Underwater sound from surface waves according to the Lighthill-Ribner theory

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The acoustic source density given by Ribner's version of the Lighthill theory is calculated for a stochastic surface gravity wavefield. A Green function gives the acoustic farfield, as a volume integral. The method eliminates some of the complications involving the moving ocean surface. For a homogeneous ocean, the farfield acoustic spectrum agrees with the spectrum of Hughes [J. Acoust. Soc. Am. 60, 1032–1039 (1976)], and at low frequencies with the spectrum of Brekhovskikh [Izv. Atmos. Ocean Phys. 2, 582–587 (1966)], after certain corrections are applied to their results. At high frequencies, the Brekhovskikh spectrum is larger by a factor $(5/4)^2$.

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INTRODUCTION

The production of underwater sound by a surface gravity wavefield has been treated by a number of authors.[1-6] In each treatment there is a perturbation series expansion in the wave-height amplitude, and from the fields at the surface a part is picked out which represents a propagating acoustic field. An acoustic wave is produced by gravity waves interacting via the nonlinear terms in the hydrodynamic equations.

Lighthill[7] develops a theory of sound production in a fluid in which quadratic nonlinearities in the hydrodynamic equations are isolated as a source term in an acoustic wave equation. In a modification of the Lighthill theory, Ribner[8] works with the overpressure relative to the pressure p_0 in a hypothetical incompressible flow, and obtains an acoustic source density $-(1/c^2)\partial^2 p_0/\partial t^2$. This has a very simple interpretation: A pressure fluctuation δp_0 produces a density fluctuation $\delta\rho = (1/c^2)\delta p_0$ in the actual compressible ocean, so the Ribner source density $-\partial^2\rho/\partial t^2$ is that of a distribution of monopoles; antireflection at the free ocean surface gives an overall dipole effect in the farfield.

We model the surface gravity wavefield as a linear stochastic superposition of plane gravity waves of the usual theory, we calculate the Ribner acoustic source density for this potential flow, and then we apply an acoustic Green function for the moving ocean to obtain an acoustic farfield. Our derivation has the advantage that it represents sound production as a volume effect; indeed, we must exercise considerable care to eliminate considerations involving the moving ocean surface. With this we are able to understand the effects of neglected terms, e.g., vorticity, higher order terms in the perturbation series, etc.

In Fig. 1 of Hughes[4] the various theoretical spectra lie 10–20 dB below the observations (1–8 Hz) in a representative case. His formula [Ref. 4, Eq. (33)] is larger than our formula (35) by a factor of 2 (the discrepancy is discussed at the end of Sec. V), so we expect our formula (35) to be 15–25 dB below the observations at these frequencies.

I. THE HYDRODYNAMIC EQUATIONS

The kinematic, dynamic, and thermodynamic equations of the ocean water are all of a piece, and describe all of the various motions identifiable as wave motions, as well as the hydrostatic behavior.[9] It is feasible and useful, however, to separate out the acoustical part of the pressure fluctuation, i.e., the part whose presence is due to the water compressibility. Such a separation necessarily involves discarding number of interactions. The method we use is derived from the Lighthill[7] theory, as modified by Ribner.[8] It gives an intuitively appealing model for the generation of sound by the gravity wave motion near the surface, and the physics of the approximations is more or less transparent. We give the derivation for completeness, since there are certain differences from the jet noise case considered by Lighthill and Ribner.

Various notations will be employed for Cartesian components, as convenient, e.g., the coordinates x,y,z may be denoted vectorially by $\mathbf{r}$ or tensorially by x_r, $r=1,2,3$, with the summation convention. The surface of the quiescent ocean is $\{z=0\}$, and z increases upwards. The unit vector in the positive z direction is denoted by $\mathbf{e}$. The equation of the instantaneous ocean surface $\mathcal{S}_t$ at time t is of the form $z-h(x,y,t)=0$, $-\infty < x, y < \infty$, where h is single valued and as smooth as necessary. The depth of the ocean does not appear in our calculations, and at time t, the ocean water occupies the region

$$\mathcal{R}_t = [(x,y,z): -\infty < z < h(x,y,t)] .$$

The velocity field of the ocean water is defined in $\mathcal{R}_t$, and is denoted by (u,v,w), or by $\mathbf{v}$ or v_r, $r=1,2,3$; arguments are usually suppressed. The water density being ρ, the continuity equation is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \quad \text{in } \mathcal{R}_t , \tag{1}$$

where ∇ is the gradient operator. The water compression rate is denoted by ϵ:

$$\epsilon = -\nabla\cdot\mathbf{v} ,$$

and the continuity equation can be rewritten as

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$$\epsilon = \frac{1}{\rho}\frac{D\rho}{Dt}, \tag{2}$$

where $D/Dt = \partial/\partial t + \nabla \cdot \mathbf{v}$ is the usual material derivative.

The Euler equation is

$$\frac{Dv_r}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_r} + \frac{\partial}{\partial x_r}(gz) - \frac{1}{\rho}\frac{\partial \sigma'_{rs}}{\partial x_s} = 0, \tag{3}$$

where g is the constant acceleration of gravity, and where the viscous deceleration is given by (Ref. 10, p. 49):

$$\frac{1}{\rho}\frac{\partial \sigma'_{rs}}{\partial x_s} = \frac{\mu}{\rho}\Delta v_r - \frac{(\mu/3)+\zeta}{\rho}\frac{\partial \epsilon}{\partial x_r}; \tag{4}$$

$\Delta = \nabla \cdot \nabla$ denotes the Laplacian. Terms involving the gradients of the dynamic viscosity coefficients μ, ζ have been omitted in (4). We take the divergence of (3) and use

$$\nabla \cdot \left(\frac{D\mathbf{v}}{Dt}\right) = \frac{\partial}{\partial x_r}\left(\frac{\partial v_r}{\partial t} + v_s \frac{\partial v_r}{\partial x_s}\right)$$
$$= -\frac{D\epsilon}{Dt} + \frac{\partial v_s}{\partial x_r}\frac{\partial v_r}{\partial x_s},$$

to obtain

$$\left(\frac{D}{Dt} - \nu'\Delta\right)\epsilon - \nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = \frac{\partial v_s}{\partial x_r}\frac{\partial v_r}{\partial x_s}, \tag{5}$$

where

$$\nu' = (4/3)\nu + \zeta/\rho \quad (\text{with } \nu = \mu/\rho),$$

is a total compressional kinematic viscosity; terms in the gradients of ν and ν' have been omitted in (5). Now we substitute from (2) for ϵ and multiply through by ρ:

$$\rho\left(\frac{D}{Dt} - \nu'\Delta\right)\frac{1}{\rho}\frac{D\rho}{Dt} - \rho\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = \Lambda \text{ in } \mathcal{R}_t, \tag{6}$$

where we have introduced

$$\Lambda = \rho\frac{\partial v_s}{\partial x_r}\frac{\partial v_r}{\partial x_s}. \tag{7}$$

Finally, we make the usual specification of the compressibility relation, namely

$$D\rho/Dt = (1/c^2)(Dp/Dt), \tag{8}$$

where $c = c(x,y,z,t)$ is the local velocity of sound. This is the assumption that if the pressure changes on a material element it undergoes adiabatic compression; cf. (Ref. 11, Eq. 4.2.32) or (Ref. 9, Eq. 5.4) (for the case of no accession of heat). We substitute this in (6) and have a preliminary form of the Lighthill equation:

$$\rho\left(\frac{D}{Dt} - \nu'\Delta\right)\frac{1}{\rho c^2}\frac{Dp}{Dt} - \rho\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = \Lambda \text{ in } \mathcal{R}_t, \tag{9}$$

with

$$\Lambda = \rho\frac{\partial v_s}{\partial x_r}\frac{\partial v_r}{\partial x_s}.$$

The only approximation in (9) is the discarding of the gradients of the viscosity coefficients, both dynamic and kinematic.

II. THE LIGHTHILL AND RIBNER EQUATIONS

In discussing (9), we borrow from the usual linear theories as necessary; we look for a linear equation for an acoustic overpressure. Equation (9) could be regarded as an inhomogeneous linear wave equation for p if the other quantities were known functions. The left-hand side would be a linear wave operator acting on p, the waves having material phase velocity c, and linear viscous damping. The right-hand side, quadratic in the velocity gradients $\nabla\mathbf{v}$, would represent an acoustic source density. Actually, of course, Λ, the coefficients, and D/Dt, all depend on all of the motions, so a variety of nonlinear interactions is involved in (9). We want to disentangle the acoustic pressure, but before this we will specify our model in more detail.

We assume that the quiescent ocean is stratified, and that phenomenological profiles $\rho_0 = \rho_0(z)$ and $c_0 = c_0(z)$ are given. We assume that both of these functions have the constant surface values ρ_s, c_s, respectively, in the mixed region where the surface gravity wave motion is nonnegligible, say, tens of meters below S_t. [Below the thermocline a value $-(1/c_0)dc_0/dz \approx (85 \text{ km})^{-1}$ is representative. The variation of ρ_0 is specified by its logarithmic derivative

$$\gamma = -\frac{1}{\rho_0}\frac{d\rho_0}{dz} = \frac{N^2}{g} + \frac{g}{c_0^2}, \tag{10}$$

where N (radians/s) is the Väisälä–Brunt frequency limit. At great depths, $\gamma \approx g/c^2 \approx (230 \text{ km})^{-1}$, and $N \to 0$, $(1/N)dN/dz \approx (1 \text{ km})^{-1}$ (Ref. 12, p. 4). In the thermocline values $N \approx (60 \text{ s})^{-1}$ and $\gamma \approx (45 \text{ km})^{-1}$ are possible; $(1/\gamma)(d\gamma/dz)$ can be large locally, with either sign (Ref. 11, p. 303.)]

The appearance of the material derivative operator in (9) is a reflection of the fact that a wave undergoes a Doppler shift when it propagates in the flowing water (Ref. 11, Sec. 4.6) (Ref. 8, Ch. III); the fractional frequency shift is bounded by $\pm M$ if the flow has Mach number M. We are going to drop from (9) the advective part $\mathbf{v} \cdot \nabla$ of D/Dt, per the following considerations. The nonlinear sound–sound terms are completely negligible for us, since the actual water flow involved in an acoustic wave at level 100 dB/μ Pa (rms amplitude 0.1 N/m^2) has Mach number $< 10^{-10}$. The Mach number in a gravity wave flow is of order (wave height)/(acoustic wavelength at same frequency), e.g., $M < 1/200$ for 20 m swell of frequency 0.1 Hz. We would expect the advection effects to be localized; in the far-field the angular distribution and phase of the sound might change to order M, but the relative change in the power spectrum should be smaller. The only nonlinear gravity wave interactions of interest are those responsible for the production of sound—our gravity wavefield will be a linear superposition of the waves of the usual theory.

Terms in the derivatives of the coefficients appear when (9) is expanded, and the coefficients themselves need to be expressed as (static + first-order fluctuation). In a linear treatment of the left-hand side of (9), at frequencies above 0.1 Hz, only one of these terms is retained, to give the correct static behavior. [In the full linear treatment of (1)–(8) the terms in $\nabla\cdot\nabla$ are first order when applied to a static quantity, and must be retained at internal wave frequencies. Also, a gravity wave with a phase velocity $g/(2\pi f)$ would be supersonic for frequencies $f < g/(2\pi c) \approx 0.001$ Hz. Compressional and buoyancy effects are strongly coupled at these frequencies, and the order of the system changes from 2 to 4 (Ref. 11, Sec. 4.2).]

In the right-hand side of (9) we replace Λ by

$$\Lambda_0 = \rho_a \frac{\partial v_s}{\partial x_r}\frac{\partial v_r}{\partial x_s},$$

arguing that $\rho \approx \rho_a$ near the surface where $\nabla \mathbf{v}$ is not negligible; the difference

$$\Lambda - \Lambda_0 = (\rho - \rho_a)\frac{\partial v_s}{\partial x_r}\frac{\partial v_r}{\partial x_s}$$

$$\approx \frac{(\text{acoustic pressure})}{c^2}\frac{\partial v_s}{\partial x_r}\frac{\partial v_r}{\partial x_s},$$

is of the same order as terms already discarded. Our approximation of (9) now takes the form

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t} - \nu'\Delta\right)\frac{\partial p}{\partial t} - \Delta p - \gamma\frac{\partial p}{\partial z} = \Lambda_0, \quad \text{in } \mathcal{R}_t .$$

A plane acoustic wave is attenuated by the viscosity term according to

$$p \sim \exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})]\ \exp[-\omega t/(2Q)] ,$$

where $Q \approx c^2/(\nu'\omega)$ varies inversely with frequency. (Q is not really a Reynolds number, since ω/k is not a material velocity.) From (Ref. 13, p. 102) the empirical value of c^2/ν' is $\approx 9.5\times10^8\ \mathrm{s}^{-1}$, and the ν' term in (9) will produce negligible effects at acoustic frequencies of interest. (E.g., at 100 Hz the power relaxation time Q/ω exceeds 40 min.) We drop the ν' term, and then have a dissipationless wave equation

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p - \gamma\frac{\partial p}{\partial z} = \Lambda_0, \quad \text{in } \mathcal{R}_t . \tag{11}$$

The pressure p in (11) is the total water pressure, and includes the static part. We need to introduce some sort of reference pressure p', defined in $\mathcal{R}_t$, such that $p - p'$ away from the gravity wave flow represents the acoustic overpressure, i.e., p' should tend to the static pressure at great depths. As will become apparent in Sec. IV, it is also very desirable to have $p - p' = 0$ on $\mathcal{S}_t$, and the complications from this occupy the rest of the present section.

The static pressure in the quiescent ocean is

$$p_s = p_a + g\int_s^0 \rho_0(z')dz', \ -\infty < z < 0,$$

where p_a is atmospheric pressure at the ocean surface, assumed uniform. The formula serves to extend the definition to $\mathcal{R}_t$ if we take $\rho_0(z') = \rho_a$ in the crests $\mathcal{R}_t \cap \{z \geq 0\}$. The extended p_s has the properties

$$\left.\begin{aligned} &\partial p_s/\partial t = 0 ,\\ &\nabla p_s = -g\rho_0\mathbf{e} ,\\ &\Delta p_s + \gamma\frac{\partial p_s}{\partial z} = 0 \end{aligned}\right\} \text{ in } \mathcal{R}_t , \tag{12}$$

$$p_s = p_a - g\rho_a h \quad \text{on } \mathcal{S}_t .$$

It follows from this that p_s satisfies the homogeneous equation in (11), so if we use p_s as the reference pressure we have

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta - \gamma\frac{\partial}{\partial z}\right)(p - p_s) = \Lambda_0 ,$$

$$p - p_s = O(1), \quad \text{as } z \to -\infty .$$

We call this the Lighthill equation for ocean acoustic waves. It suffers from the disadvantage that the boundary values of $p - p_s$ are nonvanishing on $\mathcal{S}_t$.

Following Ribner, let us use as reference pressure the pressure p_0 that would exist in a dynamically incompressible ocean. The stratification ρ_0 stays the same, but we replace the compressibility condition (8) by $D\rho/Dt = 0$. (Imagine turning $D\rho/Dt$ off and following the motion for a short time.) The equation corresponding to (9) would be

$$-\rho\nabla\cdot\left(\frac{1}{\rho}\nabla p\right) = \Lambda ,$$

but we use instead the approximation corresponding to (11), namely,

$$-\Delta p_0 - \gamma\frac{\partial p_0}{\partial z} = \Lambda_0, \quad \text{in } \mathcal{R}_t .$$

The source term should be formed from the velocity gradients of the incompressible flow. However, this will differ from the compressible Λ_0 by terms of the order of those already neglected. We make this approximation, and find that $p_1 = p - p_0$ satisfies the Ribner–Lighthill equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p_1}{\partial t^2} - \Delta p_1 - \gamma\frac{\partial p_1}{\partial z} = -\frac{1}{c_0^2}\frac{\partial^2 p_0}{\partial t^2}, \quad \text{in } \mathcal{R}_t . \tag{13}$$

The excitation term can be interpreted as a density of monopole sources: A pressure fluctuation δp_0 produces a density fluctuation $(1/c_0^2)\delta p_0$ in the actual compressible ocean, so the right-hand side can be viewed as a monopole source density $-\partial^2\rho/\partial t^2$ (Ref. 11, p. 33). The idea is due to Ribner.[8]

The boundary condition for p on $\mathcal{S}_t$ is, say

$$p = p_a - g\rho_a T\Delta_2 h \quad \text{on } \mathcal{S}_t , \tag{14}$$

where $\Delta_2 h = (\partial^2 h/\partial x^2) + (\partial^2 h/\partial y^2)$ is the usual curvature approximation of the linear theory. (The geometrical surface tension is $T \approx 7.4\times10^{-6}\ \mathrm{m}^2$.) The air above the ocean is quiet, as a first approximation; wave energy has propagated from remote sources.[14] We assume that p_0 satisfies the same condition:

$$p_0 = p_a - g\rho_a T\Delta_2 h \quad \text{on } \mathcal{S}_t , \tag{15}$$

so we will have $p_1 = 0$ on $\mathcal{S}_t$. At depths there will hold $p - p_s = O(1)$, $p_0 - p_s = o(1)$ [or $O(1)$ pseudosound], and $p_1 = p - p_0 = O(1)$ will represent the acoustic overpressure. We show later that the pseudosound term in p_0 vanishes under assumptions we make involving the surface waves.

The nuisance term $\gamma(\partial p/\partial z)$ is needed in (11) to give the correct static behavior. Since p_1 has no static part, we can eliminate this term in (13), if necessary, as follows. We define a function $B=B(z)$ in $\mathcal{R}_t$ by $B=[\rho_a/\rho_0(z)]^{1/2}$, with the extended definition of ρ_0. The values of B will differ from unity by at most 2% at depths of interest. The coefficients

$$B'/B=\tfrac{1}{2}\gamma(\approx g/c^2),$$

$$b=B''/B=\tfrac{1}{2}\gamma'+\tfrac{1}{4}\gamma^2[\approx(g/c^2)^2],$$

will be of the order indicated and will be slowly varying (except in the thermocline, where b can be larger, with either sign).

Now we put

$$p_1(x,y,z,t)=A_1(x,y,z,t)/B(z) \quad \text{in } \mathcal{R}_t;$$

this is the Lighthill transformation of (Ref. 4, Sec. 4.2). It has the property

$$\rho_0\nabla\cdot\left(\frac{1}{\rho_0}\nabla p_1\right)=\Delta p_1+\gamma\frac{\partial p_1}{\partial z}=\frac{1}{B}(\Delta A_1-bA_1),$$

so that in terms of A_1, Eq. (13) becomes the selfadjoint

$$\frac{1}{c_0{}^2}\frac{\partial^2 A_1}{\partial t^2}-\Delta A_1+bA_1=-\frac{B}{c_0^2}\frac{\partial^2 p_0}{\partial t^2} \quad \text{in } \mathcal{R}_t.$$

The effect of the b term is to change the local acoustic dispersion relation to

$$\omega/c=(k^2+b)^{1/2}.$$

The fractional change in the phase velocity is of order $(1/2)[(g/c)/\omega]^2$; this is 0.5×10^{-4} at 0.1 Hz. If we discard the b term and the dependence on B, we have a simplified version of the Ribner–Lighthill equation:

$$\frac{1}{c_0{}^2}\frac{\partial^2 p_1}{\partial t^2}-\Delta p_1=-\frac{1}{c_0{}^2}\frac{\partial^2 p_0}{\partial t^2},$$

$$p_1=O \quad \text{on } \mathcal{S}_t,$$

$$p_1=O(1), \quad \text{as } z\to-\infty. \tag{16}$$

III. THE GRAVITY WAVEFIELD

We will model the reference pressure p_0 as that of an irrotational incompressible flow: There is a velocity potential ϕ such that $\mathbf{v}=\nabla\phi$. The incompressibility condition $\nabla\cdot\mathbf{v}=0$ becomes the familiar $\Delta\phi=0$. As a Bernoulli integral we take

$$I=\rho_0\left(\frac{\partial\phi}{\partial t}+\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right)+p_0-p_s.$$

This does not quite satisfy $\nabla I=0$ (Ref. 9, Chap. IV); instead,

$$\nabla I=-\gamma\rho_0\left(\frac{\partial\phi}{\partial t}+\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right)\mathbf{e}.$$

However, we can argue that γ will vanish in the mixed region, and the other factor in ∇I will have become negligible at depths where $\gamma>0$ must be considered. We shall thus impose the normalization $I=0$ on ϕ, which is to say

$$p_0=p_s-\rho_0\left(\frac{\partial\phi}{\partial t}+\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right) \quad \text{in } \mathcal{R}_t.$$

The excitation term in (13) takes the form

$$-\frac{1}{c_0^2}\frac{\partial^2 p_0}{\partial t^2}=\frac{\rho_0}{c_0^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial\phi}{\partial t}+\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right). \tag{17}$$

The boundary values of p_0, p_s being given by (15), (12), respectively, we find that ϕ should satisfy

$$\frac{\partial\phi}{\partial t}=-g(h-T\Delta_2 h)-(\mathbf{v}\cdot\mathbf{v}/2) \quad \text{on } \mathcal{S}_t.$$

We are going to make a perturbation series expansion $\phi=\phi_1+\phi_2+\dots$ in which the expansion parameter is essentially the rms wave amplitude. First, however, we introduce the basic stochastic element. We assume that h is a stochastic process which is wide-sense stationary in x, y, t, and that h is a linear superposition of plane surface waves of the usual linear theory. That is,

$$h=(1/\sqrt{2})\int\{\exp[i(\sigma t-\boldsymbol{\kappa}\cdot\mathbf{r})]\mathcal{U}(d\boldsymbol{\kappa})$$
$$+\exp[-i(\sigma t-\boldsymbol{\kappa}\cdot\mathbf{r})]\bar{\mathcal{U}}(d\boldsymbol{\kappa})\}, \tag{18}$$

where $\mathbf{r}=(x,y,z)$ is the position vector, $\boldsymbol{\kappa}=(\kappa_x,\kappa_y,0)$ is the horizontal wave vector of a surface wavelet, with magnitude $\kappa=\|\boldsymbol{\kappa}\|=(\kappa_x^2+\kappa_y^2)^{1/2}$, and where $\sigma=\sigma(\kappa)$ is determined by the dispersion relation $\sigma^2=g\kappa(1+T\kappa^2)$.

The overbar on $\mathcal{U}$ denotes complex conjugate, and the spectral process $\mathcal{U}$ is assumed to be centered with complex orthogonal increments. That is, if we denote by $\langle\ \rangle$ the stochastic expected value

$$\langle\mathcal{U}(d\boldsymbol{\kappa})\rangle=0,$$

$$\langle\mathcal{U}(d\boldsymbol{\kappa}_1)\mathcal{U}(d\boldsymbol{\kappa}_2)\rangle=0,$$

$$\langle\mathcal{U}(d\boldsymbol{\kappa}_1)\bar{\mathcal{U}}(d\boldsymbol{\kappa}_2)\rangle=\mathcal{M}(d\boldsymbol{\kappa}_1\cap d\boldsymbol{\kappa}_2),$$

where $\mathcal{M}(d\boldsymbol{\kappa})\geq 0$ is the spectral measure of the h process:

$$\langle h^2\rangle=\int\mathcal{M}(d\boldsymbol{\kappa}). \tag{19}$$

The wave mean potential energy in a square meter column of ocean, $\mathrm{P.E.}=(g\rho_a/2)\langle h(h-T\Delta_2 h)\rangle$ (see Ref. 11, p. 228), is equal to the corresponding kinetic energy density

$$\mathrm{K.E.}=(\rho_a/2)\int_{-\infty}^{0}\langle\mathbf{v}\cdot\mathbf{v}\rangle dz, \text{ namely,}$$

$$\mathrm{P.E.}=\mathrm{K.E.}=(g\rho_a/2)\int(1+T\kappa^2)\mathcal{M}(d\boldsymbol{\kappa}).$$

Convergence of various integrals involving $\mathcal{M}(d\boldsymbol{\kappa})$ will be taken for granted. Later we will assume that there is a continuous spectral density: $\mathcal{M}(d\boldsymbol{\kappa})=f(\boldsymbol{\kappa})d\boldsymbol{\kappa}$ where $d\boldsymbol{\kappa}=d\kappa_x d\kappa_y$ is Lebesgue measure in the wave vector plane.

The acoustic intensity will involve higher-order moments of the $\mathcal{U}$ process. We assume that all third moments vanish, and that the only nonvanishing fourth moment is

$$\langle\mathcal{U}(d\boldsymbol{\kappa}_1)\mathcal{U}(d\boldsymbol{\kappa}_2)\bar{\mathcal{U}}(d\boldsymbol{\kappa}_3)\bar{\mathcal{U}}(d\boldsymbol{\kappa}_4)\rangle$$
$$=\mathcal{M}(d\boldsymbol{\kappa}_1\cap d\boldsymbol{\kappa}_3)\mathcal{M}(d\boldsymbol{\kappa}_2\cap d\boldsymbol{\kappa}_4)+\mathcal{M}(d\boldsymbol{\kappa}_1\cap d\boldsymbol{\kappa}_4)\mathcal{M}(d\boldsymbol{\kappa}_2\cap d\boldsymbol{\kappa}_3). \tag{20}$$

In other words, the moments up to order four satisfy the same relations that the moments of a centered complex orthogonal increments Gaussian process satisfy.

We expand the velocity potential $\phi=\phi_1+\phi_2+\ldots$ in powers of $\mathfrak{U},\bar{\mathfrak{U}}$; only the first two terms are obtained. The first order term ϕ_1 is the potential of the usual linear theory, satisfying

$$\Delta\phi_1=0 \quad \text{in } \mathcal{R}_t, \quad \frac{\partial\phi_1}{\partial t}=-g(h-T\Delta_2 h) \quad \text{on } \{z=0\}, \tag{21}$$

$$\phi_1\to 0, \quad \text{as } z\to-\infty .$$

Explicitly,

$$\phi_1=(1/\sqrt{2})\int(\sigma/\kappa)$$

$$\times\{i\exp[i(\sigma t-\boldsymbol{\kappa}\cdot\mathbf{r})+\kappa z]\mathfrak{U}(d\boldsymbol{\kappa})+\text{c.c.}\} \quad \text{in } \mathcal{R}_t, \tag{22}$$

where c.c. within braces will always mean the complex conjugate of the preceding term within the braces. For ϕ_1 to exist it is necessary that

$$\int(1/\kappa)\mathfrak{M}(d\boldsymbol{\kappa})$$

converge at the origin.

The second order term ϕ_2 has the properties[6]

$$\Delta\phi_2=0 \quad \text{in } \mathcal{R}_t$$

$$\frac{\partial\phi_2}{\partial t}=-h\frac{\partial^2\phi_1}{\partial z\partial t}-\frac{1}{2}\nabla\phi_1\cdot\nabla\phi_1 \quad \text{on } \{z=0\}. \tag{23}$$

We sort out the contributions to p_0 according to their degree in the $\mathfrak{U}$ process. We put $p_0=p_s+p^{\,1}+p^{\,2}+\ldots$, where the first-order term is

$$p^{\,1}=-\rho_0\frac{\partial\phi_1}{\partial t}$$

$$=(\rho_0/\sqrt{2})\int(\sigma^2/\kappa)$$

$$\times\{\exp[i(\sigma t-\boldsymbol{\kappa}\cdot\mathbf{r})+\kappa z]\mathfrak{U}(d\boldsymbol{\kappa})+\text{c.c.}\} \tag{24}$$

and the second-order is (omitting the algebra)

$$p^{\,2}=-\rho_0\left(\frac{\partial\phi_2}{\partial t}+\frac{1}{2}\nabla\phi_1\cdot\nabla\phi_1\right)=-\frac{\rho_0}{4}\int\int\{K_1\exp(\|\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2\|z)+K_2\exp[(\kappa_1+\kappa_2)z]\}\cdot(\exp\{i[(\sigma_1+\sigma_2)t-(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\cdot\mathbf{r}]\}$$

$$\times\mathfrak{U}(d\boldsymbol{\kappa}_1)\mathfrak{U}(d\boldsymbol{\kappa}_2)+\text{c.c.})-\frac{\rho_0}{4}\int\int\{K_3\exp(\|\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2\|z)+K_4\exp[(\kappa_1+\kappa_2)z]\}$$

$$\times(\exp\{i[\sigma_1-\sigma_2)t-(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2)\cdot\mathbf{r}]\}\mathfrak{U}(d\boldsymbol{\kappa}_1)\bar{\mathfrak{U}}(d\boldsymbol{\kappa}_2)+\text{c.c.}), \tag{25}$$

the coefficient abbreviations being

$$K_1=(\sigma_1+\sigma_2)^2-\frac{\sigma_1\sigma_2[\|\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2\|^2-(\kappa_1-\kappa_2)^2]}{2\kappa_1\kappa_2},$$

$$K_2=-\frac{\sigma_1\sigma_2[(\kappa_1+\kappa_2)^2-\|\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2\|^2]}{2\kappa_1\kappa_2},$$

$$K_3=(\sigma_1-\sigma_2)^2+\frac{\sigma_1\sigma_2[\|\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2\|^2-(\kappa_1-\kappa_2)^2]}{2\kappa_1\kappa_2},$$

$$K_4=\frac{\sigma_1\sigma_2[(\kappa_1+\kappa_2)^2-\|\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2\|^2]}{2\kappa_1\kappa_2}.$$

The corresponding forcing terms in the Ribner-Lighthill equation are

$$J_1=-\frac{1}{c_0^2}\frac{\partial^2 p^{\,1}}{\partial t^2},$$

$$J_2=-\frac{1}{c_0^2}\frac{\partial^2 p^{\,2}}{\partial t^2}. \tag{26}$$

The following kinematic considerations are important for understanding the sound production mechanism. Let $\mathbf{k}=(k_x,k_y,k_z)$ be the wave vector of a plane acoustic wave, of frequency $\omega=ck$ with $k=\|\mathbf{k}\|$. It is convenient to separate out the horizontal projection $\boldsymbol{\kappa}=(k_x,k_y,0)$. so that $k=(\kappa^2+k_z^2)^{1/2}$. The forcing term in (25) with horizontal phasor

$$\exp\{i[(\sigma_1+\sigma_2)t-(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\cdot\mathbf{r}]\}$$

cannot excite an acoustic component

$$\exp\{i[\omega t-\boldsymbol{\kappa}\cdot\mathbf{r}]-ik_z z\}$$

unless

$$\omega=\sigma_1+\sigma_2,$$

$$\boldsymbol{\kappa}=\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2;$$

this is a consequence of the assumed homogeneity in x,y,t. From

$$(\sigma_1+\sigma_2)/c=\omega/c=k\geq\kappa=\|\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2\|\geq|\kappa_1-\kappa_2|$$

and

$$\sigma_1^2-\sigma_2^2=g(\kappa_1-\kappa_2)[1+T(\kappa_1^2+\kappa_1\kappa_2+\kappa_2^2)], \tag{27}$$

we obtain

$$|\sigma_1-\sigma_2|\leq(g/c)[1+T(\kappa_1^2+\kappa_1\kappa_2+\kappa_2^2)]. \tag{28}$$

That is, two gravity waves at frequencies σ_1, σ_2 below the ripple region, say 13 Hz, will not interact to produce an acoustic wave at the sum frequency unless their frequency difference is less than 0.001 Hz. If the surface waves are each about 50 Hz, the difference is bounded by 0.03 Hz, so the constraint is still severe at higher frequencies.

There is a corresponding result for the difference terms. The notation being as before,

$$\omega=\sigma_1-\sigma_2,$$

$$\boldsymbol{\kappa}=\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2,$$

requires

$$(\sigma_1-\sigma_2)/c=\omega/c=k\geq\kappa=\|\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2\|\geq\kappa_1-\kappa_2$$

for the case $\sigma_1 > \sigma_2$. This and (27) give

$$\sigma_1 + \sigma_2 \leq (g/c)[1 + T(\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2)] .$$

This cannot be satisfied unless the *sum* is below 0.001 Hz or above 2×10^{12} Hz. We omit the difference frequency terms after this.

Consider now the first order term J_1 of (26). A gravity wave with horizontal phasor

$$\exp[i(\sigma_1 t - \kappa_1 \cdot \mathbf{r})]$$

will excite a plane acoustic wavelet directly only if the horizontal phasors are the same, i.e.,

$$\omega = \sigma_1 ,$$

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_1 ,$$

is the previous notation. The z component of acoustic $\mathbf{k}$ is determined by

$$k_z^2 = -\kappa_1[\kappa_1 - (g/c^2)(1 + T\kappa_1^2)] ,$$

and k_z will be imaginary at all frequencies of interest. An acoustic wave decaying exponentially with depth can be regarded as part of a nearfield. This can be reflected by the moving surface to produce a propagating field. We examine this at the end of Sec. IV; it is a small effect.

There is one part of $p_0 - p_s$ which does not decrease exponentially with depth, formally. Namely, as $z \to -\infty$,

$$p^{\,2} \approx -\rho_0 \iint \sigma_1^2\{\exp(2i\sigma_1 t)\mathfrak{U}(d\boldsymbol{\kappa}_1)\mathfrak{U}(d\boldsymbol{\kappa}_2) + \text{c.c.}\}(\to) ,$$

where $(\to)$ means that the limit $\boldsymbol{\kappa}_2 = -\boldsymbol{\kappa}_1$ is to be taken after the rms value has been calculated. (We cannot put $\boldsymbol{\kappa}_2 = -\boldsymbol{\kappa}_1$ directly in the double stochastic integral, in general, without careful examination.) The condition $\boldsymbol{\kappa}_2 = -\boldsymbol{\kappa}_1$ is that two oppositely moving plane gravity waves interfere to produce a standing wave, of course. The deep water oscillation ~$\exp(2i\sigma_1 t)$ in p_0 comes from the Bernoulli pressure drop at the surface, i.e., the surface average of $-\rho_a(\mathbf{v}\cdot\mathbf{v})/2$, propagating instantaneously to all depths. This source oscillation drives a vertically propagating acoustic wave of velocity c_0. That is, the resulting term in $p = p_0 + p_1$ should be (the amplitude is correct for constant ρ_0):

$$\hat{p} = -\rho_0 \int\int \sigma_1^2\{e^{i\theta_1}\mathfrak{U}(d\boldsymbol{\kappa}_1)\mathfrak{U}(d\boldsymbol{\kappa}_2) + \text{c.c.}\}(\to)$$

where we have used the abbreviation

$$\theta_1 = 2\sigma_1\left(t - \int_z^0 \frac{dz'}{c_0(z')}\right) ,$$

for the phase of the vertically propagating wave. Then

$$\langle \hat{p}^2 \rangle = \rho_0^2 \int\int\int\int \sigma_1^2\sigma_3^2 \cdot \langle\{e^{i\theta_1}\mathfrak{U}(d\boldsymbol{\kappa}_1)\mathfrak{U}(d\boldsymbol{\kappa}_2) + \text{c.c.}\}\cdot\{e^{i\theta_3}\mathfrak{U}(d\boldsymbol{\kappa}_3)\mathfrak{U}(d\boldsymbol{\kappa}_4) + \text{c.c.}\}\rangle(\to) = 4\rho_0^2 \int\int \sigma_1^4 \mathfrak{M}(d\boldsymbol{\kappa}_1)\mathfrak{M}(d\boldsymbol{\kappa}_2)\,\big|_{\to}$$

$$= 4\rho_0^2 \int\int \sigma_1^4 f(\boldsymbol{\kappa}_1) f(-\boldsymbol{\kappa}_1) d\boldsymbol{\kappa}_1 d(-\boldsymbol{\kappa}_1) = 0 ,$$

because the diagonal $\boldsymbol{\kappa}_2 = -\boldsymbol{\kappa}_1$ in four-dimensional Euclidean space E^4 has Lebesgue measure 0. The gist of this is that if there is a spectral density, then the probability is 0 that there will ever be any standing waves. Pseudosound will be present only when there are permanent standing waves, i.e., $\mathfrak{M}(d\boldsymbol{\kappa})$ contains paired discrete swell lines $\alpha_\pm\delta(\boldsymbol{\kappa} \pm \boldsymbol{\kappa}_s)d\boldsymbol{\kappa}$ (this is remarked by Kadota and Labianca[6]). In such a process there is *always* a standing wave at the fixed $(\boldsymbol{\kappa}_s, -\boldsymbol{\kappa}_s)$, the phase and amplitude being random. Note that the physical processes derived from h are not ergodic if $\mathfrak{M}(d\boldsymbol{\kappa})$ has nonvanishing atomic part.

IV. THE GREEN FUNCTION

We will assume from now on that the ocean is homogeneous, and ρ, c will denote the constant values of ρ_0, c_0. Our methods have generalizations in the stratified case, but we do not have closed form for the results. Moreover, the acoustic source intensity of the surface wavefield will be pretty much independent of the c_0 profile.[15]

We denote by $\Box$ the D'Alembertian wave operator for acoustic waves in the homogeneous ocean:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta ,$$

and we denote by $G = G(\mathbf{r}, t; \mathbf{r}', t')$ an acoustic Green function,

$$\Box G = \Box' G = \delta(\mathbf{r} - \mathbf{r}', t - t') ,$$

in a domain to be specified; primes denote arguments or operations on the primed variables in G. Propagation is assumed to be from $\mathbf{r}'$, t' to $\mathbf{r}$, t, so that

$$G = 0 \quad \text{if } t < t' + \frac{\|\mathbf{r} - \mathbf{r}'\|}{c} .$$

Further boundary conditions will be imposed later.

We are interested in solving an equation

$$\Box u = J \quad \text{in } \mathcal{R}_t ,$$

$$u = 0 \quad \text{on } \mathcal{S}_t ,$$

$$u = O(1) \quad \text{as } z \to -\infty ;$$

in particular, we want the rms values of u at a point well below the mixed region. Let $[t_0, t_1]$ be a time interval containing the time t of interest; later $t_0 \to -\infty$, $t_1 \to \infty$. Let $\hat{\mathcal{R}}_t$ be that part of $\mathcal{R}_t$ contained in a large ball centered at the origin. The radius L of the ball is to satisfy $L \gg \|\mathbf{r}\|$, and $L \to \infty$ later. We denote by $\mathcal{T}$ the $(\mathbf{r}, t)$ region in E^4 whose t section is $\hat{\mathcal{R}}_t$, $t_0 \leq t \leq t_1$. We apply Green's second identity to

$$u'(\Box' G) - (\Box' u')G = u'\delta - J'G ,$$

in the region $\mathcal{T}' \subset E^4$. The result is

$$u(\mathbf{r},t)=\int_{\mathcal{T}'} G(\mathbf{r},t;\mathbf{r}',t')J(\mathbf{r}',t')d\mathbf{r}'dt' + \int_{\partial\mathcal{T}'} \nu'\cdot F' d\mathfrak{a}', (\mathbf{r},t)\epsilon\mathcal{T}. \qquad (29)$$

The surface terms on the right-hand side are as follows. The boundary $\partial\mathcal{T}$ consists of ends

$$(\mathfrak{R}_{t_0}, t_0)$$

and

$$(\hat{\mathfrak{R}}_{t_1}, t_1),$$

and the sides $(\hat{\mathfrak{S}}_t, t_0 \leq t \leq t_1)$ and $(\gamma_t, t_0 \leq t \leq t_1)$, with $\hat{\mathfrak{S}}_t$ the part of $\mathfrak{S}_t$ within the ball of radius L and γ_t the part of the spherical surface of the ball within $\mathfrak{R}_t$. The integral over the part

$$(\mathfrak{R}_{t_1}, t_1)$$

of $\partial\mathcal{T}$ is noncausal and can be omitted. The outward unit normal ν to $\partial\mathcal{T}$ in E^4 has (x,y,z,t) components

$$\nu=(0,0,0,-1) \quad \text{on } (\mathfrak{R}_{t_0}, t_0),$$

$$\nu=\left(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1, -\frac{\partial h}{\partial t}\right)\Big/K \quad \text{on } (\mathfrak{S}_t, t),$$

with

$$K=\left[\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2+1+\left(\frac{\partial h}{\partial t}\right)^2\right]^{1/2},$$

$\nu=(\mathbf{n},0)$ on (γ_t, t) with $\mathbf{n}$ the unit normal in E^3 to the spherical cap γ_t.

The components of the four-dimensional flux vector $F=(\mathbf{F}, F_t)$ appearing in (29) are

$$\mathbf{F}'=-[u'(\nabla' G)-(\nabla' u')G],$$

$$F_t'=(1/c^2)[u'(\partial G/\partial t')-(\partial u'/\partial t')G].$$

The three-dimensional surface element $d\mathfrak{a}$ of $\partial\mathcal{T}$ can be parametrized as follows:

$$d\mathfrak{a}'=dx'dy'dz' \quad \text{on } (\hat{\mathfrak{R}}_{t_0}, t_0),$$

$$d\mathfrak{a}'=K'dx'dy'dt' \quad \text{on } (\hat{\mathfrak{S}}_t', t_0 \leq t' \leq t_1),$$

$$d\mathfrak{a}'=L^2 d\mathbf{n}'dt' \quad \text{on } (\gamma_{t'}, t_0 \leq t' \leq t_1).$$

(Note that K' cancels out.)

It is evidently very desirable to have the surface terms vanish. The integral over

$$(\hat{\mathfrak{R}}_{t_0}, t_0)$$

represents the influence of the remote past, and should vanish as $t_0 \to -\infty$. (E.g., turn the system on slowly with an exponential parameter which goes to 0.) The integral over $\gamma_{t'}$ represents acoustic energy entering $\mathcal{T}'$ from the sides and below, and should vanish as $L \to \infty$ if the theory is convergent. Any static and pseudosound terms are picked up by the $\gamma_{t'}$ term; the calculation is straightforward, but can be done just as well by the method of the preceding section, and we omit the details.

The boundary condition $u=0$ on $\mathfrak{S}_t$ is a fairly realistic condition on an acoustic field. (A refinement might take the water/air impedance ratio and radiation loss into the air into account.) If we impose the boundary condition

$$G=0 \quad \text{on } (\mathfrak{R}_t \times \mathfrak{S}_t)\cup(\mathfrak{S}_t \times \mathfrak{R}_{t'}),$$

we will have $F'=0$ on $\mathfrak{S}_t$. This eliminates the last of the surface terms, and we are left with

$$u=\int_{\mathcal{T}'} GJ'd\mathbf{r}'dt'.$$

Scattering of acoustic waves in the water by a moving ocean surface has been treated by Labianca and Harper in a series of papers.[16–20] Their results may be rephrased for our purposes as follows. The Green function has the form $G=G_0+G_1+G_2+\ldots$ where G_0 is the Green function for the quiescent ocean and where G_j, $j \geq 1$, is of degree j in a wave-height perturbation parameter $\langle h^2\rangle^{1/2}$. (In our setup, G_j is of degree j in $d\mathfrak{U}$ or $d\bar{\mathfrak{U}}$.) The first-order term G_1 represents Bragg reflected by the moving spatially periodic components of $\mathfrak{S}_t$. We are going to calculate pressures only to order two in $d\mathfrak{U}$. Since the source density J_2 of (26) is already of degree two in $d\mathfrak{U}$, i.e., it takes two gravity waves to generate a propagating acoustic wave, it is consistent with our approximation to omit the G_j, $j \geq 1$, terms for excitation J_2.

In the homogeneous case, the Green function for the quiescent ocean is

$$G_0=\frac{\delta(t-t'-R/c)}{4\pi R}-\frac{\delta(t-t'-\bar{R}/c)}{4\pi\bar{R}},$$

where

$$R=\|\mathbf{r}-\mathbf{r}'\|,$$

$$\bar{R}=\|\mathbf{r}-\mathbf{r}''\|,$$

with

$$\mathbf{r}''=(x',y',-z') \quad \text{if } \mathbf{r}'=(x',y',z').$$

We calculate J_2 from expression (26), and denote by $p_1{}^2$ the contribution to p_1 from J_2:

$$p_1{}^2=\int G_0 J_2' d\mathbf{r}'dt'=\frac{\rho}{16\pi c^2} \times \int\int\int \omega^2\{K_1\exp(\|\kappa_1+\kappa_2\|z')+K_2\exp[(\kappa_1+\kappa_2)z']\} \times\left[\left(\frac{e^{-i\omega R/c}}{R}-\frac{e^{-i\omega\bar{R}/c}}{\bar{R}}\right)\exp i[\omega t-(\kappa_1+\kappa_2)\cdot\mathbf{r}'] \times \mathfrak{U}(d\kappa_1)\mathfrak{U}(d\kappa_2)+\text{c.c.}\right]d\mathbf{r}', \qquad (30)$$

it being understood that ω is an abbreviation for $\omega=\sigma_1+\sigma_2$. The $d\mathbf{r}'$ integration is over a retarded time version of $\mathfrak{R}_t$, which we will not try to give explicitly. The corresponding integral over the quiescent ocean differs from the exact value by higher-order terms, however, since the integrand vanishes on $\{z'=0\}$ and the difference region has thickness $O(h)$. In other words, we may take $\{z'<0\}$ as the integration region in (30).

The z' integration can be done in (30) as it stands, but the following technique extends to the nonhomogeneous case. We are going to evaluate $\langle p_1^2\rangle$ at a point well below the mixed region. If $R=\|\mathbf{r}-\mathbf{r}'\|$ is large the spherical waves $e^{-i(\omega/c)R}$ will look like plane waves as $\mathbf{r}'$ varies over limited regions. Let $\eta=(\eta_x,\eta_y,0)$ be a point in $\{z'=0\}$; its uses will become apparent later. It is convenient to introduce the abbreviations

$$k=\omega/c\,,$$

$$\mathbf{k}=k(\mathbf{r}-\boldsymbol{\eta})/\|\mathbf{r}-\boldsymbol{\eta}\|\,;$$

that is, $\mathbf{k}$ is the direction from η to $\mathbf{r}$ renormalized to have magnitude k. If $\mathbf{r}'=\eta+(\mathbf{r}'-\eta)$ is near η,

$$\frac{e^{-ikR}}{R}=\frac{e^{-ik\|\mathbf{r}-\eta\|}}{\|\mathbf{r}-\eta\|}\circ e^{i\mathbf{k}\cdot(\mathbf{r}'-\eta)+\cdots}+\ldots\,, \tag{31}$$

where the omitted terms are of higher order in $1/R$; the expansion is valid for $\|\mathbf{r}'-\eta\|\ll\|\mathbf{r}-\eta\|$. For $\tilde{R}$ we use the same η, so simply replace z' by $-z'$.

Now we do the z' integration in (30). We make the notational change $\mathbf{r}'$ to $\mathbf{r}_1$, and abbreviate $(k_x,k_y,0)=\kappa$:

$$p_1^2=\frac{\rho}{8\pi}\int\int\left(K_1+\frac{k^2}{(\kappa_1+\kappa_2)^2+k_z^2}K_2\right)k_z$$

$$\times\,(i\exp\{i[\omega t-(\kappa_1+\kappa_2-\kappa)\cdot\mathbf{r}_1]\}$$

$$\times\exp[-ik\|\mathbf{r}-\eta\|-i\mathbf{k}\cdot\eta]\,\mathfrak{U}(d\kappa_1)\mathfrak{U}(d\kappa_2)$$

$$+\,\mathrm{c.c.})\,\frac{dx_1dy_1}{\|\mathbf{r}-\eta\|}\,. \tag{32}$$

In this, we imagine that the integration region is broken up into finite regions, squares say, and that η is piecewise constant on each region.

Now let us consider the J_1 excitation. The direct field

$$\int G_0J_1$$

vanishes exponentially with depth, and represents an acoustic nearfield. This nonpropagating field can interact with the surface to produce a propagating resultant; this term $p_1^{(1)}$ in the farfield p_1 we now calculate.

With $\mathbf{r}$ far from S_t, the term G_1 in G is determined by[20]

$$\Box'G_1=0\,,$$

$$G_1\Big|_{z'=0}=-h(x',y',t')\frac{\partial G_0}{\partial z'}\Big|_{z'=0}.$$

It is convenient to refer $\mathbf{r}'=\eta+(\mathbf{r}'-\eta)$ to a vector $\eta=(\eta_x,\eta_y,0)$, as before. We have

$$\frac{\partial G_0}{\partial z'}\Big|_{z'=0}=\frac{2u_z}{c}\,\frac{\delta'(t-t'-R/c)}{4\pi R}+\cdots,$$

where $\mathbf{u}=(\mathbf{r}-\eta)/\|\mathbf{r}-\eta\|$ is the unit vector from η to $\mathbf{r}$, and where the omitted terms are higher order in $1/R$ and will not contribute to the radiation field at $\mathbf{r}$. We express the singular function as

$$\delta'(t-t'-R/c)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}i\omega\exp[i\omega(t-t')]\,\exp[-i(\omega/c)R]d\omega\,.$$

We use the expansion (31) with the understanding that $k=\omega/c$ can now have either sign. The boundary values of G_1, to the order considered, are found to be

$$G_1\Big|_{z'=0}=-\frac{1}{4\sqrt{2}\pi^2}\int\int k_ze^{i\omega t}\frac{\exp(-ik\|\mathbf{r}-\eta\|-i\mathbf{k}\cdot\eta)}{\|\mathbf{r}-\eta\|}$$

$$\times\{ie^{it'(\omega-\sigma_1)+i(\kappa-\kappa_1)\cdot\mathbf{r}'}\mathfrak{U}(d\kappa_1)$$

$$+ie^{-it'(\omega+\sigma_1)+i(\kappa+\kappa_1)\cdot\mathbf{r}'}\bar{\mathfrak{U}}(d\kappa_1)\}d\omega\,.$$

Here, $k=\omega/c$ can have either sign, $\mathbf{k}=k\mathbf{u}$, and $\kappa=(k_x,k_y,0)$, with due regard for the sign of ω.

The bounded solution of $\Box'G=0$ with these boundary values is obtained by inserting a factor $e^{\alpha z'}$ in the integrand, with

$$\alpha=[\|\kappa-\kappa_1\|^2-(\omega-\sigma_1)^2/c^2]^{1/2}$$

for the $\mathfrak{U}(d\kappa_1)$ term and

$$\alpha=[\|\kappa+\kappa_1\|^2-(\omega+\sigma_1)^2/c^2]^{1/2}$$

for the $\bar{\mathfrak{U}}(d\kappa_1)$ term. Now we can do the dt', dz' integrations in

$$p_1^{(1)}=\int G_1J_1'\,dx'dy'dz'dt'\,,$$

with

$$J_1'=\frac{\rho}{c^2\sqrt{2}}\int\frac{\sigma_2^4}{\kappa_2}\,e^{\kappa_2z'}$$

$$\times\{\exp[i(\sigma_2t'-\kappa_2\cdot\mathbf{r}')]\mathfrak{U}(d\kappa_2)+\mathrm{c.c.}\}\,.$$

The dt' integration will introduce factors

$$2\pi\delta(\omega\pm\sigma_1\pm\sigma_2)\,,$$

and the $d\omega$ integration then fixes ω at $\pm(\sigma_1\pm\sigma_2)$. The interpretation is clear; we retain only the $\pm(\sigma_1+\sigma_2)$ terms. Also, from now on ω will be an abbreviation for $\sigma_1+\sigma_2$, as a positive quantity. The dz' integration gives a factor

$$\int_{-\infty}^{0}e^{(\kappa_2+\alpha)z'}dz'=\frac{1}{\kappa_2+\alpha}$$

for α previously described. Omitting some algebra, the result is

$$p_1^{(1)}=-\frac{\rho}{4\pi}\int\int\int\frac{\sigma_2^2}{\kappa_2}\,\{\kappa_2-[\kappa_2^2-(\sigma_2/c)^2]^{1/2}\}k_z$$

$$\times(i\exp\{i[\omega t-(\kappa_1+\kappa_2-\kappa)\cdot\mathbf{r}_1]\}$$

$$\times\exp(-ik\|\mathbf{r}-\eta\|-i\mathbf{k}\cdot\eta)\mathfrak{U}(d\kappa_1)\mathfrak{U}(d\kappa_2)$$

$$+\,\mathrm{c.c.})\,\frac{dx_1dy_1}{\|\mathbf{r}-\eta\|}\,. \tag{33}$$

It will turn out that $p_1^{(1)}$ is negligible compared to $p_1^{(2)}$.

V. THE ACOUSTIC INTENSITY

We want to calculate $\langle p_1^2\rangle$ at a point well below the surface, with $p_1=p_1^{(2)}+p_1^{(1)}$ from (32), (33). Let us multiply two copies of expression (32) together, with

$d\kappa_3$, $d\kappa_4$, $dx_2 dy_2$ as the infinitesimals in the second integrations. Taking $\langle\ \rangle$ of the product will produce a factor

$$f(\kappa_1) f(\kappa_2) d\kappa_1 d\kappa_2 ,$$

from terms $\kappa_3 = \kappa_1$, $\kappa_4 = \kappa_2$, together with "c.c." terms, and the phase factors $\exp[i(\omega t - k\|\mathbf{r} - \eta\| - \mathbf{k}\cdot\eta)]$ cancel (wide sense stationarity). There is another pair of terms from $\kappa_3 = \kappa_2$, $\kappa_4 = \kappa_1$, according to (20). The spatial phase factor in all cases has the form

$$\exp[\pm i(\kappa_1 + \kappa_2 - \kappa)\cdot(\mathbf{r}_2 - \mathbf{r}_1)] ,$$

where $\mathbf{r}_1, \mathbf{r}_2$ now lie in $\{z=0\}$.

Let us make the change of integration variables

$$\xi = \mathbf{r}_2 - \mathbf{r}_1 ,$$

$$\eta = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) ,$$

with unit Jacobian:

$$d\mathbf{r}_1 d\mathbf{r}_2 = d\xi d\eta .$$

(This is the value we choose for the previous η.) The vectors are all two-dimensional. The $d\xi$ integration produces a factor

$$\int \exp[\pm i(\kappa_1 + \kappa_2 - \kappa)]\cdot \xi d\xi = 4\pi^2 \delta(\kappa_1 + \kappa_2 - \kappa) ,$$

where the δ function is two-dimensional.

The element of area $d\eta$ subtends solid angle

$$d\Omega = \frac{d\eta_x d\eta_y}{\|\mathbf{r} - \eta\|^2} |\cos\theta| ,$$

at $\mathbf{r}$, where $\cos\theta = k_z/k$ is the z-direction cosine of the vector $\mathbf{r} - \eta$. In other words, the remaining differential $d\eta/\|\mathbf{r} - \eta\|^2$ can be expressed as

$$d\eta/\|\mathbf{r} - \eta\|^2 = k/|k_z| d\Omega ,$$

with $\Omega = \mathbf{k}/k$ the direction of $\mathbf{k}$. Altogether,

$$\langle p_1^2 \rangle = \frac{\rho^2}{4c^2} \int\int \omega^2 \Gamma^2 f(\kappa_1) f(\kappa - \kappa_1) |\cos\theta| d\Omega d\kappa_1 , \qquad (34)$$

with

$$\Gamma = \omega^2 - \frac{\sigma_1\sigma_2[\kappa^2 - (\kappa_1 - \kappa_2)^2]}{2\kappa_1\kappa_2} - \frac{\sigma_1\sigma_2 k^2[(\kappa_1 + \kappa_2)^2 - \kappa^2]}{2\kappa_1\kappa_2[(\kappa_1 + \kappa_2)^2 + k_z^2]}$$

$$- \sigma_1^2\left[1 - \left(1 - \frac{\sigma_1^2}{c^2\kappa_1^2}\right)^{1/2}\right] - \sigma_2^2\left[1 - \left(1 - \frac{\sigma_2^2}{c^2\kappa_2^2}\right)^{1/2}\right] .$$

The first term in Γ is the leading term; from

$$\sigma_1 \approx \sigma_2 \approx \omega/2 , \quad k/\kappa_1 \approx 4(g/c)/\omega , \quad |\sigma_1 - \sigma_2| = O(g/c) ,$$

one finds that the correction terms are $[(g/c)/\omega]^2$ relative to the first; this is 10^{-4} at 0.1 Hz. We drop these terms, and we replace $f(\kappa_2)$ by $f(-\kappa_1)$. The $d\Omega$ integration becomes

$$\int |\cos\theta| d\Omega = \int_0^{2\pi}\int_{\pi/2}^{\pi} |\cos\theta| \sin\theta d\theta d\phi = \pi ,$$

and the resulting formula is

$$\langle p_1^2 \rangle = \frac{\pi\rho^2}{4c^2} \int \omega^6 f(\kappa_1) f(-\kappa_1) d\kappa_1 , \qquad (35)$$

where again $\omega \approx 2\sigma_1$ is the frequency of the emitted acoustic wave and the normalization is

$$\langle h^2 \rangle = \int f(\kappa) d\kappa .$$

Dr. B. Hughes has informed the author that an extra factor of 2 was inadvertently introduced in Ref. 4, Eq. (33). (The folding of the negative frequency spectrum onto the positive frequency axis was done twice.[21]) If we take for our $f(\kappa)$ the Hughes form $X(k_1)G(\theta)$, our (35) becomes the corrected version of his (33).

Dr. Hughes also notes[21] that the corrected version of the Brekhovskikh[2] Eq. (54) should be

$$\bar{\zeta}^2 = (1/2) \int a^2(\kappa) d\kappa ,$$

essentially because Brekhovskikh uses the sine function instead of the exponential for the basic wavelets [Ref. 2, Eq. (19)]. [There are also canceling errors by a factor of 2 just before Ref. 2, Eq. (49).] If we replace $a^2(\kappa)$ in the Brekhovskikh Eq. (53) by our $2f(\kappa)$, his spectrum exceeds ours by a factor

$$\{1 + [T\kappa^2/4(1 + T\kappa^2)]\}^2 ,$$

in our notation. This factor is essentially unity below ripple frequencies (26 Hz acoustic), but increases to $(5/4)^2$ (i.e., 1.94 dB) at high frequencies.

Altogether, after the above corrections are made, the spectrum of Hughes and the present author agree, and the spectrum of Brekhovskikh agrees with these at low frequencies. The acoustic source level corresponding to (35) is discussed in Appendix II.

Let us assume that the direction-frequency spectrum of the surface gravity wavefield has the form $\Phi(\sigma)G(\theta)$; the normalization is (Ref. 22, Sec. 4.5):

$$\langle h^2 \rangle = \int_0^\infty \Phi(\sigma) d\sigma ,$$

$$\int_0^{2\pi} G(\theta) d\theta = 1 .$$

The acoustic spectrum involves the association parameter χ defined by

$$\chi = 2\pi \int_0^{2\pi} G(\theta) G(\theta + \pi) d\theta ;$$

$\chi = 1$ for uniform G, and $\chi = \pi/4$ is the value used in Ref. 4. For $\Phi(\sigma)$ we take the form

$$\Phi(\sigma) = \frac{\beta g^2}{\sigma^5} , \quad \sigma > \sigma_0 ,$$

$$= 0 , \quad \sigma < \sigma_0 ,$$

with σ_0 the wind-and-fetch dependent spectral peak and $\beta \approx 0.0123$ a dimensionless constant (Ref. 22, p. 147). Then, with $\omega = 2\sigma = 2(g\kappa)^{1/2}$ and $f_0 = \sigma_0/\pi$,

$$\langle p_1^2 \rangle = \int_{f_0}^\infty S(f) df$$

for acoustic spectral density

$$S(f) = (4\chi\rho^2 g^6 \beta^2/\pi^6 c^2)(1/f^7) .$$

{Numerically,

$$10\log[S(f)/10^{-12}] = 113.1 - 70\log f \quad \mathrm{dB}/\!/\mu\mathrm{Pa}^2/\mathrm{Hz}),$$

above the cutoff; we have used $\chi = \pi/4$.}

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APPENDIX A

Vorticity considerations

It is known that the potential flow of Sec. III leaves a nonvanishing stress on S_t when the viscosity term is included in the momentum flux tensor (Ref. 22, Sec. 3.4). This stress must be eliminated by including a vector potential term $\nabla \times \mathbf{A}$ in $\mathbf{v}$; a wave-height amplitude perturbation series has been given to second order in Ref. 6. Except for a pseudosound term proportional to $\nu\kappa^2$ (Ref. 22, Eq. 3.4.13), the vorticity field is confined to a thin layer near S_t, and the $\nabla\mathbf{v}$ components in this layer are of the same order as those in the potential flow (Ref. 22, p.48). The relative thickness of this layer (unit $1/\kappa$) is $1/(2Q)^{1/2}$, where the viscous damping coefficient for gravity waves is $Q = \sigma/(4\nu\kappa^2)$. Graphs of $1/(2Q)$ are given in (Ref. 22, Fig. 3.1) (For a clean ocean surface, $2Q \approx 100$ at $\sigma = 200$ radians/s, acoustic frequency 64 Hz. Even with oil film, $2Q \approx 50$ at 100 Hz acoustic). Since the production of sound is a volume effect, according to our treatment, vortical corrections of the acoustic intensity will be of order $1/(2Q)^{1/2}$ relative to the intensity due to potential flow. At the frequencies of interest (0.1–10 Hz) these corrections will be negligible.

APPENDIX B

The source level

An acoustic monopole in the ocean close to the surface will generate a dipole farfield, because of the antireflection at the free surface. The dipole is vertically oriented, and lies in the surface. In a homogeneous ocean, the acoustic intensity at a large distance r from the source is $p^2/(\rho c) = [3\cos^2\theta/(2\pi r^2)]W$ W/m^2, where θ is the polar angle from the dipole to the point of observation, and where W is the total acoustic power into the ocean.

Suppose next that such sources are distributed over the ocean surfaces, radiating into the ocean at a mean rate I W/m^2 which is uniform over the whole surface. The contributions from disjoint areas are assumed to be incoherent. At a point P well below the surface an element of solid angle $d\Omega$ in the upper hemisphere at P intercepts an element of area $r^2 d\Omega/\cos\theta$ on the ocean surface, where θ is now the colatitude of $d\Omega$ and r the distance from P to the surface in direction Ω. The sources in this element of area have power $Ir^2 d\Omega/\cos\theta$, and the resulting contribution at P is

$$d\,\frac{p^2}{\rho c} = \frac{3\cos^2\theta}{2\pi r^2}\cdot\frac{Ir^2 d\Omega}{\cos\theta} = I\cdot\frac{3\cos\theta}{2\pi}\,d\Omega\,.$$

The integration over the hemisphere

$$\int \cos\theta\, d\Omega = \pi$$

is elementary, and gives

$$p^2/\rho c = \tfrac{3}{2} I,$$

for the mean square pressure at P.

The factor $\frac{3}{2}$ is required by the $\cos\theta$ distribution of the flux at P, as follows. Radiation from $Ir^2 d\Omega/\cos\theta$ strikes a horizontal square meter at P obliquely, and the incident flux through a horizontal window at P is

$$\int \cos\theta\, d\!\left(\frac{p^2}{\rho c}\right) = I\int \frac{3\cos^2\theta}{2\pi}\, d\Omega = I \text{ W/(horizontal m}^2).$$

This elementary conservation of energy would be far more complicated if bottom reflection and water attenuation were taken into account[4]; reverberation could raise the level by 6–30 dB.

Comparing with (34)–(35), it is apparent that the source level producing (35) is

$$I = \frac{2}{3}\frac{p^2}{\rho c} = \frac{\pi\rho}{6c^3}\int \omega^6 f(\boldsymbol{\kappa}) f(-\boldsymbol{\kappa})\, d\boldsymbol{\kappa}\,.$$

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