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Ocean Engineering 32 (2005) 1296–1310
OCEAN ENGINEERING
www.elsevier.com/locate/oceaneng

Two sets of higher-order Boussinesq-type equations for water waves

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Received 30 April 2004; accepted 12 December 2004
Available online 25 February 2005

Abstract

Based on the classical Boussinesq model by Peregrine [Peregrine, D.H., 1967. Long waves on a beach. J. Fluid Mech. 27 (4), 815–827], two parameters are introduced to improve dispersion and linear shoaling characteristics. The higher order non-linear terms are added to the modified Boussinesq equations. The non-linearity of the Boussinesq model is analyzed. A parameter related to h/L_0 is used to improve the quadratic transfer function in relatively deep water. Since the dispersion characteristic of the modified Boussinesq equations with two parameters is only equal to the second-order Padé expansion of the linear dispersion relation, further improvement is done by introducing a new velocity vector to replace the depth-averaged one in the modified Boussinesq equations. The dispersion characteristic of the further modified Boussinesq equations is accurate to the fourth-order Padé approximation of the linear dispersion relation. Compared to the modified Boussinesq equations, the accuracy of quadratic transfer functions is improved and the shoaling characteristic of the equations has higher accuracy from shallow water to deep water.

Keywords: Boussinesq equations; Non-linear; Dispersion; Shoaling; Transfer function

1. Introduction

The classical Boussinesq equations (Peregrine, 1967) only incorporate weak dispersion and weak non-linearity, and are only valid for simulating long waves in shallow water.

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0029-8018/$ - see front matter
doi:10.1016/j.oceaneng.2004.12.004

To extend the applicable range of the equations, many new forms of Boussinesq-type equations have been developed in the past two decades. Witting (1984) presented a set of Boussinesq equations valid only for a single horizontal dimension. Madsen and Sørensen (1992) introduced a new set of Boussinesq equations with improved linear dispersion characteristics for slowly varying bathymetry in which the depth-integrated velocities are used in the momentum equations. Nwogu (1993) derived an improved Boussinesq model in which the horizontal velocity at an arbitrary depth is used as a dependent variable. Wei et al. (1995) developed the Boussinesq equations of Nwogu (1993) through improving the non-linearity accurate to $O(\mu^2)$. Beji and Nadaoka (1996) gave a new set of Boussinesq equations by adding and subtracting a dispersion term in the momentum equations. Zou (1999) derived high order Boussinesq equations by introducing an artificial velocity and mild slope assumption. Each model is different in the form and arrangement of dispersion terms. But all lead to the second-order Padé approximation of the full dispersion relation of linear waves. There are still other higher order Boussinesq models with better dispersion relation and improved non-linearity (e.g. Agnon et al., 1999; Zou, 2000; Gobbi and Kirby, 2000). Among these Boussinesq equations, a typical and relatively simple method to derive Boussinesq model was introduced by Beji and Nadaoka (1996). However, the equations only have the same weak non-linearity as the classical Boussinesq equations. In order to improve the non-linearity in Boussinesq equations, we add higher order terms accurate to the order of $O(\varepsilon\mu^2)$. It is relatively easy to improve the linear dispersion and linear shoaling characteristics compared to the improvement of the non-linear property of the Boussinesq-type equations. In this paper, a correction parameter linear to h/L_0 (where h is the water depth and L_0 is the wavelength in deep water) is introduced to improve the quadratic transfer functions in relatively deep water instead of a constant parameter in the higher non-linear term (Zou, 1999).

Two higher order Boussinesq equations are presented in this paper. One is modified Boussinesq model with higher order non-linearity in Section 2. In Section 2.1 two parameters are introduced to improve the shoaling characteristic based on the classical Boussinesq equations for varying water depth. Dispersion and shoaling characteristics of the equations are given in Section 2.2. Higher non-linear terms are added to the modified Boussinesq model in Section 2.3. A correction parameter is introduced to improve the quadratic transfer functions in relatively deep water in Section 2.4. In Section 3, the further modified Boussinesq equations are derived. Finally, conclusions are given in Section 4.

2. The derivation of the modified Boussinesq-type equations

2.1. Extended Boussinesq equations

The expression of the classical Boussinesq equations by Peregrine (1967) is

$$\eta_t + \nabla\cdot[(h+\eta)\bar{\boldsymbol{u}}] = 0 \tag{2.1}$$

$$\bar{\boldsymbol{u}}_t + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + g\nabla\eta = \frac{1}{2}h\nabla[\nabla\cdot(h\bar{\boldsymbol{u}}_t)] - \frac{1}{6}h^2\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t) \tag{2.2}$$

where $\bar{\boldsymbol{u}} = (u, v)$ is two-dimensional depth-averaged velocity vector and η is wave surface elevation; $h = h(x,y)$ is still water depth and g is gravitational acceleration; ∇ is two-dimensional gradient operator in a horizontal plane.

By an elementary addition and subtraction process, Eq. (2.2) can be written as

$$\begin{aligned}
&\bar{\boldsymbol{u}}_t + (\bar{\boldsymbol{u}} \cdot \nabla)\bar{\boldsymbol{u}} + g\nabla\eta \\
&\quad = \frac{1}{2}h(1+\alpha)\nabla[\nabla\cdot(h\bar{\boldsymbol{u}}_t)] - \frac{1}{2}\alpha h\nabla[\nabla\cdot(h\bar{\boldsymbol{u}}_t)] - \frac{1}{6}h^2(1+\gamma)\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t) \\
&\qquad + \frac{1}{6}h^2\gamma\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t)
\end{aligned} \tag{2.3}$$

where α and γ are two constant parameters to be determined.

Peregrine (1967) and Beji and Nadaoka (1996) suggested that the linear relation be used in the higher order derivative terms in the equations. Neglecting the higher order and the non-linear terms, we can obtain

$$\eta_t + \nabla\cdot(h\bar{\boldsymbol{u}}) = 0, \quad \bar{\boldsymbol{u}}_t + g\nabla\eta = 0 \tag{2.4}$$

We use the second equation of Eq. (2.4): $\bar{\boldsymbol{u}}_t = -g\nabla\eta$ to replace the terms proportional to α and γ in Eq. (2.3) and obtain

$$\begin{aligned}
&\bar{\boldsymbol{u}}_t + (\bar{\boldsymbol{u}} \cdot \nabla)\bar{\boldsymbol{u}} + g\nabla\eta \\
&\quad = \frac{1}{2}h(1+\alpha)\nabla[\nabla\cdot(h\bar{\boldsymbol{u}}_t)] + \frac{1}{2}\alpha gh\nabla[\nabla\cdot(h\nabla\eta)] - \frac{1}{6}h^2(1+\gamma)\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t) \\
&\qquad - \frac{1}{6}gh^2\gamma\nabla(\nabla^2\eta)
\end{aligned} \tag{2.5}$$

If α and γ are equal, the extended Boussinesq equations will be changed to those of Beji and Nadaoka's (1996). Hereafter, we call the equations BN model.

2.2. Dispersion and linear shoaling characteristics of the extended equations

Neglecting non-linear terms in Eqs. (2.1) and (2.5) yields the following dispersion relation

$$\frac{\omega^2}{gk} = \frac{kh\left[1 + \left(\frac{1}{2}\alpha - \frac{1}{6}\gamma\right)k^2h^2\right]}{\left[1 + \frac{1}{2}(1+\alpha)k^2h^2 - \frac{1}{6}(1+\gamma)k^2h^2\right]} \tag{2.6}$$

where ω is the angular wave frequency, $k^2 = k_x^2 + k_y^2$ and k_x, k_y are the component of the wave number vector $\mathbf{k}$ in x- and y-direction, respectively.

The expression of α and γ can be determined from the second-order Padé expansion of the linear Stokes dispersion relation $\omega^2/gk = \tanh(kh)$:

$$\frac{\omega}{gk} = \frac{kh + k^3h^3/15}{1 + 2k^2h^2/5} \tag{2.7}$$

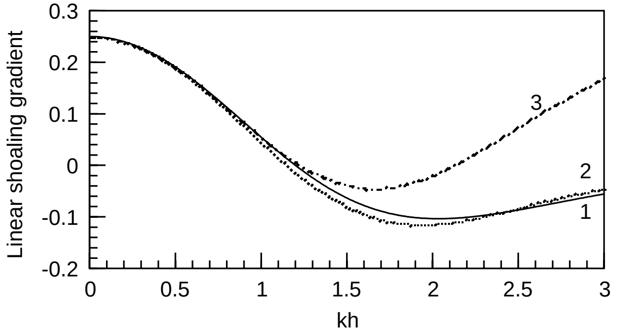

Fig. 1. Comparison of the Shoaling gradient with analytical solution. (1) Linear Stokes wave, (2) Present model, (3) BN model.

By comparison of Eq. (2.6) with Eq. (2.7), we can obtain an expression $(1/2)\alpha-(1/6)\gamma=(1/15)$. Yet the value of α and γ is still unknown. By a further comparison between the linear shoaling characteristic of the extended Boussinesq equations and that of the linear Stokes waves, the two parameters are optimized to be $\alpha=0.1308$, $\beta=-0.0076$. The detailed derivation method of the linear shoaling characteristics is referred to Schaffer and Madsen (1995). The extended equations are applicable to the water depth up to $h/L_0=0.476$ comparing with the depth $h/L_0=0.25$ for BN model. Fig. 1 shows the linear shoaling gradient comparison among the present model, BN model and the linear Stokes waves. From Fig. 1, the agreement of the present model is satisfactory.

2.3. The higher order non-linear terms

Eq. (2.5) only covers the lowest non-linearty terms. The non-linearity of Eq. (2.5) is identical to the classical Boussinesq equations. To improve the higher-order non-linearity, a strict mathematical derivation is applied here. We start the derivations with the simple case of a horizontal bottom. A Cartesian coordinate system (x,y,z) is adopted with the origin being located at the still water level and z-axis pointing vertically upward. The fluid is assumed to be incompressible and inviscid, and the flow is assumed to be irrotational. The independent and dependent, non-dimensional variables are defined as

$$x'=\frac{x}{L},\ \ y'=\frac{y}{L},\ \ z'=\frac{z}{h},\ \ t'=\frac{\sqrt{gh}}{L}t,\quad \phi'=\phi/\left(\frac{A}{h}L\sqrt{gh}\right),\ \ \eta'=\frac{\eta}{A} \tag{2.8}$$

where g is the gravitational acceleration, h is water depth, and L, A are the characteristic wave length and characteristic wave amplitude, respectively. Non-dimensional governing equations can be expressed as (the superscript is omitted in the flowing derivation)

$$\mu^2\nabla^2\phi+\phi_{zz}=0,\qquad -1<z<\varepsilon\eta \tag{2.9}$$

$$\mu^2(\eta_t+\varepsilon\nabla\phi\cdot\nabla\eta)=\phi_z,\quad z=\varepsilon\eta \tag{2.10}$$

$$\mu^2(\phi_t + \eta) + \frac{1}{2}\varepsilon(\mu^2|\nabla\phi|^2 + \phi_z^2) = 0, \quad z = \varepsilon\eta \tag{2.11}$$

$$\phi_z = 0, \quad z = -1 \tag{2.12}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.13}$$

where $\varepsilon = A/h$ and $\mu = h/L$ are the non-linearity and dispersion parameters, respectively. We assume that $O(\varepsilon) = O(\mu^2) << 1$ to study the water wave motion of Boussinesq-type Equations.

Since ϕ is an analytic function, it can be expanded as Taylor series in terms of z around seabed $z = -1$. When Eq. (2.9) and the boundary condition Eq. (2.12) are satisfied, we obtain

$$\phi(x,y,z,t) = \phi_0 - \frac{\mu^2}{2!}(z+1)^2\nabla^2\phi_0 + \frac{\mu^4}{4!}(z+1)^4\nabla^2\nabla^2\phi_0 + O(\mu^6) \tag{2.14}$$

where $\phi_0 = \phi_0(x,y)$. Substituting Eq. (2.14) into Eq. (2.11), taking the gradient ∇ of the resulting equation and applying the definition of $\boldsymbol{u}_0 \equiv \nabla\phi_0$ at seabed with the denotation of $\nabla^2\boldsymbol{u}_0 \equiv \nabla(\nabla\cdot\boldsymbol{u}_0)$ yields

$$\begin{aligned}&\boldsymbol{u}_{0t} + \varepsilon(\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 + \nabla\eta - \frac{1}{2}\mu^2 d^2\nabla^2\boldsymbol{u}_{0t} - \mu^2 d\nabla d\nabla\cdot\boldsymbol{u}_{0t} + \frac{1}{4!}\mu^4 d^4\nabla^2\nabla^2\boldsymbol{u}_{0t} \\ &\quad + \frac{1}{2}\varepsilon\mu^2\nabla[-d^2\boldsymbol{u}_0\cdot\nabla^2\boldsymbol{u}_0 + d^2(\nabla\cdot\boldsymbol{u}_0)^2] \\ &\quad = O(\mu^6, \varepsilon\mu^4)\end{aligned} \tag{2.15}$$

where d is the total water depth: $d = 1 + \varepsilon\eta$. Introducing depth-averaged velocity $\bar{\boldsymbol{u}} = 1/d\int_{-1}^{\varepsilon\eta}\nabla\phi\,\mathrm{d}z$ and using Eq. (2.14), we have

$$\boldsymbol{u}_0 = \bar{\boldsymbol{u}} + \frac{1}{3!}\mu^2 d^2\nabla^2\bar{\boldsymbol{u}} + \frac{7}{3\times 5!}\mu^4 d^4\nabla^2\nabla^2\bar{\boldsymbol{u}} + O(\mu^6) \tag{2.16}$$

Substituting Eq. (2.16) into Eq. (2.15) for $\bar{\boldsymbol{u}}_0$ yields

$$\bar{\boldsymbol{u}}_t + \varepsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta + G_1 = \frac{1}{3}\mu^2 d^2\nabla^2\bar{\boldsymbol{u}}_t + \frac{1}{45}\mu^4 d^4\nabla^2\nabla^2\bar{\boldsymbol{u}}_t + O(\mu^6, \varepsilon\mu^4) \tag{2.17a}$$

where

$$G_1 = \varepsilon\mu^2\left\{\nabla\left[\frac{d^2}{2}(\nabla\cdot\bar{\boldsymbol{u}})^2 - \frac{d^2}{3}\bar{\boldsymbol{u}}\cdot\nabla^2\bar{\boldsymbol{u}}\right] + \frac{1}{3}\mathrm{d}\eta_t\,\nabla^2\bar{\boldsymbol{u}} - \mathrm{d}\nabla\,\eta\nabla\cdot\bar{\boldsymbol{u}}_t\right\} \tag{2.17b}$$

The fourth spatial derivative in Eq. (2.17a) would bring difficulty to the numerical computation. This problem can be solved by neglecting the $O(\varepsilon\mu^2,\mu^4))$ terms in Eq. (2.17a)

$$\bar{\boldsymbol{u}}_t + \varepsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \frac{1}{3}\mu^2 d^2\nabla^2\bar{\boldsymbol{u}}_t + O(\varepsilon\mu^2, \mu^4) \tag{2.18}$$

The fourth spatial derivative in Eq. (2.17a) can be expressed by the left-hand size terms of the Eq. (2.18) and then we can get

$$\bar{\boldsymbol{u}}_t + \varepsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta + G_2 = \frac{1}{3}\mu^2 d^2\nabla^2\bar{\boldsymbol{u}}_t + \frac{1}{15}\mu^2 d^2\nabla^2(\bar{\boldsymbol{u}}_t + \nabla\eta) + \mathrm{O}(\varepsilon\mu^4\cdot\mu^6) \tag{2.19a}$$

and

$$G_2 = \varepsilon\mu^2\left\{\nabla\left[\frac{d^2}{2}(\nabla\cdot\bar{\boldsymbol{u}})^2 - \frac{d^2}{3}\bar{\boldsymbol{u}}\cdot\nabla^2\bar{\boldsymbol{u}} - \frac{d^2}{30}\nabla^2(\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}})\right] + \frac{1}{3}\mathrm{d}\eta_t\,\nabla^2\boldsymbol{u} - \mathrm{d}\nabla\eta\nabla\cdot\bar{\boldsymbol{u}}_t\right\} \tag{2.19b}$$

The expressions $(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} = \frac{1}{2}\nabla(\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}}) - \bar{\boldsymbol{u}}\times\nabla\times\bar{\boldsymbol{u}}$ and $\nabla\times\bar{\boldsymbol{u}} = \mathrm{O}(\varepsilon\mu^2)$ are used in the above derivation.

The continuity equation can be derived by substituting Eq. (2.14) of the potential ϕ into Eq. (2.10) and we can obtain

$$\eta_t + \nabla\cdot(\mathrm{d}\bar{\boldsymbol{u}}) = 0 \tag{2.20}$$

The term η_t in the non-linearity terms G can be rewritten by using Eq. (2.20), Eqs. (2.20) and (2.19) with dimensions can be written as

$$\eta_t + \nabla\cdot(\mathrm{d}\bar{\boldsymbol{u}}) = 0 \tag{2.21}$$

$$\bar{\boldsymbol{u}}_t + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + g\nabla\eta + G = \frac{d^2}{3}\nabla^2\bar{\boldsymbol{u}}_t + \frac{h^2}{15}\nabla^2(\bar{\boldsymbol{u}}_t + g\nabla\eta) \tag{2.22a}$$

and

$$G = \nabla\left\{\frac{h^2}{3}\left[(\nabla\cdot\bar{\boldsymbol{u}})^2 - \bar{\boldsymbol{u}}\cdot\nabla^2\bar{\boldsymbol{u}} - \frac{1}{10}\nabla^2(\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}})\right]\right\} - \nabla\eta\,\mathrm{d}\nabla\cdot\bar{\boldsymbol{u}}_t \tag{2.22b}$$

or

$$G = \nabla\left\{\frac{d^2}{3}\left[(\nabla\cdot\bar{\boldsymbol{u}})^2 - \bar{\boldsymbol{u}}\cdot\nabla^2\bar{\boldsymbol{u}} - \frac{1}{10}\nabla^2(\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}})\right]\right\} - \nabla\eta\left[\mathrm{d}\nabla\cdot\bar{\boldsymbol{u}}_t - \frac{1}{3}\nabla\cdot\bar{\boldsymbol{u}}\nabla\cdot(\mathrm{d}\bar{\boldsymbol{u}})\right] - \frac{1}{3}(\nabla\eta\cdot\bar{\boldsymbol{u}})\nabla(\mathrm{d}\nabla\cdot\bar{\boldsymbol{u}}) \tag{2.22c}$$

Comparing Eq. (2.22a) with Eq. (2.5) and adding higher non-linear terms $\mathrm{O}(\varepsilon\mu^2)$ to Eq. (2.5) we obtain

$$\bar{\boldsymbol{u}}_t + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + g\nabla\eta + G = \frac{1}{2}h(1+\alpha)\nabla[\nabla\cdot(h\bar{\boldsymbol{u}}_t)] + \frac{1}{2}\alpha gh\nabla[\nabla\cdot(h\nabla\eta)] - \frac{1}{6}h^2(1+\gamma)\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t) - \frac{1}{6}gh^2\gamma\nabla(\nabla^2\eta) \tag{2.23a}$$

$$G=\nabla\left\{\frac{h^2}{3}\left[(\nabla\cdot\bar{\boldsymbol{u}})^2-\bar{\boldsymbol{u}}\cdot\nabla^2\bar{\boldsymbol{u}}-\frac{3}{2}\left(\frac{\alpha}{2}-\frac{\gamma}{6}\right)\nabla^2(\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}})\right]\right\}-\frac{2}{3}h\eta\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t)$$
$$-h\nabla\eta\nabla\cdot\bar{\boldsymbol{u}}_t \tag{2.23b}$$

In order to improve the non-linear property, a correction parameter $\beta=12.1h/L_0$ is introduced in the higher non-linear term:

$$G=\nabla\left\{\frac{h^2}{3}\left[(\nabla\cdot\bar{\boldsymbol{u}})^2-\bar{\boldsymbol{u}}\cdot\nabla^2\bar{\boldsymbol{u}}-\frac{3}{2}\beta\left(\frac{\alpha}{2}-\frac{\gamma}{6}\right)\nabla^2(\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}})\right]\right\}-\frac{2}{3}h\eta\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t)$$
$$-h\nabla\eta\nabla\cdot\bar{\boldsymbol{u}}_t \tag{2.23c}$$

2.4. Transfer function of super and sub harmonics

For a bichromatic wave train, the following first order equations of (2.1) and (2.23) is assumed to be

$$\eta_1=A_1\cos(k_1x-\omega_1t)+A_2\cos(k_2x-\omega_2t) \tag{2.24}$$

By the first order equations of (2.1) and (2.23), we can obtain

$$u_1=\frac{\omega_1}{k_1h}A_1\cos(k_1x-\omega_1t)+\frac{\omega_2}{k_2h}A_2\cos(k_2x-\omega_2t) \tag{2.25}$$

where A_i, ω_i, k_i ($i=1,2$) are the amplitude, wave frequency and wave number. From Eqs. (2.1)–(2.23) for constant depth, we can obtain the second order equations

$$\eta_2=A_1A_2G_2^{\pm}(\omega_1,\omega_2)\cos(k_{\pm}x-\omega_{\pm}t)+\frac{1}{2}A_1^2G_2^{+}(\omega_1,\omega_2)\cos 2(k_1x-\omega_1t)$$
$$+\frac{1}{2}A_2^2G_2^{+}(\omega_2,\omega_2)\cos 2(k_2x-\omega_2t) \tag{2.26}$$

where $k_{\pm}=k_1\pm k_2$, $\omega_{\pm}=\omega_1+\omega_2$ $G_2^{\pm}=(\omega_1,\omega_2)$ is the quadratic transfer function with the following expression

$$G_2^{\pm}(\omega_1,\omega_2)=\frac{1}{h}(g_{\mathrm{BB}}^{\pm}+g_{\mathrm{se}}^{\pm}+g_{\eta}^{\pm}+\beta g_{\mathrm{dis}}^{\pm}) \tag{2.27a}$$

where

$$g_{\mathrm{BB}}^{\pm}=\frac{k_{\pm}}{2D_{\pm}}\left[\omega_{\pm}\left(1+\frac{2}{5}k_{\pm}^2h^2\right)\left(\frac{\omega_1}{k_1}+\frac{\omega_2}{k_2}\right)+k_{\pm}\left(1+\frac{1}{15}k_{\pm}^2h^2\right)\frac{\omega_1\omega_2}{k_1k_2}\right] \tag{2.27b}$$

$$g_{\mathrm{se}}^{\pm}=\mp\frac{k_{\pm}^2h^2}{2D_{\pm}}\left(1+\frac{1}{15}k_{\pm}^2h^2\right)\left(\frac{\omega_1\omega_2}{k_1k_2}\right)\left[k_1k_2\mp\frac{1}{3}(k_1^2+k_2^2)-\frac{1}{3}k_1k_2\right] \tag{2.27c}$$

$$g_{\eta}^{\pm}=-\frac{k_{\pm}}{2D_{\pm}}\left(1+\frac{1}{15}k_{\pm}^2h^2\right)\left[k_1\omega_2^2\pm k_2\omega_1^2+\frac{2}{3}(k_1\omega_1^2\pm k_2\omega_2^2)\right] \tag{2.27d}$$

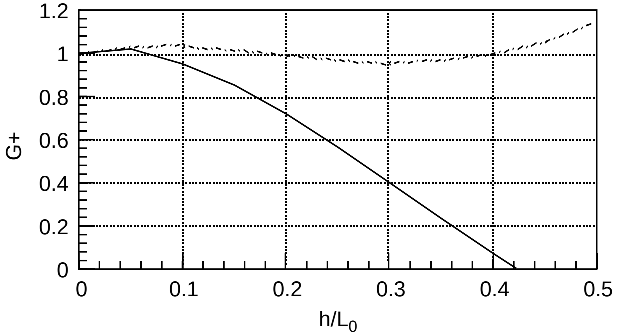

Fig. 2. Ratio of super harmonic $G_2^+(\omega_1,\omega_2)$ to Stokes solution $G+(\omega_1,\omega_2)$ $\bar{G}+=G_2^+(\omega_1,\omega_2)/G+(\omega_1,\omega_2)$, $\omega_2-\omega_1=0.1\bar{\omega}$, $\bar{\omega}=(\omega_1+\omega_2)/2$. Eqs. (2.1) and (2.23) with $\beta=1$ (solid); Eqs. (2.1) and (2.23) $\beta=12.1h/L_0$ (dash-dotted).

$$g_{\rm dis}^{\pm}=\frac{k_{\pm}^2h^2}{30D_{\pm}}\left(1+\frac{1}{15}k_{\pm}^2h^2\right)\left(\frac{\omega_1\omega_2}{k_1k_2}\right) \tag{2.27e}$$

$$D_{\pm}=-ghk_{\pm}^2\left(1+\frac{1}{15}k_{\pm}^2h^2\right)+\omega_{\pm}^2\left(1+\frac{2}{5}k_{\pm}^2h^2\right) \tag{2.27f}$$

Figs. 2 and 3 are the comparisons between the present model and the Stokes solution. From the figures, the super harmonic of the present model with $\beta=1.0$ deviates from the Stokes solution quickly with the increasing of water depth. But when introducing $\beta=12.1h/L_0$, even for $h/L_0=0.5$, the super harmonic can well agree with the Stokes solution and the maximum error is 14%. As for the sub harmonic, the present model can well agree with the Stokes solution up to the water depth $h/L_0=0.5$.

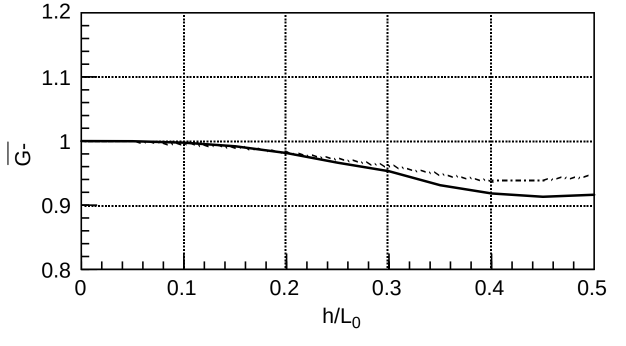

Fig. 3. Ratio of sub harmonic $G_2^-(\omega_1,\omega_2)$ to Stokes solution $G-(\omega_1,\omega_2)$ $\bar{G}-=G_2^-(\omega_1,\omega_2)/G-(\omega_1,\omega_2)$, $\omega_2-\omega_1=0.1\bar{\omega}$, $\bar{\omega}=(\omega_1+\omega_2)/2$. Eqs. (2.1) and (2.23) with $\beta=1$ (Solid); Eqs. (2.1) and (2.23) $\beta=12.1h/L_0$ (dash-dotted).

3. Further improvement of the modified Boussinesq-type equations

3.1. Derivation of the equations

The dispersion of the modified Boussinesq-type equations with two parameters is only to the accuracy of $O(\mu^4)$ and the maximum range of the shoaling characteristics is only $h/L=0.476$. To further improve the dispersion and linear shoaling characteristic of the equations, a new velocity variable $\tilde{\boldsymbol{u}}$ is introduced to replace the depth-averaged velocity $\bar{\boldsymbol{u}}$

$$\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}} - a_1\mu^2h^2\nabla^2\bar{\boldsymbol{u}} - a_2\mu^2h\nabla^2(h\bar{\boldsymbol{u}}) - b_1\mu^4h^4\nabla^2\nabla^2\bar{\boldsymbol{u}} - b_2\mu^4h\nabla^2\nabla^2(h^3\bar{\boldsymbol{u}}) \tag{3.1}$$

where a_1, a_2, b_1 and b_2 are four parameters to be determined. Eq. (3.1) is accurate to $O(\mu^4)$, which is the same in order of approximation as the velocity for the modified Boussinesq-type equations in Eqs. (2.1), (2.23a) and (2.23b). To simplify the equations, we consider a mildly sloping seabed and employ the mild slope assumption $\nabla h = O(\mu^2)$.

Substituting Eq. (3.1) into the non-dimensional expression of Eqs. (2.1) and (2.23) leads to

$$\begin{aligned}&\eta_t + \nabla\cdot[(h+\varepsilon\eta)\tilde{\boldsymbol{u}}]\\&\quad= a_1\mu^2\nabla\cdot[(h+\varepsilon\eta)h^2\nabla^2\tilde{\boldsymbol{u}}] + a_2\nabla\cdot[(h+\varepsilon\eta)h\nabla^2(h\tilde{\boldsymbol{u}})]\\&\qquad+ b_1\mu^4\nabla\cdot(h^5\nabla^2\nabla^2\tilde{\boldsymbol{u}}) + b_2\mu^4\nabla\cdot[h^2\nabla^2\nabla^2(h^3\tilde{\boldsymbol{u}})] + O(\varepsilon^2\mu^2,\mu^6)\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\tilde{\boldsymbol{u}}_t + \varepsilon(\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{u}} + \nabla\eta + G\\&\quad= (a_1 - \frac{1}{6})\mu^2h^2\nabla(\nabla\cdot\tilde{\boldsymbol{u}}_t) + \left(a_2+\frac{1}{2}\right)\mu^2h\nabla[\nabla\cdot(h\tilde{\boldsymbol{u}})]\\&\qquad+ -\frac{\gamma}{6}\mu^2h^2\nabla[\nabla\cdot(\tilde{\boldsymbol{u}}_t+\nabla\eta)] + \frac{\alpha}{2}\mu^2h\nabla[\nabla\cdot(h\tilde{\boldsymbol{u}}_t + h\nabla\eta)]\\&\qquad+ \left[b_1 - \frac{1}{3}(a_1+a_2)\right]\mu^4h^4\nabla^2\nabla^2\tilde{\boldsymbol{u}}_t + b_2\mu^4h\nabla^2\nabla^2(h^3\tilde{\boldsymbol{u}}_t) + O(\varepsilon^2\mu^2,\mu^6)\end{aligned} \tag{3.3a}$$

$$\begin{aligned}G &= \frac{1}{3}\varepsilon\mu^2\nabla\left\{h^2\left[(\nabla\cdot\tilde{\boldsymbol{u}})^2 - (1+3(a_1+a_2))\tilde{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\tilde{\boldsymbol{u}}) - \frac{3}{2}\left(\frac{\alpha}{2}-\frac{\gamma}{6}\right)\nabla^2(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}})\right]\right\}\\&\quad-\frac{2}{3}\varepsilon\mu^2h\eta\nabla(\nabla\cdot\tilde{\boldsymbol{u}}_t) - \varepsilon\mu^2h\nabla\eta\nabla\cdot\tilde{\boldsymbol{u}}_t\end{aligned} \tag{3.3b}$$

In order to remove the fourth spatial derivatives in the above equations, neglecting the $O(\varepsilon\mu^2,\mu^4)$ terms in Eqs. (3.2), (3.3a) and (3.3b) gives

$$\eta_t + \nabla\cdot[(h+\varepsilon\eta)\tilde{\boldsymbol{u}}] = (a_1+a_2)\mu^2h^3\nabla\cdot(\nabla^2\tilde{\boldsymbol{u}}) + O(\varepsilon\mu^2,\mu^4) \tag{3.4}$$

$$\eta_t + \nabla\cdot[(h+\varepsilon\eta)\tilde{\boldsymbol{u}}] = (a_1+a_2)\mu^2\nabla\cdot[\nabla^2(h^3\tilde{\boldsymbol{u}})] + O(\varepsilon\mu^2,\mu^4) \tag{3.5}$$

$$\tilde{\boldsymbol{u}}_t + \varepsilon(\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{u}} + \nabla\eta = \left(\frac{1}{3} + a_1 + a_2\right)\mu^2 h^2 \nabla(\nabla\cdot\tilde{\boldsymbol{u}}_t) + \mathrm{O}(\varepsilon\mu^2, \mu^4) \tag{3.6}$$

$$\tilde{\boldsymbol{u}}_t + \varepsilon(\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{u}} + \nabla\eta = \left(\frac{1}{3} + a_1 + a_2\right)\mu^2 \nabla[\nabla\cdot(h^2\tilde{\boldsymbol{u}}_t)] + \mathrm{O}(\varepsilon\mu^2, \mu^4) \tag{3.7}$$

Substituting Eqs. (3.4)–(3.7) into the fourth spatial derivatives in Eqs. (3.2), (3.3a) and (3.3b) and using the relation $(\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{u}} = (1/2)\nabla(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}) - \tilde{\boldsymbol{u}}\times\nabla\times\tilde{\boldsymbol{u}}$ and $\nabla\times\tilde{\boldsymbol{u}} = \mathrm{O}(\varepsilon\mu^2)$ gives the flowing equations expressed in dimensional form:

$$\eta_t + \nabla[(h+\eta)\tilde{\boldsymbol{u}}] = a_1\nabla\cdot[(h+\eta)h^2\nabla(\nabla\cdot\tilde{\boldsymbol{u}})] + a_2\nabla\cdot[(h + \eta)h\nabla(\nabla\cdot(h\tilde{\boldsymbol{u}}))] + \frac{b_1}{a}\nabla\cdot\{h^2\nabla[\eta_t + \nabla\cdot((h+\eta)\tilde{\boldsymbol{u}})]\} + \frac{b_2}{a}\nabla\cdot\{\nabla[h^2\eta_t + h^2\nabla\cdot((h+\eta)\tilde{\boldsymbol{u}})]\} \tag{3.8}$$

$$\tilde{\boldsymbol{u}}_t + (\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{u}} + g\nabla\eta + G = \left(a_1 - \frac{1}{6}\right)h^2\nabla(\nabla\cdot\tilde{\boldsymbol{u}}_t) + \left(a_2 + \frac{1}{2}\right)h\nabla[\nabla\cdot(h\tilde{\boldsymbol{u}}_t)] + c_1 h^2\nabla[\nabla\cdot(\tilde{\boldsymbol{u}}_t + g\nabla\eta)] + c_2\nabla[\nabla\cdot(h^2\tilde{\boldsymbol{u}}_t + gh^2\nabla\eta)] + \frac{\alpha}{2}h\nabla[\nabla\cdot(h\tilde{\boldsymbol{u}}_t + gh\nabla\eta)] \tag{3.9a}$$

The expression of G is accurate to $\mathrm{O}(\varepsilon\mu^2)$

$$G = \frac{1}{3}\nabla\left\{h^2\left[(\nabla\cdot\tilde{\boldsymbol{u}})^2 - (1+3a)\tilde{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\tilde{\boldsymbol{u}}) - \frac{3}{2}\left(c + \frac{\alpha}{2}\right)\nabla^2(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}})\right]\right\} - \frac{2}{3}h\eta\nabla(\nabla\cdot\tilde{\boldsymbol{u}}_t) - h\nabla\eta\nabla\cdot\tilde{\boldsymbol{u}}_t \tag{3.9b}$$

$$a = a_1 + a_2,\ \ b = b_1 + b_2,\ \ c = c_1 + c_2;\quad c_1 = -\gamma/6 + (b_1 - a/3)/(a + 1/3),\ \ c_2 = b_2/(a+1/3) \tag{3.9c}$$

Eqs. (3.8), (3.9a) and (3.9b) are the new form of the higher order Boussinesq equations in terms of the new velocity vector $\tilde{\boldsymbol{u}}$.

3.2. Properties of the new form of the equations

In this section, we will determine the six parameters contained in the new derived equations by investigating the dispersion, non-linear characteristics and shoaling property of the equations.

3.2.1. Linear dispersion characteristic

Using the method introduced by Schaffer and Madsen (1995), we can obtain the following dispersion relation for Eqs. (3.8) and (3.9):

$$C^2 = \frac{\omega^2}{k^2} = gh\frac{[1+(a+b/a)k^2h^2][1+(c+\alpha/2)k^2h^2]}{[1+(b/a)k^2h^2][1+(1/3+a+c+\alpha/2)k^2h^2]} \tag{3.10}$$

Comparing with the following expression of the expansion of the linear Stokes dispersion

$$(C^S)^2 = gh\frac{\tanh kh}{kh} = gh\frac{1+(1/9)k^2h^2+(1/945)k^4h^4}{1+(4/9)k^2h^2+(1/63)k^4h^4} + O(k^{10}h^{10}) \tag{3.11}$$

the value a,b/a and $c+\alpha/2$ can be obtained by choosing the parameters (a, b/a, $c+\alpha/2$) in Eq. (3.10) to match fourth-order Padé expansion in Eq. (3.11), and the following four sets of the solutions can be obtained

$$\left(a,\frac{b}{a},c+\alpha/2\right) = \begin{Bmatrix} -\frac{1}{6}+\frac{\sqrt{805}}{630}+\frac{\sqrt{133}}{63},\frac{2}{9}-\frac{\sqrt{133}}{63},\frac{1}{18}-\frac{\sqrt{805}}{630} \\ -\frac{1}{6}-\frac{\sqrt{805}}{630}+\frac{\sqrt{133}}{63},\frac{2}{9}-\frac{\sqrt{133}}{63},\frac{1}{18}+\frac{\sqrt{805}}{630} \\ -\frac{1}{6}+\frac{\sqrt{805}}{630}-\frac{\sqrt{133}}{63},\frac{2}{9}+\frac{\sqrt{133}}{63},\frac{1}{18}-\frac{\sqrt{805}}{630} \\ -\frac{1}{6}-\frac{\sqrt{805}}{630}-\frac{\sqrt{133}}{63},\frac{2}{9}+\frac{\sqrt{133}}{63},\frac{1}{18}+\frac{\sqrt{805}}{630} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.061, 0.039, 0.011 \\ -0.0287, 0.039, 0.101 \\ -0.305, 0.405, 0.011 \\ -0.395, 0.405, 0.101 \end{Bmatrix} \begin{Bmatrix} \text{Group } A \\ \text{Group } B \\ \text{Group } C \\ \text{Group } D \end{Bmatrix} \tag{3.12}$$

Using each of the parameter sets given in Eq. (3.12), the new equations will yield the dispersion Eq. (3.11), but as discussed in Section 3.2.2, only one set is appropriate for giving accurate super and sub harmonic transfer function.

The results of the celerity ratio C/C^s and the corresponding group velocity ratio C_g/C_g^s are given in Table 1. We can see that when $h/L_0 \le 0.5$, the error is less than 1%, even when $h/L_0 = 1.0$, the maximum error of C is only 2% and that of C_g is only 11%.

3.2.2. Transfer function of super and sub harmonics

When solving the Eqs. (3.8) and (3.9) for constant water depth by the similar way in Section 2.4, we can obtain the following expressions:

Table 1
Scaled celerity, group velocity and the shoaling gradients

h/L_0	0	0.2	0.4	0.5	0.7	0.8	1.0
C/C^s	1.000	1.000	1.000	1.000	1.004	1.007	1.020
C_g/C_g^s	1.000	1.000	1.001	1.003	1.011	1.043	1.102
α^s	0.2498	0.0484	0.0840	0.0471	0.0090	0.0035	0.0005
α^5	0.2498	0.0511	0.0908	0.0524	0.0084	0.0034	0.0171

$$G_2^{\pm}(\omega_1,\omega_2)=\frac{1}{h}(g_{\mathrm{BB}}^{\pm}+g_{\mathrm{se}}^{\pm}+g_{\eta}^{\pm}+g_{\mathrm{dis}}^{\pm}+\frown g_{\eta}^{\pm}) \quad (3.13a)$$

$$g_{\mathrm{BB}}^{\pm}=\frac{k_{\pm}}{2D_{\pm}}\left[\omega_{\pm}f_3(k_{\pm})\left(\frac{\omega_1}{k_1'}+\frac{\omega_2}{k_2'}\right)+k_{\pm}f_1(k_{\pm})\frac{\omega_1\omega_2}{k_1'k_2'}\right], \quad (3.13b)$$

$$g_{\mathrm{se}}^{\pm}=\mp\frac{k_{\pm}^2h^2}{2D_{\pm}}f_1(k_{\pm})\left(\frac{\omega_1\omega_2}{k_1'k_2'}\right)\left[k_1k_2\mp\frac{1}{3}(1+3a)(k_1^2+k_2^2)-\frac{1}{3}k_1k_2\right], \quad (3.13c)$$

$$g_{\eta}^{\pm}=-\frac{k_{\pm}h^2}{2D_{\pm}}f_1(k_{\pm})\left[\frac{k_1k_2}{k_2'}\omega_2^2\pm\frac{k_2k_1}{k_1'}\omega_1^2+\frac{2}{3}\left(\frac{k_1^2}{k_1'}\omega_1^2\pm\frac{k_2^2}{k_2'}\omega_2^2\right)\right], \quad (3.13d)$$

$$g_{\mathrm{dis}}^{\pm}=\frac{c+\alpha/2}{2}\frac{k_{\pm}^4h^2}{D_{\pm}}f_1(k_{\pm})\left(\frac{\omega_1\omega_2}{k_1'k_2'}\right), \quad (3.13e)$$

$$\hat{g}_{\eta}^{\pm}=a\frac{\omega_{\pm}k_{\pm}h^2}{2D_{\pm}}f_3(k_{\pm})\left(\frac{k_1^2}{k_2'}\omega_1+\frac{k_2^2}{k_2'}\omega_2\right)+\left(\frac{a}{b}\right)\frac{\omega_{\pm}k_{\pm}^3h^2}{2D_{\pm}}f_3(k_{\pm})\left(\frac{\omega_1}{k_1'}+\frac{\omega_2}{k_2'}\right) \quad (3.13f)$$

where

$$f_1(k)=1+\left(a+\frac{b}{a}\right)k^2h^2,\quad f_2(k)=1+(c+\alpha/2)k^2h^2,\quad f_3(k)$$
$$=1+\left(\frac{1}{3}+a+c+\alpha/2\right)k^2h^2,\quad f_4(k)=1+\left(\frac{b}{a}\right)k^2h^2,\quad k_i'$$
$$=k_if_1(k_i)/f_4(k_i)\ \ (i=1,2),\quad D_{\pm}$$
$$=-ghk_{\pm}^2f_1(k_{\pm})f_2(k_{\pm})+\omega_{\pm}^2f_3(k_{\pm})\cdot f_4(k_{\pm}) \quad (3.13g)$$

When a, b/a, $c+\alpha/2$ are different, then $G_{\pm}(\omega_1,\omega_2)$ will be different. Figs. 4 and 5 give the different results with different value of a, b/a, $c+\alpha/2$. From the figures, we know only the value in Group B can agree well with the Stokes solution compared with the other three groups, so we will choose Group B as the final solution. Compared with the modified Boussiensq equations (with $\beta=1$), we can find that the equations in this section have a better super transfer function for the range of $h/L_0<0.3$, but the sub harmonic transfer function is only applicable to $h/L_0<0.3$, which is less accurate compared with modified Boussinesq equations. If we apply the method used in Section 2.4, with a correction parameter introduced, the sub harmonic transfer function is only applicable to $h/L_0<0.5$. The maximum error is 13%. The sub harmonic transfer function is still only applicable to $h/L_0<0.3$.

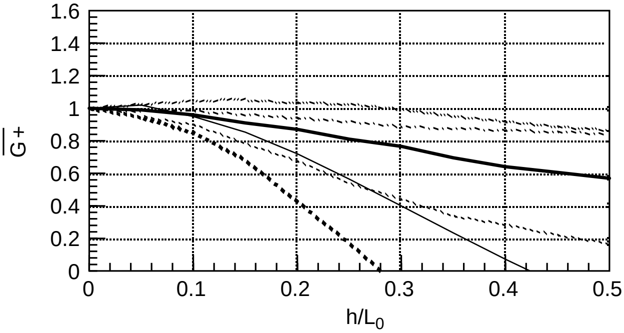

Fig. 4. Ratio of super harmonic $G_2^+(\omega_1,\omega_2)$ to Stokes solution $G+(\omega_1,\omega_2)$ $\bar{G}+=G_2^+(\omega_1,\omega_2)/G+(\omega_1,\omega_2)$, $\omega_2-\omega_1=0.1\bar{\omega}$, $\bar{\omega}=(\omega_1+\omega_2)/2$. ⋯ Group A; — Group B with $\beta=1$; - - - Group C; -·-·- Group D; -·····- Group B with $\beta=12.1h/L_0$; — Eqs. (2.1) and (2.23) with $\beta=1$.

3.2.3. Linear shoaling characteristic

Using the method introduced by Schaffer and Madsen (1995) in Eqs. (3.8) and (3.9a), we obtain

$$\alpha_1\frac{A_x}{A}+\alpha_2\frac{k_x}{k}+\alpha_3\frac{h_x}{h}=0 \tag{3.14}$$

From Eq. (3.10) we can obtain the following expression:

$$\frac{k_x}{k}=\alpha_4\frac{h_x}{h} \tag{3.15}$$

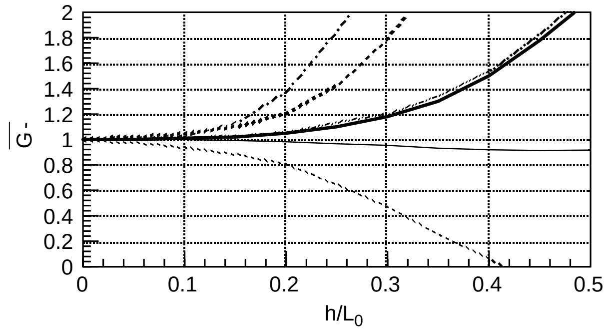

Fig. 5. Ratio of sub harmonic $G_2^-(\omega_1,\omega_2)$ to Stokes solution $G-(\omega_1,\omega_2)$ $\bar{G}-=G_2^-(\omega_1,\omega_2)/G-(\omega_1,\omega_2)$, $\omega_2-\omega_1=0.1\bar{\omega}$, $\bar{\omega}=(\omega_1+\omega_2)/2$. ⋯ Group A; — Group B with $\beta=1$; - - - Group C; -·-·- Group D; –··–··– Group B with $\beta=12.1h/L_0$; — Eqs. (2.1) and (2.23) with $\beta=1$.

where

$$\alpha_1 \equiv (1-3\beta_1\bar{k}^2)\Lambda_2\Lambda_5 g_1 + (1-3\beta_6\bar{k}^2)\Lambda_1^2 g_2 + 2\beta_5\Lambda_1\Lambda_2\bar{k}^2 g_1 + 2\beta_2\Lambda_1\Lambda_5\bar{k}^2 g_1,$$

$$\alpha_2 \equiv -(1-3\beta_1\bar{k}^2)[1+(\beta_2-3\beta_1)\bar{k}^2+\beta_1\beta_2\bar{k}^4]\Lambda_5\Lambda_6 + (\beta_5\Lambda_2 g_1 - 3\beta_6\Lambda_1 g_2 - 4\beta_2\beta_5\bar{k}^2 g_1)\Lambda_1\bar{k}^2 + \beta_2\Lambda_1\Lambda_5\bar{k}^2 g_1 - 2\beta_5(1-3\beta_1\bar{k}^2)\Lambda_2\bar{k}^2 g_1 - 3\beta_1\Lambda_2\Lambda_5\bar{k}^2 g_1,$$

$$\alpha_3 \equiv (1-3\beta_1\bar{k}^2)[2(\beta_1\Lambda_2-\beta_2\Lambda_1)\bar{k}^2 - \Lambda_1\Lambda_2]\Lambda_5\Lambda_6 + \Lambda_2\Lambda_3\Lambda_5 g_1 + [(\beta_7-2\beta_5)\Lambda_1\Lambda_2 g_1 - \beta_8\Lambda_1^2 g_2 + 4\beta_5(\beta_1\Lambda_2-\beta_2\Lambda_1)\bar{k}^2 g_1]\bar{k}^2 + \beta_4\Lambda_1\Lambda_5\bar{k}^2 g_1$$

$$\alpha_4 \equiv (g_1 g_2 + \hat{g})/(2g_1 g_2 + \hat{g}), \quad \hat{g} \equiv 2\bar{k}^2[(\beta_2\Lambda_5+\beta_5\Lambda_2)g_1 - (\beta_1\Lambda_6+\beta_6\Lambda_1)g_2],$$
$$\bar{k} \equiv kh, \quad \Lambda_i \equiv 1-\beta_i\bar{k}^2 \ (i=1,8), \quad g_1 \equiv \Lambda_1\Lambda_6, \quad g_2 \equiv \Lambda_2\Lambda_5.$$

From Eqs. (3.14) and (3.15) we have

$$\frac{A_x}{A} = -\alpha^5\frac{h_x}{h} \tag{3.16a}$$

$$\alpha^5 = \frac{\alpha_3-\alpha_2\alpha_4}{\alpha_1} \tag{3.16b}$$

where α^5 is the linear shoaling characteristics of the Eqs. (3.8) and (3.9). Properly choosing the parameters a_2, b_2 and α, we can make α^5 match the Stokes solution α^{s} from shallow water to deep water. And the expression of the Stokes solution α^{s} is

$$\alpha^{\mathrm{s}} = \frac{P}{(1+P)^2}\left[1+\frac{1}{2}P(1-\cosh 2kh)\right] \tag{3.17}$$

where $P = 2kh/\sinh 2kh$

$$\frac{1}{2\pi}\int_0^{2\pi}(\alpha^5-\alpha^s)^2 d(kh) = \min \tag{3.18}$$

Eq. (3.18) means that the mean square error α^5 over the water depth range of $0 < h/L_0 \leq 1.0$ will be minimal. So we can obtain

$$\left(a_2, \frac{b_2}{a}, \alpha\right) = (-0.0220, -0.0191, 0.2386) \tag{3.19}$$

Other parameters are

$$(a_1, a_2, b_1, b_2, c_1, c_2) = (-0.0067, -0.0220, -0.0017, 0.0005, -0.020, 0.0016) \tag{3.20}$$

In fact, c_2 is not an independent parameter. It is determined by $c_2 = b_2/(a+1/3)$. The comparisons of α^5 and the Stokes solution α^s are listed in Table 1. From the table, we can see that α^5 matches α^{s} well from shallow water to deep water. The shoaling property is more accurate over the wide range of $0 < h/L_0 < 1.0$.

4. Conclusions

From the classical Boussinesq equations, we derived two sets of Boussinesq equations. The dispersion of the modified Boussinesq-type equations is accurate to $O(\mu^4)$ and that of further enhancement of the modified Boussinesq equations is accurate to $O(\mu^8)$. The shoaling characteristics of the two different Boussinesq equations are different: the former is applicable to $h/L_0 < 0.476$ and the latter is to $h/L_0 < 1$.

The super and sub harmonic transfer function of the two set Boussinesq equations are different. The latter is more accurate in super harmonic transfer function for $h/L_0 < 0.3$, but the former is more accurate in sub harmonic transfer function for $h/L_0 < 0.5$ with the parameter β being 1. If we choose $\beta = 12.1h/L_0$ for the modified Boussinesq equations, both the super and sub harmonic transfer function will be accurate over wide range of $h/L_0 < 0.5$. The maximum error is 14%. If we choose the parameter $\beta = 12.1h/L_0$ for the Boussinesq equations with six parameters, the super harmonic transfer function will be accurate over wide range of $h/L_0 < 0.5$, and the maximum error is 13%. But the sub harmonic transfer function will be accurate over $h/L_0 < 0.3$, and the maximum error is 20%.

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