{\cos \theta} \quad (5.7)$$

and

$$a = a' \left(\frac{dw}{dw'} \frac{\cos \theta'}{\cos \theta} \right)^{1/2}. \quad (5.8)$$

Equation (5.7) is a generalization of the well-known formula (Burnside 1914) for the change in amplitude of waves approaching a coast normally; on setting $\theta = \theta' = 0$, we obtain Burnside's equation (ii). Since, in general $\cos \theta > \cos \theta'$, we deduce that waves approaching a coast obliquely are amplified less than those approaching normally.

To facilitate numerical calculation, we have plotted in figure 5 graphs of

$$\frac{w'}{w} = \tanh wh \quad (5.9)$$

and

$$r = (1 + 2wh \operatorname{cosech} 2wh)^{-1} \quad (5.10)$$

as functions of $w'h = \sigma^2 h/g$. The amplitude function

$$\left(\frac{dw}{dw'}\right)^{1/2} = (rw/w')^{1/2} \quad (5.11)$$

is also shown. When $w'h$ is large, these all tend to unity.

When $w'h$ is small, we have

$$\frac{w'}{w} \sim (w'h)^{1/2}, \quad \nu \sim \frac{1}{2}, \quad \left(\frac{dw}{dw'}\right)^{1/2} \sim \frac{(w'h)^{-1/4}}{\sqrt{2}} \quad (5.12)$$

(as was first shown by Green, 1838). However, the present linear approximation is only valid so long as

$$a \ll \frac{4}{3} w^2 h^3 \quad (5.13)$$

(Stokes 1847, Ursell 1953), which implies that

$$a' \ll \frac{4\sqrt{2}}{3} (w'h)^{5/4} h \frac{\cos \theta}{\cos \theta'}. \quad (5.14)$$

If this condition is not satisfied, the waves will depart from the sinusoidal form, and may become solitary waves or may break.

6. NUMERICAL EXAMPLE

Suppose that waves of mean period 10 sec approach the shore initially at an angle of 60° to the normal:

$$\sigma = \frac{2\pi}{10} \text{ sec}^{-1}, \quad w' = \frac{\sigma^2}{g} = 0.012 \text{ ft}^{-1}, \quad \theta' = 60^\circ.$$

When the waves have reached water of depth $h = 20$ ft we have $w'h = 0.25$ and so, from figure 5,

$$\frac{w'}{w} = 0.48, \quad r = 0.54, \quad \left(\frac{dw}{dw'}\right)^{1/2} = 1.07.$$

The angle θ between the wave crests and the shore is given by

$$\sin \theta = 0.48 \sin 60^\circ = 0.42, \quad \theta = 24^\circ.$$

Let the initial spread in wave-number be one-quarter of the mean wave-number, and the initial spread in direction be 15° :

$$\nu' = 0.25, \quad \gamma' = 0.26.$$

Then, assuming that $q' = 0$, we have from (4.12)

$$\nu = 0.135, \quad \gamma = 0.088, \quad q = 0.68.$$

Therefore, the angular spread is reduced by a factor $\gamma/\gamma' = 0.30$.

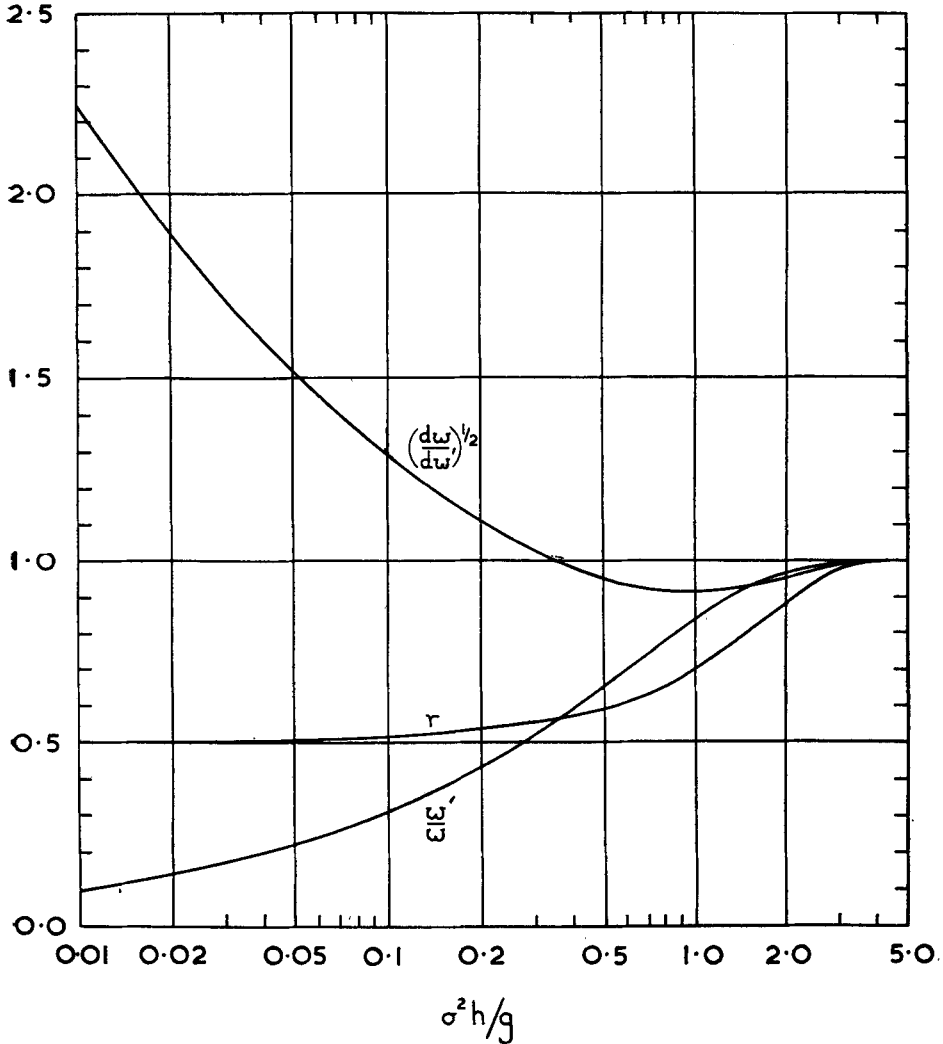


Figure 5. Graphs of the functions w'/w , r and $(dw/dw')^{1/2}$.

Taking account of the reduction in mean wavelength ($w'/w = 0.48$), we see that the mean length of the crests is multiplied by 1.6, i.e. increased by 60%.

From (3.10), we find for the angle of skewness

$$\beta = 27^\circ,$$

and from (5.8),

$$a = 0.79a'.$$

Thus the r.m.s. wave amplitude is reduced by 21%, although if the waves approached the coast normally it would be increased by 7%.

Equation (5.14) shows that, for the analysis to be valid, the amplitude a' in deep water must be small compared with 11 ft.

7. A MIXED SPECTRUM

So far we have supposed that the distribution of energy is narrowly restricted in wave-number and direction. It is interesting, however, to consider the changes due to refraction when the sea consists of a long-crested swell superposed on short-crested local waves. Under these conditions, the swell may be expected to be amplified more than the local waves, producing an increase in the length of the crests.

Let the energy function E consist of two parts, E_1 and E_2 , which represent respectively a narrow band of swell, of total energy μ_1 and a local, fairly short-crested, sea of total energy μ_2 .

For simplicity, we suppose that both systems of waves are approaching the coast normally.

The average number of zero-crossings along a line parallel to the crests will be denoted simply by N (so that $N = N_{\min}$). It can be shown (see I) that

$$\mu N^2 = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta^2 E \, d\xi d\eta. \quad (7.1)$$

So, if N_1 and N_2 denote the corresponding numbers for the two component systems,

$$\left. \begin{aligned} \mu N^2 &= \mu_1 N_1^2 + \mu_2 N_2^2, \\ \mu &= \mu_1 + \mu_2, \end{aligned} \right\} \quad (7.2)$$

giving

$$N^2 = \frac{\mu_1 N_1^2 + \mu_2 N_2^2}{\mu_1 + \mu_2}, \quad (7.3)$$

or, if a_1, a_2 are the corresponding r.m.s. wave amplitudes, .

$$N^2 = \frac{a_1^2 N_1^2 + a_2^2 N_2^2}{a_1^2 + a_2^2}. \quad (7.4)$$

The amplitudes and wave-numbers are transformed (approximately) as follows:

$$\left. \begin{aligned} N_1 &= N'_1, & a_1 &= \left(\frac{dw_1}{dw'_1} \right)^{1/2} a'_1, \\ N_2 &= N'_2, & a_2 &= \left(\frac{dw_2}{dw'_2} \right)^{1/2} a'_2, \end{aligned} \right\} \quad (7.5)$$

where w_1, w'_1, w_2, w'_2 are the values of w and w' for the two groups. These relations enable us to determine N in terms of the initial values of a_1, a_2, N_1 and N_2 .

For example, suppose that the mean periods of the two bands are 12 sec and 6 sec, respectively, and their r.m.s. angular widths are 10° and 20° . Then we have

$$\begin{aligned} w'_1 &= 8.5 \times 10^{-3} \text{ ft}^{-1}, & N' &= 4.7 \times 10^{-4} \text{ ft}^{-1}, \\ w'_2 &= 3.4 \times 10^{-2} \text{ ft}^{-1}, & N'_2 &= 3.8 \times 10^{-3} \text{ ft}^{-1}, \end{aligned}$$

and, if the wave amplitudes are initially equal, the crest-length is given by

$$N'^2 = \frac{1}{2}(N_1^2 + N_2^2) = 7.2 \times 10^{-6} \text{ ft}^{-2}.$$

By the time the waves have reached water of depth 10 ft. the amplitudes of the swell and the local sea are increased by 1.35 and 1.01 respectively, and we find

$$N^2 = 6.3 \times 10^{-6} \text{ ft}^{-2}.$$

Hence the mean crest-length is multiplied by 1.07, i.e. increased by 7%.

8. CONCLUSIONS

We have shown that, when a band of waves which has a fairly narrow spectrum in both wavelength and direction enters shallow water, the following changes occur.

(1) The mean wavelength decreases and the direction of the crests becomes more nearly parallel to the shore.

(2) The crests become 'staggered', i.e. the envelope of the waves is in a different direction to the direction of the crests.

(3) The mean crest-length may either increase or decrease, usually the former: if the waves are incident normally, the average crest-length remains unaltered. The ratio of crest-length to wavelength almost always increases.

(4) The number of waves in a group is increased, being multiplied by 2 when the waves reach shallow water.

(5) The amplitude of the waves first diminishes and then increases, being proportional to $h^{-1/4}$ in shallow water. The increase is less for waves approaching obliquely.

If the spectrum of the waves is not narrow, or if more than one narrow band is present, we may expect that the longer waves will eventually be amplified more than the shorter ones. Since the longer waves usually have a much greater crest-length than the shorter waves, this leads to an increase in the crest-length.

The above conclusions suggest two possible reasons for the frequently observed change in crest-length as the waves approach the shore. The first is the simple effect of refraction, as described in § 4 and § 5. The second is the relative amplification of a long band of swell, which might be hidden in deeper water off-shore.

Our quantitative analysis is applicable only so long as the waves are fairly low (see equation (5.14)), and does not apply to waves in shallow water

which have steepened at their crests, still less to breaking waves. For these, some of the effects described may still be true qualitatively, but non-linear effects such as those pointed out by Jeffreys (1924) must also be taken into account.

APPENDIX

Some of the effects described in the present paper are clearly illustrated by the aerial photograph shown in plate 1, for which I am indebted to Mr W. W. Williams. This shows a very regular swell approaching a coastline in a gradually diminishing depth of water. Not only do the mean wavelength and direction vary, but it will be seen that the wave crests arrange themselves in an échelon pattern with an angle of skewness that becomes very pronounced inshore. Lines of low wave amplitude may be traced visually over limited distances, and are not unlike those in figure 3 (a).

REFERENCES

- BARBER, N. F. 1950 Ocean waves and swell. Lecture published by Institution of Civil Engineers, London.
- BURNSIDE, W. 1915 On the modification of a train of waves as it advances into shallow water, *Proc. Lond. Math. Soc.* (2), **14**, 131.
- GREEN, G. 1838 On the motion of waves in a variable canal of small depth and width, *Trans. Camb. Phil. Soc.* **6**, 457.
- JEFFREYS, H. 1924 On water waves near the shore, *Phil. Mag.* (6), **48**, 44.
- LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.
- LONGUET-HIGGINS, M. S. 1956 a The statistical analysis of a random moving surface, *Phil. Trans. A* (in press).
- LONGUET-HIGGINS, M. S. 1956 b On the transformation of a continuous spectrum by refraction, *Proc. Camb. Phil. Soc.* (in press).
- STOKES, G. G. 1847 On the theory of oscillatory waves, *Trans. Camb. Phil. Soc.* **8**, 441. *Mathematical and Physical Papers*, **1**, 197. Cambridge University Press.
- STONELEY, R. 1935 The refraction of a wave group, *Proc. Camb. Phil. Soc.* **31**, 360.
- URSELL, F. 1953 The long-wave paradox in the theory of gravity waves, *Proc. Camb. Phil. Soc.* **49**, 685.

M. S. Longuet-Higgins, The refraction of sea waves in shallow water, Plate I.



Plate 1. Aerial photograph of regular swell approaching a coastline obliquely.

