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# On the mass, momentum, energy and circulation of a solitary wave. II†

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By accurate calculation it is found that the speed $F$ of a solitary wave, as well as its mass, momentum and energy, attains a maximum value corresponding to a wave of *less* than the maximum amplitude. Hence for a given wave speed $F$ there can exist, when $F$ is near its maximum, two quite distinct solitary waves.

The calculation is made possible, first, by the proof in an earlier paper (I)† of some exact relations between the momentum and potential energy, which enable the coefficients in certain series to be checked and extended to a high order; secondly, by the introduction of a new parameter $\omega$ (related to the particle velocity at the wave crest) whose range is exactly known; and thirdly by the discovery that the series for the mass $M$ and potential energy $V$ in powers of $\omega$ can be accurately summed by Padé approximants. From these, the values of $F$ and of the wave height $\epsilon$ are determined accurately through the exact relations $3V = (F^2-1)M$ and $2\epsilon = (\omega+F^2-1)$.

The maximum wave height, as determined in this way, is $\epsilon_{max} = 0.827$, in good agreement with the values found by Yamada (1957) and Lenau (1966), using completely different methods. The speed of the limiting wave is $F = 1.286$. The maximum wave speed, however, is $F_{max} = 1.294$, which corresponds to $\epsilon = 0.790$.

The relation between $\epsilon$ and $F$ is compared to the laboratory observations made by Daily & Stephan (1952), with reasonable agreement. An important application of our results is to the understanding of how waves break in shallow water. The discovery that the highest solitary wave is not the most energetic helps to explain the qualitative difference between plunging and spilling breakers, and to account for the marked intermittency which is characteristic of spilling breakers.

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## 1. Introduction

In this paper we continue and extend the study of solitary waves on water of uniform depth which was begun in an earlier paper (Longuet-Higgins 1974, to be referred to as paper (I)). In that paper some new relations were proved connecting the mass, momentum and energy of a solitary wave, and these were used to derive a simple but close approximation to the form of the solitary wave of maximum amplitude. In the present paper we turn our attention to the calculation of waves having less than the maximum amplitude.

Since the first approximate theories given by Boussinesq (1871) and Rayleigh (1876), which were valid strictly for waves of small amplitude, higher approximations have been suggested by McCowan (1891), Weinstein (1926), Long (1956), Laitone (1960), Grimshaw (1971) and Fenton (1972). All these rely on expansions in powers of a small parameter, essentially the wave height. In particular Fenton (1972) has carried such an expansion numerically to the ninth order, obtaining accurate results (enhanced by the use of Shanks transforms) up to amplitudes $\epsilon$ within about 10 % of the maximum.

Although further progress along these lines is possible (and will be described) one would nevertheless hardly expect that a power series in $\epsilon$ would give accurate results right up to the maximum wave amplitude. At this value of $\epsilon$, the surface develops a sharp crest with an angle of 120°, which must correspond to a singularity on or outside the radius of convergence.

Other authors, including Nekrasov (1921), Milne-Thomson (1964, 1968), Lenau (1966), Byatt-Smith (1970) and Strelkoff (1971), have derived integral equations for the wave profile. Numerical solutions for waves of less than the maximum amplitude, have been given by Schwitters (1966), Thomas (1967) and Byatt-Smith (1970). Here again, however, difficulties are encountered when the wave amplitude approaches its maximum value. These difficulties arise from the large curvatures near the crest, and our present ignorance as to the analytical structure of the flow in the neighbourhood of the crest (see Grant 1973).

In the present paper we attack the problem again by means of series expansions, but with the addition of some new weapons to our armoury. First, in §§ 2–4, we extend the series approximations of Fenton (1972) as far as the practical limit of computation, which, with the available word-lengths, should be about the fifteenth power of $\epsilon$. Beyond the ninth approximation the coefficients are found to become irregular, but we are able to show, by means of the identities derived in (I), that the irregularities are not due to rounding errors. In fact the identities are checked to a high degree of accuracy, indicating that the irregular behaviour of the coefficients is indeed significant (see § 3).

We next show (in § 4) that by a change of independent variable from $\epsilon$ to $\gamma = F^2 - 1$, where $F$ is the Froude number, the irregularities in the series for the momentum and energy are removed. This appears to be due to the singular behaviour of $\epsilon$ as a function of $\gamma$ in the neighbourhood of $\epsilon = \epsilon_{\max}$. Physically, there is a sharp increase in the wave-height as the maximum amplitude is approached. However, no such sudden increase is found in the total mass $M$, so that the momentum and the energies are all smooth functions of $\gamma$.

With this change of variable the new coefficients behave sufficiently regularly for the maximum value of $\gamma$ to be estimated from a Domb–Sykes plot (see figure 1). The extended series can then be summed, with the help of Shanks transforms or Padé approximants, up to $F^2$ equal to about 1.60, but not beyond (see figures 2 and 3).

The next step is to introduce a third variable $\omega$, related to the particle speed at the wave crest. Unlike either of the two variables $\epsilon$ and $\gamma$, the range of variation of $\omega$ is precisely known. When expressed in powers of $\omega$, the series for $M$, $T$ and $V$ appear initially less regular than for the series in powers of $\gamma$. Nevertheless the summation by Padé approximants leads to rapidly converging results. Moreover from $M$ and $V$ it is possible to calculate $F^2$ with a comparable degree of accuracy, and hence also the wave height $\epsilon$ (see figure 4).

The maximum wave height, as determined in this way, is in remarkably good agreement with the values found by Yamada (1957) and Lenau (1966) by completely different and independent methods.

The calculations also yield the unexpected result that the wave speed $F$ attains a maximum value at a certain wave amplitude less than the maximum. Hence, there is a certain range of speeds near the maximum, where two quite distinct solitary waves can exist with the same wave speed.

Similar results are found also for the total mass $M$ and for the kinetic and potential energies.

These results are discussed in § 7. To many readers they will appear less extraordinary when it is recalled that Fenton (1972) already had found indications of a maximum in the value of the total 'drift' $\delta$ (which is numerically equal to the mass $M$), and that Schwartz (1974) has found a comparable result for *periodic* waves in water of uniform depth, namely that the Stokes parameter (roughly the amplitude of the first harmonic) is not a monotonically increasing function of the wave height, but also attains a maximum for waves of less than the maximum height.

In § 6 we compare our theoretical relation between $\epsilon$ and $F$ with the laboratory observations by Daily & Stephan (1952). Over the range of $\epsilon$ in which the observations were made there is satisfactory agreement. An important application of our results is to the breaking of waves in shallow water. The fact that the highest solitary wave is not also the most energetic helps to explain the qualitative difference between plunging and spilling breakers, and to account for the marked intermittency which has been observed in spilling breakers (Longuet-Higgins & Turner 1974).

## 2. The small-amplitude approximation

Consider a solitary wave of arbitrary amplitude $a$, travelling with velocity $c$ in water of undisturbed depth $h$ (see figure 1 of (I)). The form of the wave is known to depend on a single parameter, which may be taken to be either the relative wave amplitude $\epsilon = a/h$ or the Froude number $F = c/\surd(gh)$, where $g$ denotes the acceleration of gravity. Throughout this paper we shall choose units of length and time so that

$$g = h = 1, \quad \epsilon = a, \quad F = c. \tag{2.1}$$

Let rectangular coordinates be chosen in a frame of reference travelling with the waves, so that the motion is independent of the time. With the origin beneath the wave crest at the level of the undisturbed fluid, let the $x$ axis be in the direction of wave propagation and the $y$ axis vertically upwards.

Fenton (1972) has given a solution for the surface elevation $\eta$ in the form of a series:

$$\eta(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} b_{ij}\, \alpha^{2i} \operatorname{sech}^{2j}(\alpha x) \tag{2.2}$$

in which the coefficients $b_{ij}$ are constants and $\alpha$ is a parameter tending to zero as the wave amplitude tends to zero. In fact setting $x = 0$ we have

$$\epsilon = \eta(0) = \sum_{i=1}^{\infty} \alpha^{2i} \sum_{j=1}^{i} b_{ij}. \tag{2.3}$$

The Froude number $F$ is expressible in terms of $\alpha$ by Stokes's relation

$$F^2 = \frac{\tan 2\alpha}{2\alpha}. \tag{2.4}$$

By inverting the series (2.3), both $\alpha$, $\eta$ and $F^2$ may be expressed as power series in $\epsilon$. In this way we find, for instance, that

$$\left.\begin{aligned} F^2 &= \sum_{n=0}^{\infty} A_n \epsilon^n, \\ \alpha &= \frac{\surd 3}{2} \sum_{n=0}^{\infty} B_n \epsilon^{n+\frac{1}{2}}, \end{aligned}\right\} \tag{2.5}$$

where the leading coefficients $A_0$ and $B_0$ are equal to unity. Also the total mass $M$ and the total kinetic and potential energies $T$ and $V$ (see (I)) are given by

$$\left.\begin{aligned} M &= \frac{4}{\surd 3} \sum_{n=0}^{\infty} C_n \epsilon^{n+\frac{1}{2}}, \\ T &= \frac{4}{3\surd 3} \sum_{n=0}^{\infty} D_n \epsilon^{n+\frac{3}{2}}, \\ V &= \frac{4}{3\surd 3} \sum_{n=0}^{\infty} E_n \epsilon^{n+\frac{3}{2}}, \end{aligned}\right\} \tag{2.6}$$

where again the leading coefficients in each series are equal to 1.

Fenton (1972) carried the expansions of $\alpha$, $F^2$ and the mean drift $\delta$ (which is equivalent to $M$) as far as the ninth order, i.e. to terms in $\epsilon^9$. With the aid of Shanks transforms, satisfactory convergence was obtained up to about $\epsilon = 0.5$ and perhaps further. However, two puzzling features of the expansion were apparent. First in the expression for the mean drift $\delta$ (or mass $M$) the Shanks transforms did not converge well. Secondly, the coefficients in the series for $\delta$ and $\alpha$ in terms of $\epsilon$ began to show certain irregularities at the higher orders.

## 3. Extension and verification of the series

To obtain greater accuracy, and at the same time to reduce the possibility of rounding errors, we first recalculated the coefficients for $M$, $T$, $V$ and $F^2$ as far as the fourteenth order, with the aid of the CDC 6600 at London University, where a precision of 29 decimal places is readily available. The resulting values (verified on the IBM 370 at Cambridge) are shown in table 1.

Table 1. Coefficients in the expansions of $F^2$, $M$, $T$ and $V$ in powers of $\epsilon$

| $n$ | $F^2$ | $M$ | $T$ | $V$ | $R_1$ | $R_2$ |
|---|---|---|---|---|---|---|
| 0 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | $3\times10^{-29}$ | 0 |
| 1 | 1.000000 | .375000 | .725000 | .325000 | $-8\times10^{-29}$ | 0 |
| 2 | −.050000 | −.280312 | −.449062 | −.341920 | $3\times10^{-28}$ | $5\times10^{-27}$ |
| 3 | −.042857 | −.108169 | −.194540 | −.144510 | $2\times10^{-25}$ | $-1\times10^{-26}$ |
| 4 | −.034286 | −.063088 | −.113816 | −.090043 | $-3\times10^{-24}$ | $-2\times10^{-24}$ |
| 5 | −.031519 | −.053078 | −.100767 | −.076775 | $4\times10^{-22}$ | $2\times10^{-23}$ |
| 6 | −.029278 | −.042063 | −.079892 | −.061986 | $3\times10^{-20}$ | $-2\times10^{-21}$ |
| 7 | −.026845 | −.039924 | −.078847 | −.062097 | $2\times10^{-18}$ | $-2\times10^{-19}$ |
| 8 | −.030263 | −.035031 | −.068734 | −.050015 | $-2\times10^{-16}$ | $-2\times10^{-17}$ |
| 9 | −.021935 | −.040232 | −.086790 | −.076875 | $4\times10^{-15}$ | $2\times10^{-15}$ |
| 10 | −.048229 | −.008036 | .007800 | .044564 | $-8\times10^{-12}$ | $-3\times10^{-14}$ |
| 11 | .051809 | −.212733 | −.605472 | −.675052 | $2\times10^{-9}$ | $6\times10^{-11}$ |
| 12 | −.506790 | 1.381975 | 4.155444 | 4.667953 | $-3\times10^{-8}$ | $-1\times10^{-8}$ |
| 13 | 3.4666 | −13.4419 | −40.0928 | −43.6958 | $4\times10^{-5}$ | $3\times10^{-7}$ |
| 14 | −31.64 | 148.52 | 443.41 | 475.94 | $4\times10^{-3}$ | $-3\times10^{-4}$ |

It can be seen that up to the ninth order the values given by Fenton (1972) are verified except for $M$, where there were errors in the fifth or sixth decimal places. However, it will also be apparent that beyond the ninth order the coefficients, though previously tending to diminish, now begin to increase rapidly and to alternate in sign.

The first question to be asked is whether the higher coefficients are significant or are merely the result of rounding errors. As a check we used two identities, the first,

$$3V = (F^2-1)M, \tag{3.1}$$

being due to Starr (1947). Both this and the second identity:

$$T-V = \int_1^{F^2} (T/F^2)\,\mathrm{d}F^2 \tag{3.2}$$

are proved in paper I. Each side of these equations was expanded in powers of $\epsilon$ and the coefficients on the two sides were compared. The differences in the coefficients are shown in the last two columns of table 1, from which it can be seen that the identities are well verified. Hence it appears that the coefficients are indeed significant, to the degree of precision given.

## 4. Change of variable

What then is the reason for the irregular behaviour of the coefficients in table 1? It appeared to us possible that irregularities were due to the singular behaviour of the parameter $\epsilon$ at or near the value of $F$ corresponding to the wave of maximum amplitude. For the limiting wave itself is known to have a sharp-angled crest, with an interior angle of 120° (Stokes 1880). Shortly before the maximum amplitude is attained, there could be a large change in $\epsilon$ corresponding to only a relatively small increase in the speed of the wave. If this were so, it might be expected that expansions of *smoothly* varying quantities in terms of a *singular* parameter $\epsilon$ might well behave in an irregular way.

To test this conjecture the series for $M$, $T$ and $V$ were reformulated in powers of the new parameter

$$\gamma = F^2 - 1, \tag{4.1}$$

which like $\epsilon$ vanishes for waves of small amplitude. Thus we now have

$$\left.\begin{aligned} \epsilon &= \sum_{n=0}^{\infty} a_n \gamma^{n+1}, \\ \alpha &= \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} b_n \gamma^{n+\frac{1}{2}}, \end{aligned}\right\} \tag{4.2}$$

and

$$\left.\begin{aligned} M &= \frac{4}{\sqrt{3}} \sum_{n=0}^{\infty} c_n \gamma^{n+\frac{1}{2}}, \\ T &= \frac{4}{3\sqrt{3}} \sum_{n=0}^{\infty} d_n \gamma^{n+\frac{3}{2}}, \\ V &= \frac{4}{3\sqrt{3}} \sum_{n=0}^{\infty} e_n \gamma^{n+\frac{3}{2}}. \end{aligned}\right\} \tag{4.3}$$

The resulting coefficients for $\epsilon$, $M$, $T$ and $V$ are shown in table 2. It will be seen that all the coefficients for $M$, $T$ and $V$ beyond $n = 1$ are now of the same sign, and moreover vary smoothly with $n$. The coefficients for $V$ are in fact identical with those for $M$, in accordance with equation (3.1).

In figure 1 we have plotted the *ratios* $c_n/c_{n-1}$ of the coefficients in the series (4.3) for $M$. These are plotted against $n^{-1}$, as in a Domb–Sykes plot. It can be seen that the odd ratios now lie on a smooth curve, and the even ratios on another smooth curve, both tending towards the same limit as $n \to \infty$. As is well known, the limit of $c_n/c_{n-1}$, if it exists, is equal to the reciprocal of the radius of convergence $\gamma_0$ of the series (4.3).

Extrapolation to $1/n = 0$ by means of the rational function

$$\frac{P_0+P_1/n+P_2/n^2}{1+Q_1/n+Q_2/n^2} \tag{4.4}$$

fitted to the points $n = 4, 6, 8, 10$ and $12$ leads to the value $1/\gamma_0 = 1.462$, hence $F^2 = 1.684$, while a similar extrapolation from the points $n = 5, 7, 9, 11$ and $13$ leads to $1/\gamma_0 = 1.562$, hence $F^2 = 1.640$. In figure 1 the broken line is drawn to the

TABLE 2. COEFFICIENTS IN THE EXPANSIONS OF $\epsilon$, $M$, $T$ AND $V$ IN POWERS OF $\gamma$

| $n$ | $\epsilon$ | $M$ | $T$ | $V$ |
|---|---|---|---|---|
| 0 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| 1 | .050000 | .400000 | .800000 | .400000 |
| 2 | .047857 | −.228571 | −.285714 | −.228571 |
| 3 | .045625 | −.093714 | −.112762 | −.093714 |
| 4 | .049653 | −.066543 | −.071461 | −.066543 |
| 5 | .056030 | −.065298 | −.072131 | −.065298 |
| 6 | .064568 | −.065873 | −.069568 | −.065873 |
| 7 | .081683 | −.074001 | −.078924 | −.074001 |
| 8 | .093366 | −.084548 | −.088451 | −.084548 |
| 9 | .141814 | −.101693 | −.106586 | −.101693 |
| 10 | .086752 | −.123754 | −.128555 | −.123754 |
| 11 | .641255 | −.155288 | −.161156 | −.155288 |
| 12 | −2.95101 | −.196263 | −.202767 | −.196263 |
| 13 | 29.716 | −.2536 | −.2616 | −.2536 |
| 14 | −317. | −.34 | −.35 | −.34 |

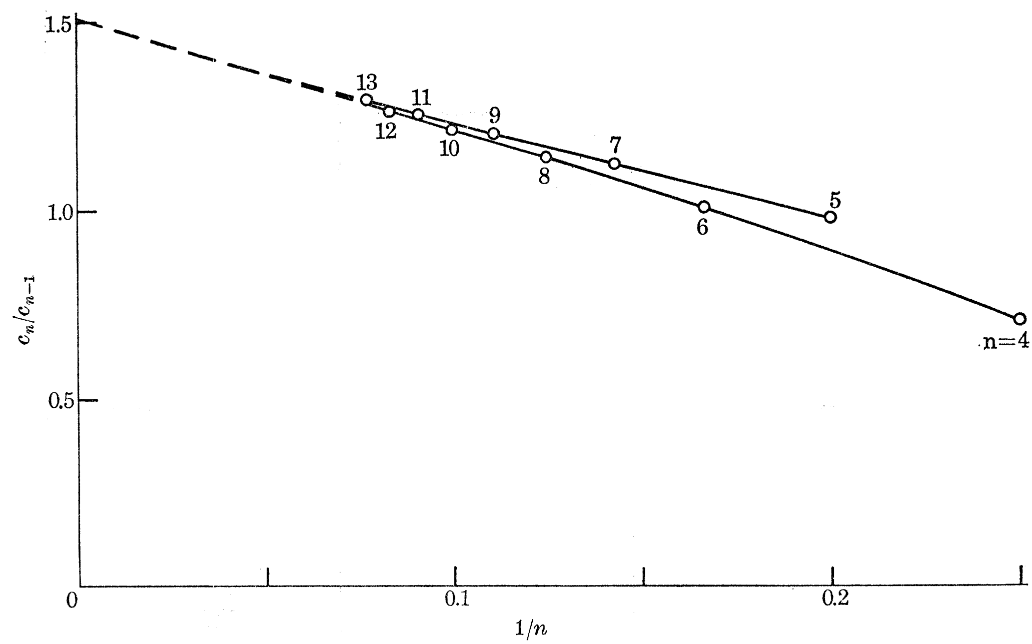


FIGURE 1. A Domb–Sykes plot of the ratios $c_n/c_{n-1}$ of successive coefficients in the series for the mass $M$ in powers of $\gamma$.

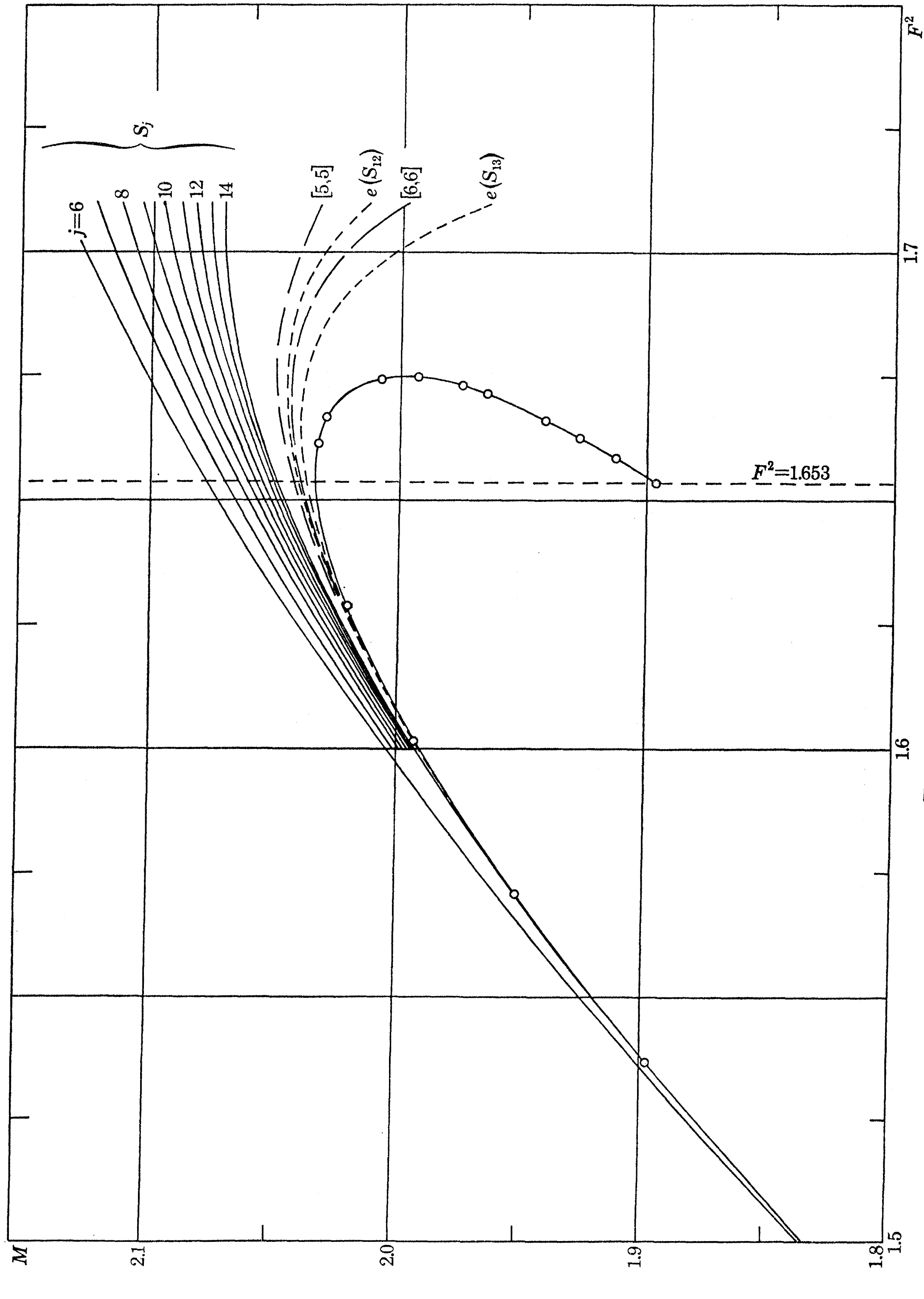


FIGURE 2. For legend see opposite.

mean value $1/\gamma_0 = 1.511$, corresponding to $F^2 = 1.662$. However the difference between the two extrapolations is a measure of the uncertainty of this estimate.

Figure 1 also gives some indication of the nature of the singularity on the circle of convergence. For if near $\gamma_0$

$$M \sim M_0 + K|\gamma - \gamma_0|^p, \tag{4.5}$$

where $M_0$, $K$ and $p$ are constants, then as $n \to \infty$

$$\frac{c_n}{c_{n-1}} \sim \frac{1}{\gamma_0}\left(1 - \frac{p+1}{n}\right). \tag{4.6}$$

Therefore

$$p+1 = -\gamma_0 \lim_{n\to\infty} \frac{\mathrm{d}(c_n/c_{n-1})}{\mathrm{d}(1/n)}. \tag{4.7}$$

In figure 1, the gradient of the asymptote near $1/n = 0$ is about 2.94, giving $p+1 = 1.95$ and so $p = 0.95$. This being close to unity suggests that there is at most a weak singularity – possibly only logarithmic – at $\gamma = \gamma_0$.

A Domb–Sykes plot of the coefficients for $V$ and $T$ gives almost identical results with those for $M$. By contrast, the ratios of the coefficients for $\epsilon$, as seen from the second column of table 2, behave in a quite irregular manner, after about $n = 9$. There is no well-determined radius of convergence, and it appears that the series in powers of $\gamma$ is at best asymptotic.

When we attempt to sum the series for $M$ and $\epsilon$ in powers of $\gamma$ we find (see figure 2) that for $M$ the partial sums

$$S_j = \frac{4}{\sqrt{3}} \sum_{n=0}^{j} c_n \gamma^{n+\frac{1}{2}} \quad (j = 1, 2, \ldots, 14)$$

converge well up to about $F^2 = 1.60$, but not beyond. Some improvement can be obtained with the use of Shanks (1955) transforms:

$$e(S_j) = \frac{S_{j-1}S_{j+1} - S_j^2}{S_{j-1} + S_{j+1} - 2S_j}$$

as in Fenton (1972), and by Padé approximants $[N, N]$ (see Baker 1965) up to about $F^2 = 1.63$, but not beyond. The series for $V$ and $T$ behave in a similar fashion.

As for $\epsilon$ (see figure 3) the Shanks transforms $e(S_{12})$ and $e(S_{13})$ appear at first to give results accurate up to $F^2 = 1.70$ but the Padé approximants $[N, N]$ do not agree with the Shanks transforms. In fact the Padé approximants clearly diverge when $F^2 > 1.68$.

The reasons for this behaviour will shortly become clear.

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FIGURE 2. Convergence of the power series (4.3) for $M$ in powers of $\gamma$, in the range

$1.5 < F^2 < 1.75.$

The $S_j$ indicate partial sums to $j$ terms, $e(S_j)$ indicate the Shanks transforms and $[N, N]$ the Padé approximants. The open circles represent values calculated from the series for $M$ and $F^2$ in powers of $\omega$ (see § 5).

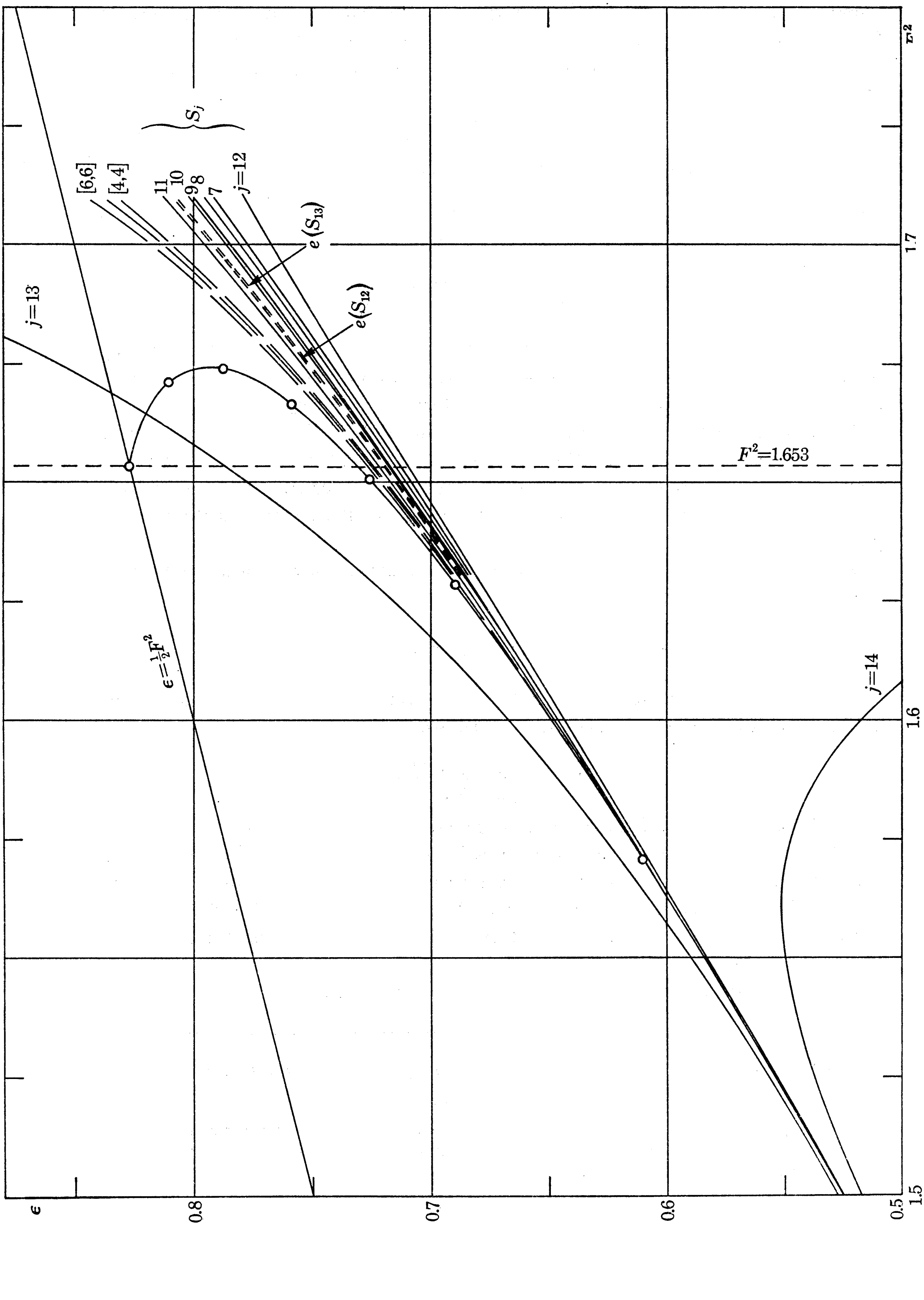


FIGURE 3. For legend see opposite.

## 5. Expansions in powers of $\omega$

A difficulty in using either $\gamma$ or $\epsilon$ as expansion parameters is that the precise range of these variables is not yet well determined (see I). On the other hand, one quantity whose range is precisely known is $U$, the value of the velocity at the wave crest (in the frame of reference moving with the wave speed). In the wave of limiting height $U$ vanishes, and for waves of low amplitude $U$ tends to $\sqrt{(gh)}$. Thus the new parameter

$$\omega = 1 - U^2/gh \tag{5.1}$$

varies between 0 and 1 as the wave amplitude varies from 0 to its (unknown) limiting value $\epsilon_{\max}$.

Application of Bernoulli's theorem shows at once that

$$\omega = 1 - (F^2 - 2\epsilon). \tag{5.2}$$

Thus we have simply

$$\omega = 2\epsilon - \gamma \tag{5.3}$$

and from (5.2) and (2.5), since $A_0 = A_1 = 1$, we have

$$\omega = \epsilon - \sum_{n=2}^{\infty} A_n \epsilon^n. \tag{5.4}$$

Inverting this series we obtain

$$\epsilon = \omega + \sum_{n=2}^{\infty} G_n \omega^n \tag{5.5}$$

say. By substitution in equations (2.5) and (2.6) we can now obtain also $F^2$, $M$, $T$ and $V$ in powers of $\omega$.

The coefficients of the new series are shown in table 3. At first sight they appear similar to those in table 1, that is to say they tend to diminish in magnitude up to about $n = 9$, and then to increase sharply and oscillate in sign. Nevertheless we find that by this simple change of variable the convergence is now radically improved. Thus, the $[N, N]$ Padé approximants for $M$, $V$ and $T$ are found to converge remarkably rapidly, as can be seen from table 4. Even when $\omega = 1$, one can rely on at least three significant figures and, generally, more. The approximants for $\epsilon$ are not so regular. However, from the values of $M$ and $V$ we can calculate $\gamma$ using the relation

$$\gamma = 3V/M \tag{5.6}$$

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Figure 3. Convergence of the power series (4.2) for $\epsilon$ in powers of $\gamma$, in the range

$$1.5 < F^2 < 1.75.$$

The $S_j$ indicate partial sums to $j$ terms, $e(S_j)$ indicate the Shanks transforms and $[N, N]$ the Padé approximants. The open circles represent values calculated from the series for $M$ and $F^2$ in powers of $\omega$ (see § 5).

derived from equation (3.1), and then $\epsilon$ from

$$\epsilon = \tfrac{1}{2}(\gamma+\omega) \tag{5.7}$$

by (5.3).

As a check we can calculate the values of $L = T - V$ and it is found that the relation

$$F^2\,\mathrm{d}L = T\,\mathrm{d}F^2 \tag{5.8}$$

derived from equation (3.2), is indeed well verified.

The limiting value of $\gamma$ (as can be found from table 4) is equal to 0.653, which is quite consistent with the radius of convergence of the series for $M(\gamma)$, estimated earlier. Moreover, the corresponding values of $F^2$ and $\epsilon$, namely

$$F^2 = 1.653, \quad \epsilon_{\max} = 0.827 \tag{5.9}$$

Table 3. Coefficients in the expansions of $\epsilon$, $M$, $T$ and $V$ in powers of $\omega$

| $n$ | $\epsilon$ | $M$ | $T$ | $V$ |
|---|---|---|---|---|
| 0 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| 1 | −.050000 | .350000 | .650000 | .250000 |
| 2 | −.037857 | −.327679 | −.595536 | −.438393 |
| 3 | −.024196 | −.106652 | −.216039 | −.148777 |
| 4 | −.017886 | −.041461 | −.079560 | −.058696 |
| 5 | −.012495 | −.026995 | −.054047 | −.036427 |
| 6 | −.007442 | −.010688 | −.018356 | −.011596 |
| 7 | −.009214 | −.008586 | −.017636 | −.015099 |
| 8 | .002612 | −.000784 | .001473 | .009990 |
| 9 | −.025296 | −.012157 | −.036005 | −.050047 |
| 10 | .089517 | .057553 | .173921 | .222498 |
| 11 | −.524074 | −.384976 | −1.161020 | −1.357498 |
| 12 | 3.80580 | 3.05844 | 9.1826 | 10.2853 |
| 13 | −34.07 | −28.75 | −86.23 | −94.18 |
| 14 | 364. | 317. | 950. | 1021. |

Table 4. Padé approximants $[N, N]$ for $M$, $T$ and $V$ at two high values of $\omega$

| $N$ | $M$ | $T$ | $V$ | |
|---|---|---|---|---|
| 1 | 2.64704 | .948117 | .776295 | $\omega = 0.95$ |
| 2 | 2.00866 | .566769 | .455148 | |
| 3 | 1.97105 | .542441 | .442753 | |
| 4 | 1.96612 | .539911 | .440575 | |
| 5 | 1.96454 | .538266 | .439735 | |
| 6 | 1.96447 | .537943 | .439488 | |
| 7 | 1.96432 | .537928 | .439444 | |
| 1 | 2.72686 | 1.030925 | .839691 | $\omega = 1.00$ |
| 2 | 1.95986 | .548482 | .436380 | |
| 3 | 1.90715 | .512723 | .418231 | |
| 4 | 1.89958 | .508686 | .414792 | |
| 5 | 1.89698 | .505855 | .413384 | |
| 6 | 1.89689 | .505269 | .412954 | |
| 7 | 1.89664 | .505241 | .412875 | |

agree closely with Yamada's (1957) estimate $\epsilon_{max} = 0.828$ derived from a direct calculation of the profile of the highest wave, and Lenau's (1966) result

$$\epsilon_{max} = 0.827$$

derived independently from the solution of an integral equation.†

The behaviour of $M$, $T$, $V$, $\gamma$ and $\epsilon$ as functions of $\omega$ is shown graphically in figure 4. When $\omega$ is small, these quantities are evidently all increasing functions of $\omega$.

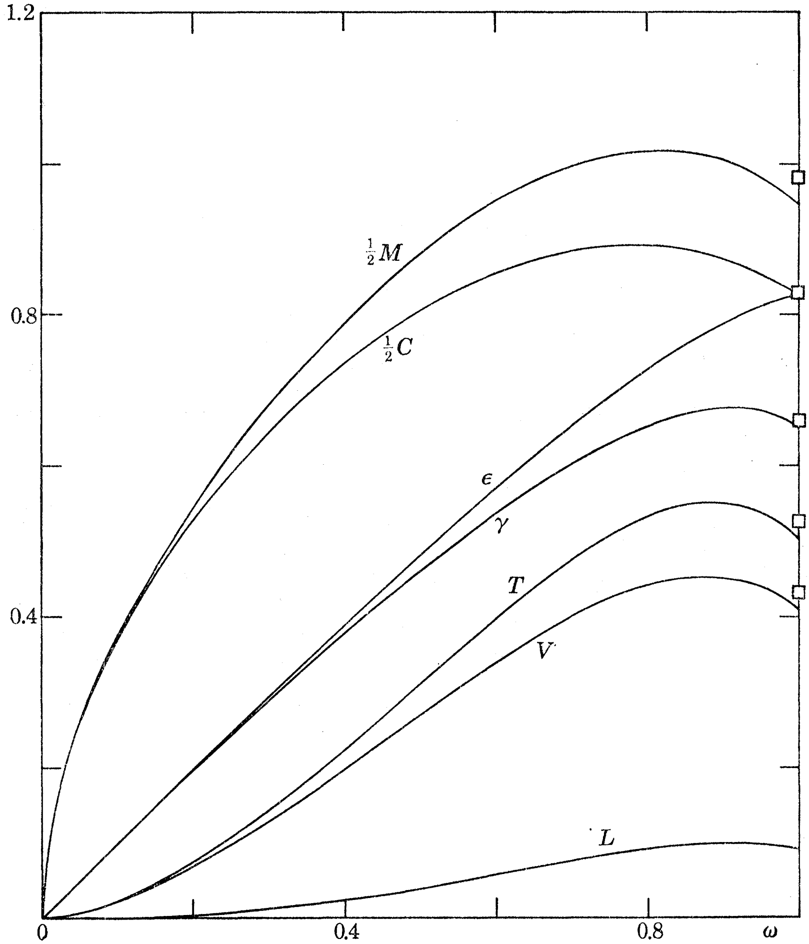


FIGURE 4. The mass $M$, circulation $C$, wave height $\epsilon$, kinetic energy $T$, potential energy $V$, Lagrangian $L$ and $\gamma = F^2 - 1$, where $F$ is the non-dimensional wave speed, as functions of the parameter $\omega$. The plotted points denote the rough estimates for the wave of limiting amplitude given in paper I.

† Lenau also pointed out that this is practically the same value as can be obtained from a solution due to Packham (1952), who approximated the pressure condition at the free surface, replacing $\sin\theta$ by $\frac{1}{3}l \sin 3\theta$. If in Packham's solution one sets $l = 1$ one obtains

$$\epsilon_{max} = \sqrt{27}/2\pi = 0.8270$$

(see also Davies 1952).

The most striking feature of the diagram is that while $\epsilon$ is increasing throughout the entire range, both $M$, $T$, $V$ and $\gamma$ have *maxima* between $\omega = 0.80$ and $\omega = 1.00$. The maximum in $M$ was suspected by Fenton (1972). On the other hand the maxima in $V$, $T$ and $\gamma$ are more surprising, and may have some important physical consequences (see §§ 8 and 9).

Using $\omega$ as a parameter, the variables $M$ and $\epsilon$ can now be plotted as functions of $F^2$ (see figures 2 and 3). The previously found behaviour of the series for $M$ and $\epsilon$ in powers of $\gamma$ can now be explained. For, beyond a certain value of $F^2$, both $M$ and $\epsilon$ are double-valued functions of $F^2$. The ordinary partial sums $S_j$ cannot

Table 5. Calculated values of $M$, $T$, $V$, $\gamma$, $\epsilon$, $F$, $L$, $C$ and $I$ for particular values of the parameter $\omega$

| $\omega$ | $M$ | $T$ | $V$ | $\gamma$ | $\epsilon$ | $F$ | $L$ | $C$ | $I$ |
|---|---|---|---|---|---|---|---|---|---|
| .05 | 0.5250 | .00887 | .00870 | .04971 | .04986 | 1.02456 | .00017 | .5206 | .5379 |
| .10 | 0.7534 | .02578 | .02484 | .09891 | .09946 | 1.04829 | .00094 | .7406 | .7898 |
| .15 | 0.9344 | .04845 | .04593 | .14746 | .14873 | 1.07120 | .00252 | .9105 | 1.0009 |
| .20 | 1.0906 | .07603 | .07100 | .19531 | .19765 | 1.09333 | .00503 | 1.0533 | 1.1924 |
| .25 | 1.2299 | .10792 | .09935 | .24234 | .24617 | 1.11460 | .00857 | 1.1772 | 1.3708 |
| .30 | 1.3563 | .14354 | .13041 | .28845 | .29423 | 1.13510 | .01313 | 1.2866 | 1.5395 |
| .35 | 1.4715 | .18231 | .16359 | .33352 | .34176 | 1.15478 | 0.1872 | 1.3835 | 1.6993 |
| .40 | 1.5765 | .22361 | .19833 | .37741 | .38871 | 1.17363 | .02528 | 1.4692 | 1.8503 |
| .45 | 1.6717 | .26671 | .23399 | .41991 | .43496 | 1.19160 | .03272 | 1.5443 | 1.9920 |
| .50 | 1.7572 | .31081 | .26991 | .46081 | .48040 | 1.20864 | .04090 | 1.6095 | 2.1236 |
| .55 | 1.8326 | .35499 | .30532 | .49981 | .52491 | 1.22467 | .04967 | 1.6646 | 2.2443 |
| .60 | 1.8975 | .39819 | .33939 | .53658 | .56829 | 1.23959 | .05880 | 1.7097 | 2.3521 |
| .65 | 1.9510 | .43915 | .37117 | .57074 | .61036 | 1.25329 | .06798 | 1.7444 | 2.4452 |
| .70 | 1.9923 | .47646 | .39956 | .60166 | .65082 | 1.26557 | .07690 | 1.7684 | 2.5214 |
| .75 | 2.0203 | .50844 | .42332 | .62860 | .68930 | 1.27617 | .08512 | 1.7814 | 2.5782 |
| .80 | 2.033 | .53313 | .4410 | .6506 | .7253 | 1.2848 | .0921 | 1.782 | 2.612 |
| .85 | 2.030 | .5482 | .4510 | .6665 | .7583 | 1.2909 | .0972 | 1.771 | 2.621 |
| .90 | 2.008 | .5509 | .4512 | .6742 | .7871 | 1.2939 | .0997 | 1.746 | 2.598 |
| .95 | 1.964 | .5379 | .4394 | .6711 | .8108 | 1.2927 | .0985 | 1.707 | 2.539 |
| 1.00 | 1.897 | .5052 | .413 | .653 | .827 | 1.286 | .0924 | 1.653 | 2.440 |

possibly converge beyond the radius of convergence: $F^2 = 1.653$. Even the Padé approximants $[N, N]$, which *can* converge beyond this point, are necessarily single-valued, and cannot yield values for $M$ and $\epsilon$ beyond the point where $F^2$ takes its maximum value.

Figure 2 also suggests a reason for the rather higher estimates of $\epsilon_{max}$ made by Byatt-Smith (1970) and Fenton (1972). These estimates were based on an extrapolation of the relation between $\epsilon$ and $F$ for waves of substantially less than the maximum amplitude, and so did not take account of the sharp curvature of the actual curve near $\epsilon = \epsilon_{max}$.

The maximum value of $F$ we find to be 1.294, corresponding to $\epsilon = 0.790$. It is perhaps significant that Byatt-Smith was unable to obtain convergence in his integral equation for any value of $F$ greater than 1.293. This would be consistent with a natural expectation that in the neighbourhood of a stationary value of $F$

the integral equation would not have a unique solution, and hence would not converge.

We have also examined the Padé approximants to $M$ and $F^2$ derived from the series in powers of $\epsilon$. However, the approximants derived from the $\omega$-series converge more rapidly than those from the $\epsilon$-series, in both cases. In every case, for both $M$ and $F^2$, the Padé approximants converge more rapidly and regularly than do the Shanks transforms.

For practical use, we have given in table 5 the values of $M$, $T$, $V$, $\epsilon$ and $\gamma$ at equal intervals of $\omega$. Also given in table 5, and shown graphically in figure 4, are the values of the *circulation* $C$, defined by

$$C = \int_{-\infty}^{\infty} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{s} = [\phi]_{-\infty}^{\infty} \tag{5.10}$$

(see I; $\boldsymbol{u}$ and $\phi$ refer to the motion with respect to a stationary frame of reference, in which $\boldsymbol{u} \rightarrow 0$ at infinity). To calculate $C$ we used the relations

$$\left.\begin{aligned} 2T &= F(I-C), \\ I &= FM, \end{aligned}\right\} \tag{5.11}$$

given in (I), from which

$$C = MF - 2T/F. \tag{5.12}$$

Evidently $C$ has a maximum at about $\omega = 0.775$. The limiting value of $C$, from table 5, is 1.653. Hence we can state that for the limiting wave

$$C = F^2 \tag{5.13}$$

very nearly. We conjecture that this relation is exact.

## 6. Discussion

The most unexpected of our findings, namely the maxima in $M$, $T$, $V$ and $F^2$, may be made to seem more reasonable in the following way.

When the wave height $\epsilon$ is a little less than the maximum, the crest will be rounded, so that the wave profile is rounded and convex, in at least a short horizontal interval enclosing the crest. On the other hand, in the limiting wave, the surface profile consists of two *concave* arcs, and the surface elevation immediately falls away with horizontal distance from the crest. Hence, if as the wave amplitude approaches its limit, the crest 'flips up' locally, the limiting wave profile may intersect the profile of a wave having slightly smaller amplitude, and it is easy to see that the total volume $M$ of the limiting wave can indeed be slightly less.

Further, the kinetic energy $T$ and potential energy $V$, both of which can be expressed as integrals over the *volume* of the wave, can also be slightly reduced, for the same reason.

Now we may also define $\bar{\eta}$, the weighted average of $\eta$ taken over the energetic part of the wave, by

$$\bar{\eta} = \int_{-\infty}^{\infty} \eta^2\,\mathrm{d}x \Big/ \int_{-\infty}^{\infty} \eta\,\mathrm{d}x \tag{6.1}$$

and by the same argument it is not difficult to see that $\bar{\eta}$ may actually diminish as the limiting wave is approached. But $\bar{\eta}$ is also related to the wave speed $F^2$, for from (6.1) and (3.1) we have

$$\bar{\eta} = 2V/M = \tfrac{2}{3}(F^2 - 1). \tag{6.2}$$

Therefore a slight decrease in $\bar{\eta}$ implies also a decrease in the speed $F$.

Another view of this phenomenon is to be gained from the remark that in a solitary wave the speed $F$ is related to the exponent $(-2\alpha)$ in the outskirts of the wave profile by Stokes's equation (2.4). As the wave amplitude $\epsilon$ increases, $F$ also increases at first, hence the total width of the wave diminishes. But in the final stages of growth, as $\epsilon$ approaches $\epsilon_{max}$, a slightly wider 'base' is required to support the highest waves and so $F$ must finally *decrease* slightly.

It would be instructive to draw the actual sequence of wave profiles, from the wave of lowest amplitude to the highest wave. It would then necessarily be found that whereas at the lower amplitudes each profile intersects the preceding profile at only one point in the interval $0 < x < \infty$, the highest wave intersects its neighbours in two points in $0 < x < \infty$.

However, as pointed out in § 5, the wave profile $\eta(x)$, and particularly the height $\eta(0)$ of the wave crest, is not determined so accurately by the use of series expansions as are the integral properties represented by $M$, $T$ and $V$, which have been used for the calculation of $F$. We therefore leave the accurate determination of the surface profile for a future study.

## 7. The total energy

Figures 5 and 6 show the mass $M$, the momentum $I$ and the total energy $E = T + V$ as functions of the wave height $\epsilon$. It is remarkable that the curves for $E$ and $\frac{1}{2}M$ practically touch, not far from the maximum value of $F$. We can, however, show that if the curves do indeed touch then the point of contact is not exactly at the maximum of $F$. For from equations (3.1) and (3.2) we have in general

$$\left.\begin{aligned} 3\,\mathrm{d}V &= (F^2-1)\,\mathrm{d}M + M\,\mathrm{d}F^2, \\ F^2(\mathrm{d}T - \mathrm{d}V) &= T\,\mathrm{d}F^2 \end{aligned}\right\} \tag{7.1}$$

and if the curves in figure 5 touch, then

$$\mathrm{d}T + \mathrm{d}V = \tfrac{1}{2}\mathrm{d}M. \tag{7.2}$$

From the last three equations we may eliminate $\mathrm{d}T$ and $\mathrm{d}V$ to give

$$\frac{\mathrm{d}F^2}{\mathrm{d}M} = \frac{4F^2 - 7}{6T - 4F^2M} = 0.0299, \tag{7.3}$$

which does not vanish. On the contrary, from figure 5 we see that

$$\frac{\mathrm{d}M}{\mathrm{d}\epsilon} \doteqdot 1.68$$

so

$$\frac{\mathrm{d}F^2}{\mathrm{d}\epsilon} = \frac{\mathrm{d}F^2}{\mathrm{d}M}\frac{\mathrm{d}M}{\mathrm{d}\epsilon} \doteqdot 0.050.$$

The horizontal distance $\Delta\epsilon$ to the maximum of $F^2$ is thus given by

$$\Delta\epsilon = \frac{dF^2}{d\epsilon}\bigg/\frac{d^2F^2}{d\epsilon^2} \doteqdot \frac{0.050}{57.1} = 0.0009,$$

which is small, but consistent with our calculations.

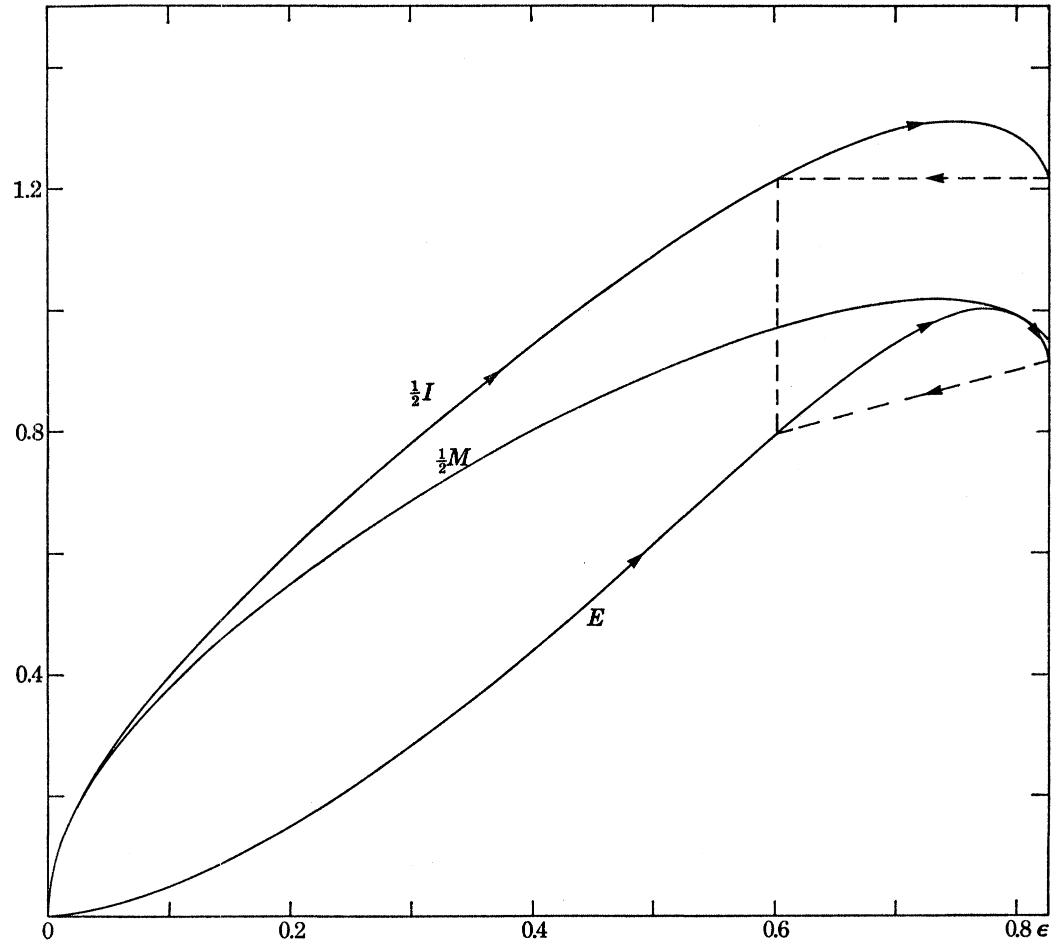


FIGURE 5. The momentum $I$, mass $M$ and total energy $E$ shown as functions of the normalized wave height $\epsilon$.

In general it appears from figure 5 that

$$M > 2(T+V). \tag{7.4}$$

On substituting for $T$ and $V$ from equations (3.1) and (5.11) we have

$$M > (F^2M - FC) - \tfrac{2}{3}(F^2-1)\,M;$$

that is,

$$\tfrac{5}{3}(F^2-1)\,M > FC \tag{7.5}$$

or

$$5V > FC. \tag{7.6}$$

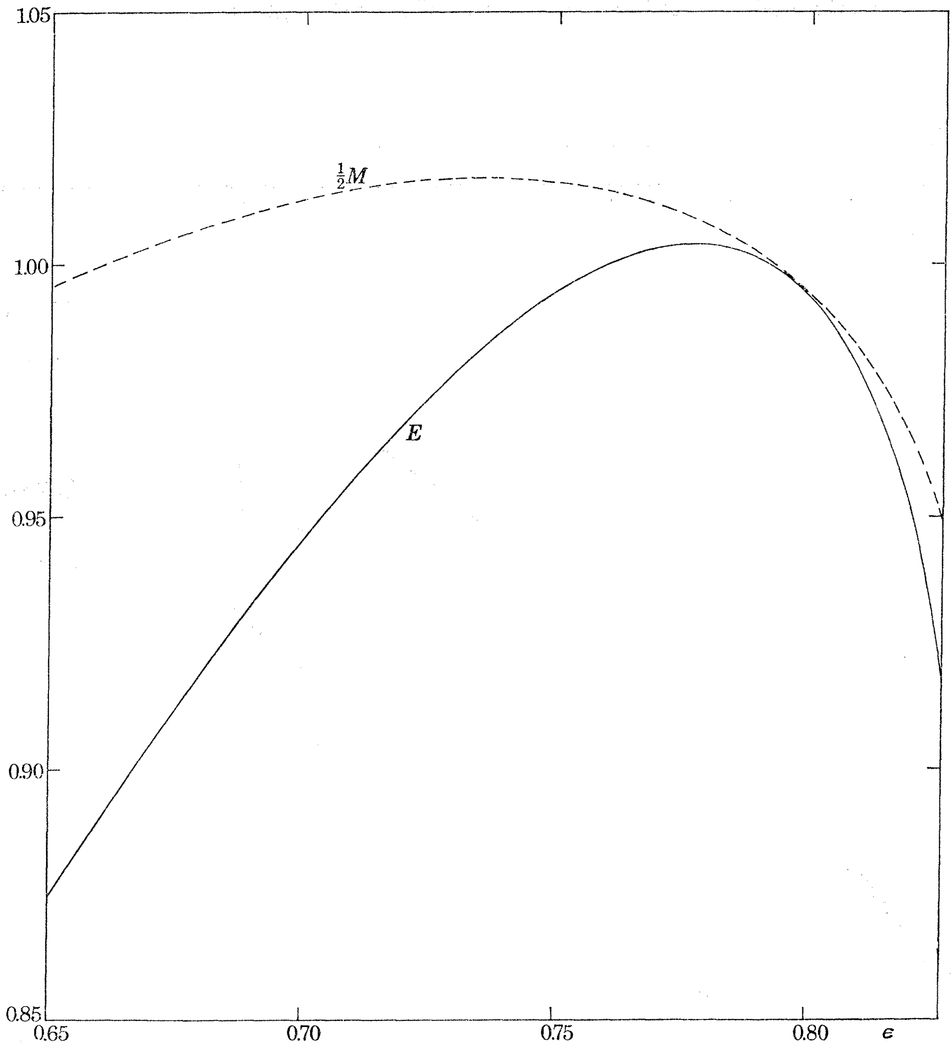


FIGURE 6. The mass $M$ and energy $E$ over the upper range of $\epsilon$.

## 8. Comparison with observation

The best modern observations of speed and surface profile of solitary waves are probably those of Daily & Stephan (1951, 1952) who were able to make measurements over the range of wave amplitudes corresponding to $0 < \epsilon < 0.62$, or about three-quarters of the range of $\epsilon$. They were not able to obtain wave amplitudes very close to the theoretical maximum. In figure 7 their experimental plot of $F$ against $\epsilon$ is reproduced, together with our theoretical curve. Also shown are two earlier approximations. Evidently the observations agree better with our theoretical curve than with either of the two approximations. However, there appears to be a

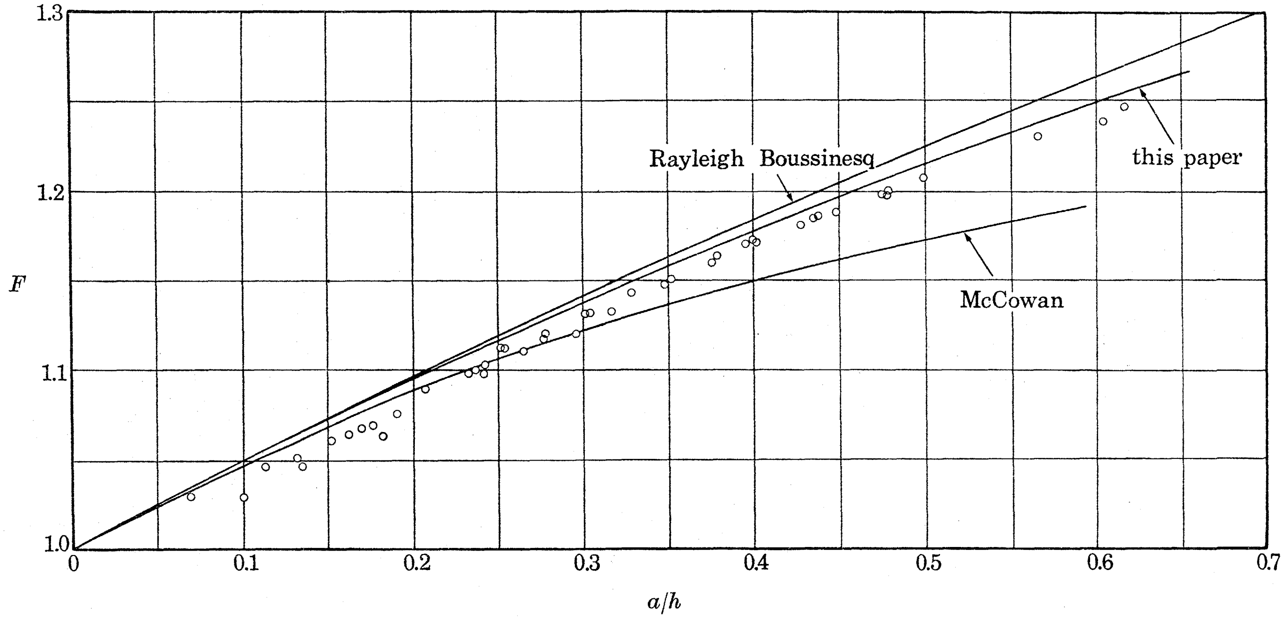


FIGURE 7. (After Daily & Stephen 1952.) Observed values of the non-dimensional speed $F = c/\sqrt{(gh)}$ in relation to the non-dimensional wave height $\epsilon = a/h$, compared with the approximations of Boussinesq (1871) and McCowan (1891), and with the present calculations.

systematic tendency for the observed wave height to exceed slightly the theoretical wave height, for a given value of $F$. This may be due partly to the presence of a viscous boundary-layer at the bottom, but also may be due to a general tendency for unsteady motions to be associated with augmented heights. In experiments with solitary waves on shoaling beaches, Ippen & Kulin (1955) found maximum amplitudes far in excess of the corresponding amplitude in water of uniform depth, generally by a factor of 2 or 3. It is possible that in uniform depth unsteadiness due to energy dissipation may produce a similar effect.

Daily & Stephan (1952) did not report observations of waves with amplitudes greater than 0.62. This could be due simply to their method of wave generation. For, if it is true that a plunger tends to generate waves of a given mass $M$ and impulse $I$ depending on the stroke of the plunger, then, over a range in which $M$ and $I$ are nearly stationary with respect to $\epsilon$, it will be hard indeed for the apparatus to select a unique wave.

Furthermore since from figure 5 the increment of the total energy $E$ is also small beyond $\epsilon = 0.65$, we see that only a slight loss of energy can appreciably reduce the wave amplitude to a point lower down the energy curve. In small-scale experiments in a channel of uniform depth, and with side walls, such losses are appreciable (see Ippen & Kulin 1955). Hence the higher waves are more likely to be obtained by generating waves of moderate amplitude, which are then made to propagate into a gradually narrowing channel, or into water of very gradually diminishing depth, as discussed in the next section.

## 9. On the breaking of waves in shallow water

From figures 5 and 6 it is clear that the total energy $E$ (or in dimensional terms $E/gh^3$) has a maximum at about $\epsilon = 0.778$. The presence of this maximum has some interesting and far-reaching consequences for the way in which solitary waves break as they enter shoaling water.

Most observers (Mason 1952; Iverson 1952; Ippen & Kulin 1955) have distinguished two main types of breaking wave: on the one hand the 'plunging' breaker, in which the forward face of the wave grows markedly steeper than the rear slope, and finally the crest falls violently into the forward face; on the other hand the 'spilling' breaker, in which the crest remains practically symmetrical and a quasi-steady whitecap, or roller, forms on the forward face. The height of plunging breakers, at the moment of plunging, is noticeably variable, and ranges from $0.9h$ to about $3.0h$, depending on the slope of the beach (see Ippen & Kulin 1955). The height of symmetrical spilling breakers, on the other hand, is much less, lying between about $0.65h$ and $0.85h$.

Now a solitary wave entering slowly shelving water will, if it is not breaking, have only a small dissipation of energy, arising mainly from friction at the bottom. Its total energy $Egh^3$ (in dimensional units) will therefore remain almost constant. As the depth $h$ decreases, the non-dimensional energy $E$ will accordingly increase.

The wave will therefore tend to evolve along the curve of figure 5, with both $\epsilon$ and $E$ increasing at first.

However, when the maximum value of $E$ is attained the wave can no longer remain symmetrical. It must therefore become unsymmetrical and possibly unsteady. If it is evolving slowly, as on a gentle slope, it may develop small instabilities on the forward face of the wave, and in this way lose enough energy to enable it to proceed *down* the declining energy curve towards the symmetrical wave of *maximum* amplitude. At this point, or close to it, there may momentarily be a balance between the energy lost through the instabilities on the forward face (which in large waves will develop into a whitecap) and the relative energy $E$ gained through diminishing depth of water. However, if the dissipation is sufficient, the energy will fall below the threshold value $0.918gh^3$ and the wave will jump back to a point on the lower part of the energy curve. Most probably, since some energy may be lost whereas momentum is conserved over a short distance, the wave will jump to a point on the energy curve where the impulse $I$ is close to its limiting value 2.44.† This occurs when the wave height $\epsilon$ is about 0.64. The wave will then grow again and the process repeat itself. This hypothesis is well in accordance with the marked intermittency noticed in spilling breakers by Longuet-Higgins & Turner (1974).

If on the other hand the beach slope is relatively steep, then the solitary wave, on arriving at the maximum value of $E$, may not be able to dissipate sufficient energy soon enough to enable it to descend the energy curve towards the wave of maximum amplitude. In this case it evolves rapidly into an unsymmetrical wave, and ultimately becomes a plunging breaker. Because the wave is not restricted to a symmetrical form, its height can ultimately be much greater than the limiting value of 0.827 for the symmetrical wave.

## 10. Conclusions

By accurate calculation, and the use of integral quantities such as the mass and potential energy, we have determined the behaviour of the wave speed and other parameters of the solitary wave, as functions of the wave height $\epsilon$, over the entire range $0 < \epsilon < 0.827$. We have shown that the speed $F$, the mass $M$ and the potential and kinetic energies, all have maxima within the range of wave amplitudes.

There is good agreement between our theoretical values of $F$ and the best available observations of solitary waves in water of uniform depth, which however extend only as far as $\epsilon = 0.62$. The absence of observations of very high waves in water of uniform depth can be accounted for by the reduction in wave amplitude associated with only a very slight loss of energy.

Considerable light can be thrown on the breaking of solitary waves in shallow water by the single fact that the highest wave is not the most energetic. To obtain greater energy the wave must become asymmetrical and probably unstable. The

† Any additional mass $M$ can always be accommodated in the fringes of the wave, without altering the mean level.

growth of the asymmetry and the nature of the instabilities which seem to develop on the forward face of the wave would repay further study, both theoretical and experimental. On the smaller scales, however, such instabilities are bound to be influenced strongly by surface tension, which we have so far ignored, as having increasingly less effect when the wavelength is enlarged.

The reasons given in § 6 for the existence of an energy maximum for solitary waves must also apply to *periodic* gravity waves, including waves in deep water. Thus we may expect to find similar phenomena for breaking in deep water, particularly some instabilities on the forward face of each wave and a marked intermittency in spilling breakers. The detailed discussion of waves in deep water is left for a separate study.

*Note added on* 6 *May* 1974. Since completing our calculations it has come to our notice that Witting (1974) has arrived at a similar conclusion regarding the asymptotic nature of the series (2.2), but by a quite different approach. His conclusion is based in part on an examination of the numerical behaviour of the coefficients in an expansion of $(x+iy)$ about the point $(\phi+i\psi) = \infty$ (where $\phi$ is the velocity potential). He suggests that in general an expansion in integral powers is not complete. (It must be admitted, however, that asymptotic series can give highly accurate results.) In the special case of the *highest* wave, his numerical results suggest that the series (2.2) may be complete, and that, if so, then $\epsilon_{max} = \sqrt{27}/2\pi = 0.827...$. Thus in this case his conclusions independently confirm our own.

We are indebted to Dr L. W. Schwartz for comments on a first draft of this paper.

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