Normalized and Equilibrium Spectra of Wind Waves in Lake Michigan

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ABSTRACT

An empirical spectral equation for fetch-limited deep-water wind waves

$$S(\omega)=(0.4g^2/\mathrm{Fo}^{\frac{1}{4}}\omega^5)\exp[-5.5\times10^3(g/U_*\mathrm{Fo}^{\frac{1}{3}}\omega)^4]$$

was derived by applying similarity analysis to wind and wave data recorded at the Lake Michigan Research Tower near Muskegon, Mich., during the autumn of 1967. The field data indicates that both the equilibrium range coefficient β in $S(\omega)=\beta g^2\omega^{-5}$ and the dimensionless peak-frequency parameter, $\omega_m U_*/g$, vary with Fo, where $\mathrm{Fo}=gF/U_*^2$ is the dimensionless fetch parameter with respect to fetch F and friction velocity U_*. The equation produces reasonably good results in estimating actual wave spectra, provided sufficient duration is achieved in the wind field. The equation also indicates that a fully-developed state will not be reached at a steady wind speed as the very low-frequency waves grow continuously with increasing fetch.

1. Introduction

Two major spectral equations for wind waves developed by Bretschneider (1959) and Pierson and Moskowitz (1964) are of the form

$$S(\omega)=p\omega^{-5}\exp(-q\omega^{-4}), \tag{1}$$

where $S(\omega)$ is the frequency spectrum of the water surface displacements at a fixed location with respect to the radial frequency ω, and p and q are dimensional parameters dependent on the wind field. Bretschneider's equation was obtained from an analytic function for the joint probability distribution of wave heights and periods and requires a specification of a height statistic and a period statistic. Pierson and Moskowitz's equation, on the other hand, requires only knowledge of the wind to estimate p and q which was derived for fully-developed seas using the similarity theory of Kitaigorodskii (1962).

In connection with the study of Great Lakes wind waves at the Lake Survey Center, a different scheme of similarity analysis has been employed to formulate an appropriate fetch-limited spectral equation. The data used in this study were recorded at the Lake Michigan Research Tower near Muskegon, Mich. (Fig. 1) during the autumn of 1967. The proposed equation incorporates a simple normalization of the wave spectra with two empirical relations. As these empirical relations include Lake Michigan data as well as laboratory and oceanic observations, the results are believed to be applicable to all not fully-developed, fetch-limited, deep-water wind waves with sufficient wind duration.

2. Data

The measurements of surface waves and wind profiles are essentially the same as those described in an earlier paper (Liu, 1970). The wind speed profile was recorded at anemometer heights of 4, 8, 12 and 14 m above the mean water surface. The friction velocity U_* was determined by fitting the hourly averaged wind data to the logarithms of height. Although the data cover a range of air-water temperature differences of −6 to 6C, the effects of atmospheric statability have not been considered. The logarithmic law appeared to be adequate for describing wind profiles within this range. The deviations between measured data and fitted curves are generally 2% or less.

The wave data were measured with a staff-relay gage and continuously recorded on magnetic tapes. Sample wave records 8–10 min long, selected to coincide with wind data, were reproduced in analog form on strip charts and digitized at a sampling interval of 0.5 sec. Wave spectra were computed from digitized data using the Blackman and Tukey method (1958) with approximately 40 degrees of freedom for each spectral estimate.

3. Analysis and discussion

a. Normalization of spectra

Applications of similarity analysis to correlate wave spectra with generation factors have been used extensively. The analysis establishes the dependence between various normalized nondimensional parameters and thereby leads to a possible universal function. One of many schemes used in studying the similarity characteristics of wind wave spectra is the simple normalization

$$S(\omega)\omega_m/\sigma^2=F(\omega/\omega_m), \tag{2}$$

where ω_m is the frequency at which $S(\omega)$ is a maximum, $F(\omega/\omega_m)$ a dimensionless function that is universal for the spectrum of wave energy, and σ^2 the variance of

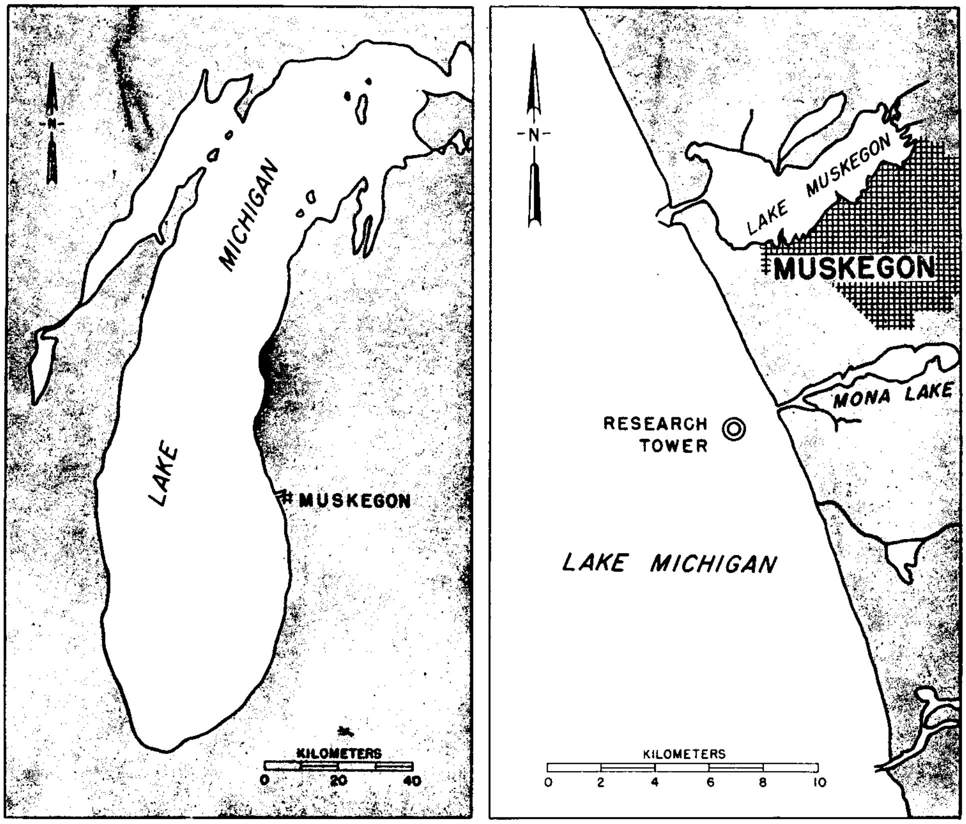


FIG. 1. Location of Lake Michigan Research Tower.

the water surface displacements which is related to the spectral function by

$$\sigma^2=\int_0^\infty S(\omega)d\omega. \quad (3)$$

The normalization scheme (2) is an effective way to demonstrate the similarity characteristics of wave spectra obtained from a wide variety of sea conditions. Hidy and Plate (1966), Colonell (1966), Mitsuyasu (1968), and Kononkova *et al.*, (1970) have all applied this method to their respective wave studies with variable degrees of success. In connection with studies of wind waves in Lake Michigan, Liu (1968, 1970) successfully used (2) to describe both deep and shallow-water wave spectra. Fig. 2 shows examples of the deep water case from the earlier applications. The 11 normalized spectra plotted in this figure clearly display the behavior of similarity.

In the present study, 62 sets of computed wave spectra have been normalized by (2). These wave records are for wind speeds varying from 5–15 m sec^{-1}, including both growing and decaying conditions. In all cases, the normalized spectra bear close resemblance to those plotted in Fig. 2 and indicate that the results can be described by a universal spectral form.

In an effort to seek a quantitative form for the dimensionless function $F(\omega/\omega_m)$, values of $S(\omega)\omega_m/\sigma^2$ were averaged with respect to the corresponding ω/ω_m. The resulting averages and their standard deviations are plotted in Fig. 3. The solid line represents the best-fit equation given by

$$S(\omega)\omega_m/\sigma^2=4.08(\omega/\omega_m)^{-5}\exp[-1.02(\omega/\omega_m)^{-4}]. \quad (4)$$

Eq. (4) is slightly different from the similar equation of an earlier study (Liu, 1970), in which coefficients were inferred through a less precise curve-fitting technique with more limited data.

Comparing the solid line with the data points in Fig. 3 indicates that (4) closely fits the data for the range of dimensionless frequencies ω/ω_m between 0.7 and 5.0. As this range covers the dominant part of a wind wave spectrum, the deviations at the high- and low-frequency ends of the spectrum are generally insignificant. Thus, Eq. (4) gives a suitable representation for the normalized wave spectrum.

While (4) was obtained empirically in the present study, the form of the equation is certainly not original. Bretschneider (1963) proposed the following form of one-dimensional linear wave spectrum:

$$S(\omega)=S(\omega_m)(\omega/\omega_m)^{-m}\exp\{(-m/n)[(\omega/\omega_m)^{-n}-1]\}. \quad (5)$$

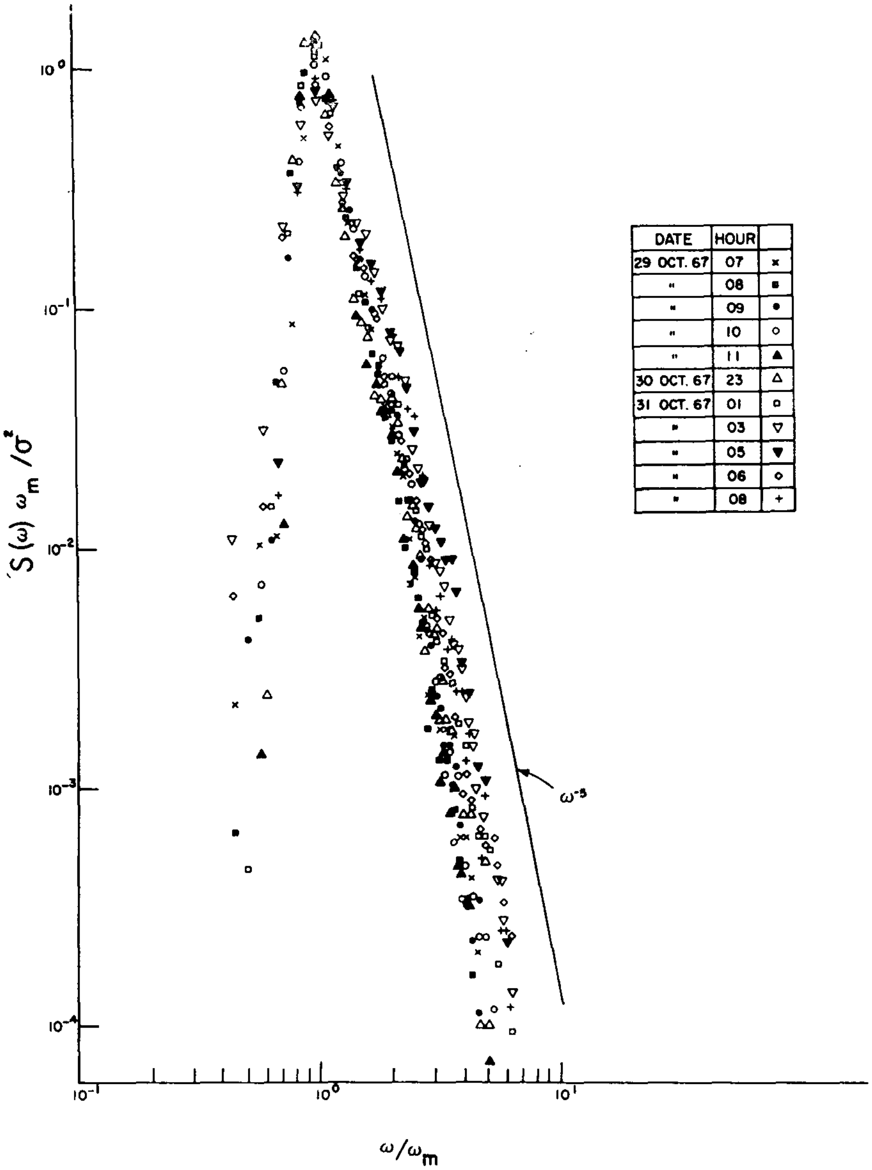


FIG. 2. Examples of normalized wave spectra.

Using (3), the parameter $S(\omega_m)$ can be found as

$$S(\omega_m)=\left\{\frac{n(m/n)^{(m-1)/n}}{\Gamma[(m-1)/n]}\right\}(\sigma^2/\omega_m)\exp(-m/n), \quad (6)$$

where Γ is the gamma function. For $m=5$ and $n=4$, Eq. (5) becomes

$$S(\omega)=5(\sigma^2/\omega_m)(\omega/\omega_m)^{-5}\exp[-1.25(\omega/\omega_m)^{-4}], \quad (7)$$

which is nearly the same as (4). There is insufficient information at present to evaluate differences between the coefficients of the theoretical and empirical equations.

Defining the wave parameters in terms of the statistical moments of the spectrum, Strekalov (1965) also derived the same spectral form as (4) or (5) but which differs from (4) by a scale factor of approximately 3.

b. The equilibrium spectrum

One of several interesting features obtained from study of (4) is the establishment of the "equilibrium range" of a wave spectrum, first derived by Phillips (1958a). Over the equilibrium range where the wave frequencies are greater than ω_m, the spectrum can be represented by

$$S(\omega)=\beta g^2\omega^{-5}, \quad \text{for} \quad \omega>\omega_m, \quad (8)$$

where g denotes the acceleration due to gravity and β a dimensionless coefficient. Earlier workers have con-

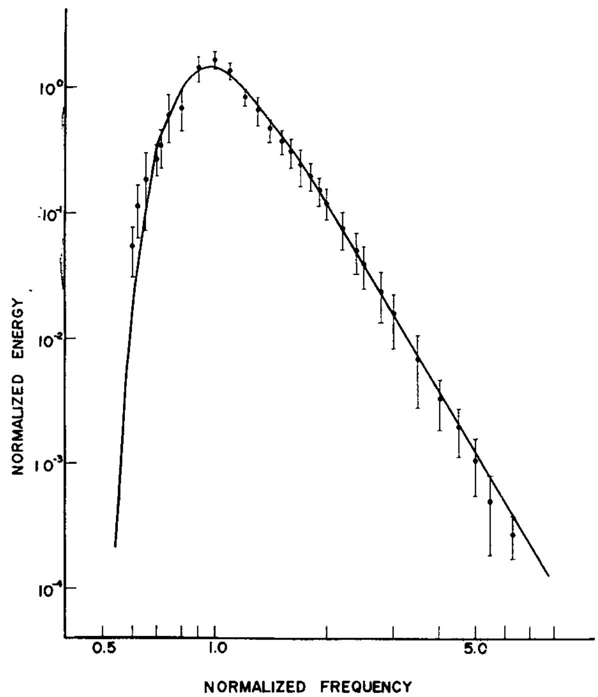


FIG. 3. Averages and standard deviations of 62 sets of normalized wave spectra. The best-fit curve is given by Eq. (4).

sidered β to be an absolute constant, although experimental results differ between investigators. Observing that the values of β decrease with increase of the nondimensional parameter gF/U^2, where F is the fetch and U a representative wind speed, Longuet-Higgins (1969) questioned the constancy of β.

For $\omega \gg \omega_m$, (4) becomes

$$S(\omega)=4.08(\sigma^2\omega_m{}^4/g^2)g^2\omega^{-5}, \tag{9}$$

and comparing (8) and (9)

$$\beta=4.08(\sigma^2\omega_m{}^4/g^2). \tag{10}$$

Thus, β is, in fact, dependent on the variance and peak frequency of the spectrum, and σ and ω_m are in turn dependent on wind speeds as well as fetch and duration.

A corroboration of the dependency of β on the wave generating factors was given by the experimental study of Mitsuyasu (1969). Following the theoretical results of Longuet-Higgins (1969) and defining the dimensionless fetch by $gF/U_*{}^2$, Mitsuyasu deduced the relation

$$1/\beta=21.0\log(gF/U_*{}^2)-34.5. \tag{11}$$

A plot of the present data together with Mitsuyasu's results, however, disagrees with (11). Fig. 4 shows that a better empirical relationship is given by

$$\beta=0.4(gF/U_*{}^2)^{-\frac{1}{4}}. \tag{12}$$

It should be pointed out that only the available deep-water wave data with wind blowing across the lake were plotted in Fig. 4. A few sets of data with offshore wind did not follow the same relationship. The fetch F is given by the distance in the direction of wind from the recording station to the opposite lake shore. While the scattering in the figure may render uncertainty for the exact values of coefficient and exponent in (12), the linear dependence between $\log\beta$ and $\log(gF/U_*{}^2)$ appears to be quite definite.

c. The development of spectral peak frequency

With the high frequency part of the wave spectrum well established, it is also of interest to examine the front face, or the low-frequency part of, the spectrum. The portion of the spectrum when frequencies are less than ω_m can be characterized by the regular shift of the front faces toward lower frequencies as wind speed or fetch or duration increases. As the location of the front face depends on the location of ω_m, a parameter introduced by Kitaigorodskii (1962), $\omega_m U_*/g$, is of particular significance. The correlation of this dimensionless frequency parameter with the dimensionless fetch parameter has been studied by a number of investigators (Volkov, 1968; Mitsuyasu, 1969; Kononkova *et al.*, 1970). Fig. 5 synthesizes these previously published results with the present deep-water data. The solid line that fits the data best is given by

$$\omega_m U_*/g=8.57(gF/U_*{}^2)^{-\frac{1}{3}}. \tag{13}$$

In spite of various methods of measuring and determining the parameters with different accuracies, these

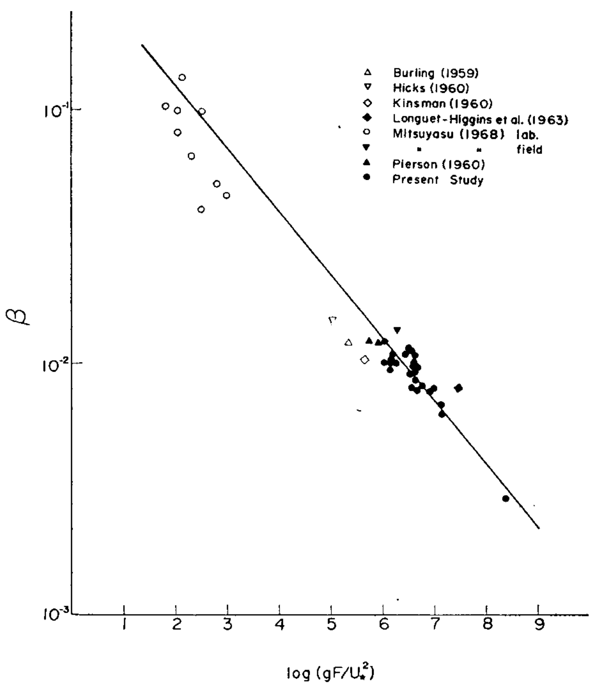


FIG. 4. The equilibrium range constant β vs the dimensionless fetch parameter $gF/U_*{}^2$. Data points of various investigators are from Mitsuyasu (1969).

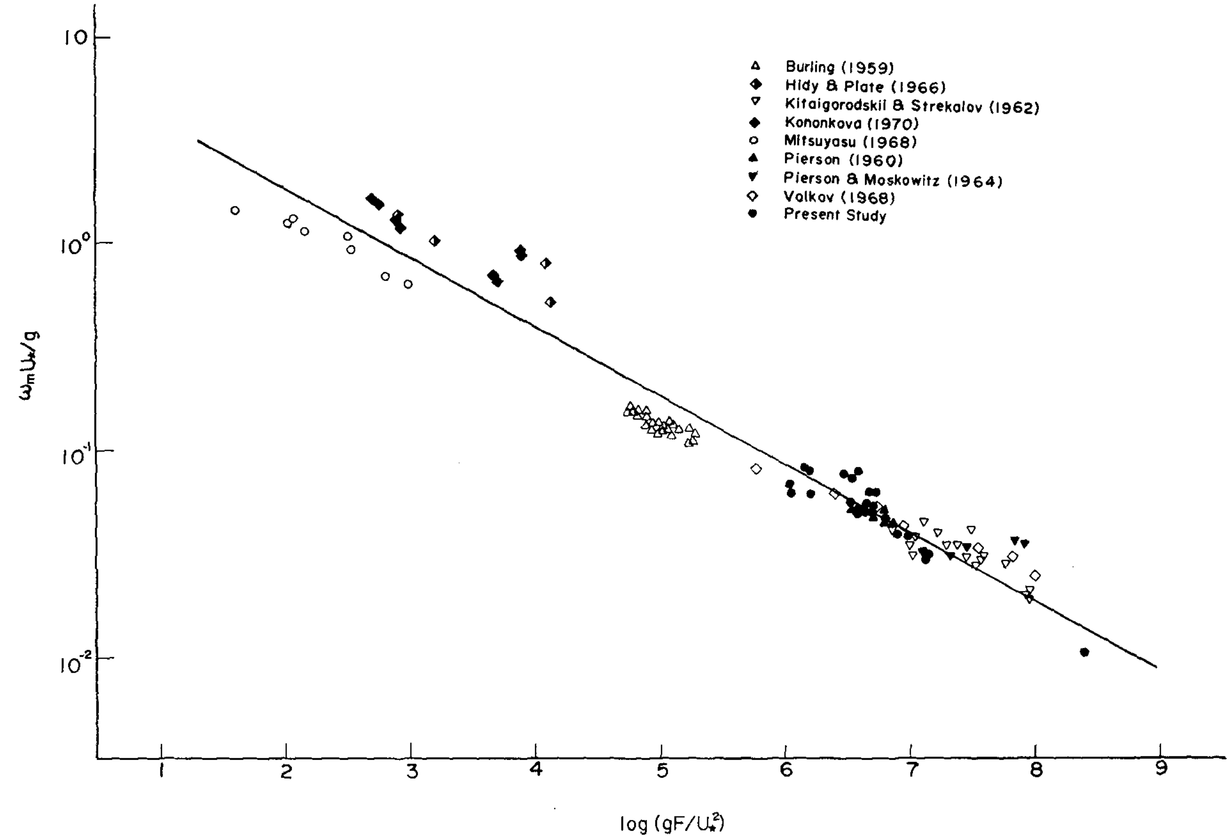


FIG. 5. The dimensionless peak frequency $\omega_m U_*/g$ vs the dimensionless fetch parameter gF/U_*^2. Data points of various investigators are from Volkov (1968), Mitsuyasu (1969) and Kononkova *et al.* (1970).

experimental data are in quite good agreement with each other. The relation given by (13) can therefore be considered as sufficiently universal.

d. The determination of U_*

Effective application of the parameters used in (12) and (13) requires that U_* be known input data. To determine U_* from a given wind speed U_{10} at 10 *m* requires knowledge of the drag coefficient. Many studies have contributed to the establishment of the relationship between the drag coefficient, $C_{10}=(U_*/U_{10})^2$, and U_{10}. However, no universal relation has yet been found. In the present study, following the suggestion of Longuet-Higgins (1969), an attempt has been made to correlate C_{10} with the dimensionless fetch parameter gF/U_*^2. The result, shown in Fig. 6, gives an approximate relation

$$C_{10}=(gF/U_*^2)^{-2/5}. \qquad (14)$$

From (14) and the definition of C_{10}, it follows that

$$U_*=U_{10}(U_{10}^2/gF)^{\frac{1}{8}}. \qquad (15)$$

Corresponding to the deep-water wave data of Figs. 4 and 5, the wind data used in plotting Fig. 6 are derived from wind fields with relatively constant or increasing wind speed blowing over the same direction. The single point at $gF/U_*^2=2.5\times10^8$, which affirms the power law relation, however, is given from a decayed wave spectrum with reduced wind speed.

e. A generalized spectral equation

Combining (10), and (13) with (4), a generalized form can be written as

$$S(\omega)=(ag^2/\mathrm{Fo}^{\frac{1}{4}}\omega^5)\exp[-b(g/U_*\mathrm{Fo}^{\frac{1}{3}}\omega)^4], \qquad (16)$$

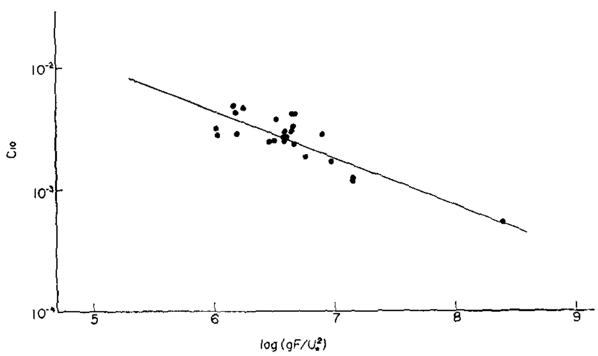
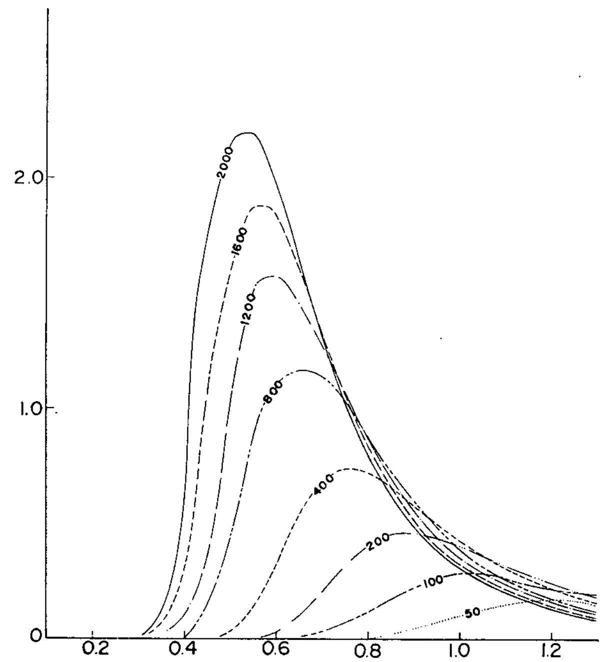


FIG. 6. The drag coefficient C_{10} vs the dimensionless fetch parameter gF/U_*^2.

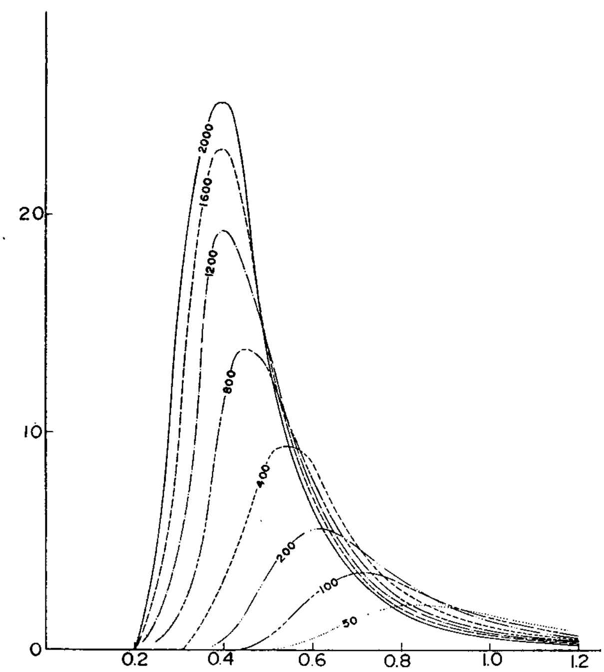
where $a=0.4$, $b=5.5\times10^3$ and $\mathrm{Fo}=gF/U_*^2$. Given U_* and F using any set of consistent units [e.g., $S(\omega)$, $\mathrm{m^2\,sec}$; F, m; U_5, $\mathrm{m\,sec^{-1}}$; ω, $\mathrm{rad\,sec^{-1}}$; $g=9.8\ \mathrm{m\,sec^{-2}}$], a fetch-limited wave spectrum can readily be determined from (16). This equation is quite similar to the form of the Pierson and Moskowitz (1964) equation. However, as Fo does not tend to be constant at any stage, (16) does not asymptotically approach a fully-developed state. This seemingly surprising implication is by no means unexpected, the following discussion by Phillips (1966) being germane:

> "... although the idea of a 'fully aroused sea' is a familiar one in the literature on wave forecasting and may provide an approximate description to certain observations, there seems to be no reason in principle why a truly asymptotic state should exist at all. If the wind continues to blow, the very low-frequency waves will continue to grow, albeit slowly."

An illustration of the spectral growth with respect to fetch (km) computed from (16) for 10 m wind speeds of 10, 20 and 30 m sec^{-1} is shown in Figs. 7a–c. The U_*'s were computed using (15). As a result of the inconstancy of β, there is no envelope for the equilibrium range. The spectra are quite wide at early stages of the development, while they become narrower and the front face steeper as the wind speeds or fetches increase. All of these behaviors agree with the general characteristics of observed spectral growth of wind waves.



a.



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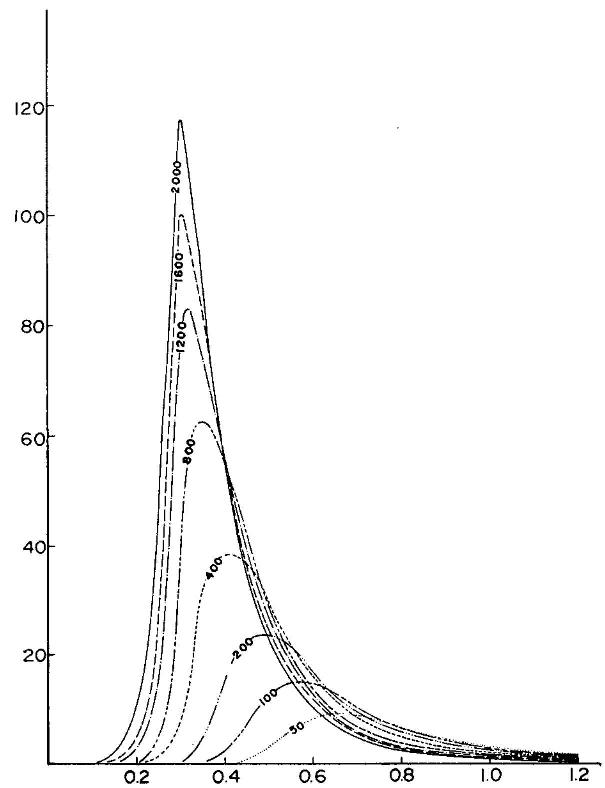

c.

FIG. 7. Spectral growth with respect to fetch (km) as computed from Eq. (15) for 10 m wind speeds of 10, 20 and 30 m sec^{-1}, a.-c., respectively. The ordinate and abscissa (deleted to permit reduction) are energy (m^2 sec) and frequency (rad sec^{-1}), respectively.

A further evaluation of the applicability of (16) is made by comparing computed with actual wave spectra. Figs. 8 and 9 show examples of this comparison. The closed circles are spectral values obtained from original data. The solid lines are spectra computed from (16) with the given U_* and F. Fig. 8 shows an episode of four spectra during the period between 2230 (all times EST) 20 October 1967 and 0330 on 21 October. The agreement between the computed and the original spectra is quite encouraging. With the exception of the last case, the computed spectra are well within 80% confidence limit of the original; the proposed equation has closely predicted the shape and width of the spectra and thus the variance or the area under the spectra. There is some deviation near the peak values but the fits for the front faces and the equilibrium ranges are fairly good. The last spectrum in the group (0330 on 21 October) shows that the proposed equation does not represent situations where the wind undergoes a rapid decay in time, the decay of the spectrum being less rapid.

In the episode of Fig. 8, the winds had blown in the same westerly direction for more than 6 hr prior to 2230 on 20 October. In the episode of Fig. 9, on the other hand, the winds were mostly offshore before 2330 on 24 October and switched over to the southwest with much higher intensity thereafter. A comparison of the computed and actual spectra in Fig. 9 shows that this lack of duration in the given wind field reduces the applicability of (16). In all four cases the proposed equation greatly overestimates the actual wave process.

Although duration is an equally important parameter, it is not included in the present study. A similar duration-limited spectral equation may be obtained from (16) by applying the dynamic equivalence relation $F=Dc/2$, between fetch F and duration D (Phillips, 1958b), where c is the phase speed of waves and $c/2$ the deep water group velocity. However, in dealing with actual data both duration and phase speed are not as clearly defined as are the fetches, and further study on duration-limited cases is therefore omitted.

4. Concluding remarks

A one-dimensional, fetch-limited spectral equation for deep-water wind waves has been developed empirically. The equation yields good results for a given wind field with sufficient duration. As the empirical relations used in deriving this equation were obtained from Lake Michigan data as well as laboratory and oceanic observations, the equation can be used as a general equation for describing deep-water wind waves.

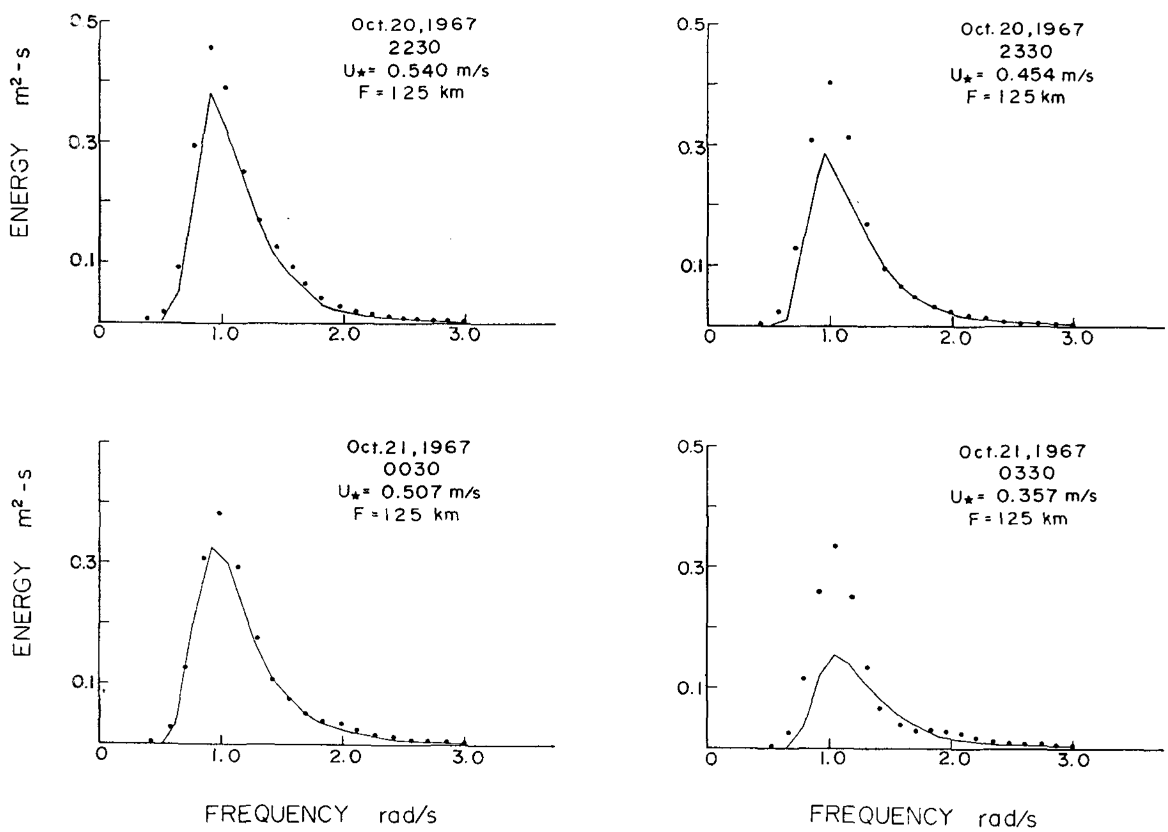

Fig. 8. Comparisons between computed and observed spectra. Closed circles denote spectral values of original data and the solid lines the spectra computed from Eq. (15).

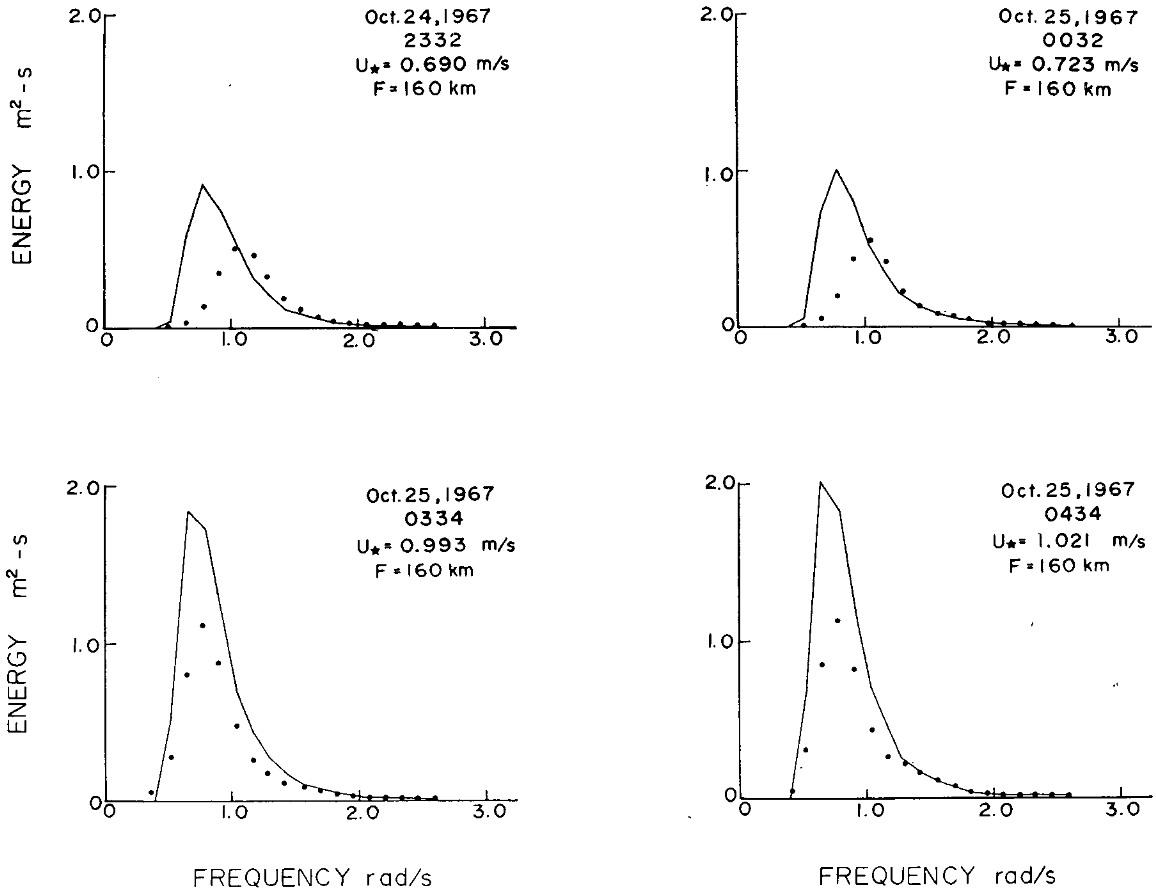

FIG. 9. Same as Fig. 8.

Directionality has not been considered in this study. The Lake Survey Center is planning to measure directional wave spectra to develop a spreading function that can be incorporated into the present equation for more general application.

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