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Reflection and transmission at the ocean/sea-ice boundary

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Abstract

The scattering of water waves by the edge of a semi-infinite ice sheet in a finite depth ocean is solved using the residue calculus technique. We consider both the case where the obliquely incident plane wave is from the open sea region and the complementary problem where the wave is incident from the ice-covered region. Exact solutions to these problems are obtained, equivalent to those that can be obtained if the Wiener–Hopf technique is used. Contrary to popular belief, the solutions are easy to evaluate numerically.

1. Introduction

The effect of a thin sheet of sea ice, modelled as an elastic plate, on the propagation of surface gravity waves in the ocean has been the subject of extensive study. A classic problem is that of a plane wave obliquely incident from an open ocean of constant finite depth on an ice sheet in the form of a half-plane. This problem was solved using the Wiener–Hopf technique by Evans and Davies [1]. In their report Evans and Davies wrote of part of the solution process "Unfortunately, the determination of the constants . . . presents enormous computational difficulties . . . " and ever since their appears to have been a general feeling that the Wiener–Hopf solution to this problem is cumbersome and impractical. Actually this is not the case and numerical computations based directly on Evans and Davies' formulation, incorporating some judicious algebraic simplifications, have been performed in [2].

The Wiener–Hopf technique was also used for the normal incidence and infinite depth version of the above problem in [3] though no numerical results were presented. More recently it has been shown that this problem can be solved by first formulating it as a singular integral equation, and computations of the reflection and transmission coefficients have been reported [4].

In an attempt to overcome some of the perceived drawbacks in Evans and Davies' formulation, Balmforth and Craster [5] presented an analysis which, as well as incorporating a more general model for the ice flexure, aimed to make the numerical calculation of results more straightforward. This involved the evaluation of certain integrals via quadrature which seems to us a backward step since, as shown in [2] and as demonstrated in this paper, the evaluation of the solution from the explicit exact solution is actually straightforward and extremely efficient. Computations

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based on the explicit Wiener–Hopf solution have also been presented recently [6–8], for the case of normal incidence in both finite and infinite water depth.

An alternative to the Wiener–Hopf approach is to use mode-matching. The velocity potentials in the open ocean and ice-covered regions are expanded in appropriate eigenfunctions found through separation of variables and then the two expansions are matched at their common boundary. The first numerical results for the problem tackled by Evans and Davies were obtained in this way in [9] (computations for normal incidence were reported in [10]) though the matching process used (the minimisation of a certain error integral) is somewhat unsatisfactory. A similar approach was taken in [11]. Despite the presence of an exact solution, improved mode-matching analyses continue to be developed, e.g. [12,13].

The aim of this paper is twofold. On the one hand we aim to derive the exact solution to Evans and Davies' problem in a form which will demonstrate that the computation of results from it is a straightforward matter, and on the other we will show that one does not need to use the elaborate Wiener–Hopf machinery to generate it. The method that we use is the so-called residue calculus technique described in [14] and adapted for water-wave problems in [15]. One sets up the problem exactly as in the mode-matching approach, but instead of solving the matrix equation that is generated by the matching numerically, a complex function is constructed in such a way that the unknowns of the problem correspond to the residues at the function's poles. These residues are then easily evaluated. This technique has been used to solve the simpler problem in which the ice sheet is replaced by a rigid dock in [16], where it formed part of the solution to the finite dock problem. We note that approximate reflection and transmission coefficients for a finite ice sheet have been determined in terms of the semi-infinite ones (in the case of normal incidence) in [17].

We begin in Section 2 by formulating the boundary-value problem to be solved and setting up the various depth eigenfunctions that will be used. The particular non-dimensionalisation that we have adopted [18] has the advantage that it allows tank tests to be designed to have flexural responses that are equivalent to field tests, though we will not dwell on these issues here. In Section 3 the classic problem of Evans and Davies is solved using residue calculus theory and then in Section 4 we solve the complementary problem in which a wave is incident from the ice-covered region. In both cases it will be shown that there are substantial simplifications if attention is restricted to normal incidence. Finally, in Section 5, we show how these two scattering problems are related and discuss the computation of the solution.

2. Formulation

The boundary-value problem under consideration can be non-dimensionalised in a variety of different ways. We introduce a characteristic length scale ℓ_c and a characteristic time scale t_c, defined by

$$\ell_c = \left(\frac{D}{\rho g}\right)^{1/4}, \qquad t_c = \left(\frac{\ell_c}{g}\right)^{1/2}. \tag{2.1}$$

Here ρ is the water density, g is the acceleration due to gravity, and D is the effective flexural rigidity of the ice sheet, related to the effective Young's modulus E via $D = Eh^3/12(1-\nu^2)$, h being the thickness of the ice sheet and ν being Poisson's ratio for sea ice (taken to be 0.3). All the variables which appear below have been non-dimensionalised with respect to these quantities. The length scale ℓ_c was shown in [19] to remove all the physical parameters from the governing equations for static flexure of floating ice and in [18] it was shown that it was also appropriate for dynamic responses. One of the consequences of this non-dimensionalisation is that the flexural response of the ice sheet is most significant close to non-dimensional frequency $\omega = 1$.

Cartesian coordinates are chosen so that the undisturbed free surface lies in the (x, y)-plane and z points vertically upwards. The elastic plate covers the region $x > 0$ and if we seek solutions which are time harmonic with angular frequency ω the boundary-value problem we wish to solve is [2]

$$(\nabla_{xz}^2 - l^2)\Phi = 0, \quad -H < z < 0, \tag{2.2}$$

$$\omega^2\Phi = \Phi_z \quad \text{on } z = 0, \ x < 0, \tag{2.3}$$

$$\omega^2\Phi = ((\partial_x^2 - l^2)^2 + 1 - \delta)\Phi_z \quad \text{on } z = 0, \ x > 0, \tag{2.4}$$

$$\Phi_z = 0 \quad \text{on } z = -H. \tag{2.5}$$

Here $\delta = m\omega^2$, where m is the mass per unit area of the ice sheet, non-dimensionalised by $\rho\ell_c$. For this linear boundary-value problem to be valid we must have $m \ll 1$. The conditions at the ice edge, assumed free, are

$$c_2 - \nu l^2 c_0 = 0, \qquad c_3 - \nu_1 l^2 c_1 = 0, \tag{2.6}$$

representing zero bending moment and zero shear stress, respectively, where

$$c_i = \lim_{x\to 0+} \left.\frac{\partial^{i+1}\Phi}{\partial z \partial x^i}\right|_{z=0}, \tag{2.7}$$

and $\nu_1 = 2 - \nu$. To complete the specification of the problem we need appropriate radiation conditions and then the velocity potential is given by $\mathrm{Re}[\Phi(x, z)\,\mathrm{e}^{\mathrm{i}(ly-\omega t)}]$.

We begin by defining an orthogonal set of functions which are the appropriate depth eigenfunctions for the region $x < 0$. Thus

$$\phi_n(z) = N_n^{-1}\cos k_n(z + H), \qquad N_n^2 = \frac{1}{2}H\left(1 + \frac{\sin 2k_n H}{2k_n h}\right), \quad n \ge 0, \tag{2.8}$$

where k_n are the solutions to the water-wave dispersion relation

$$\omega^2 + k_n \tan k_n H = 0. \tag{2.9}$$

Here $k_0 = -\mathrm{i}k$ $(k > 0)$ is purely imaginary and k_n, $n \ge 1$ are real and positive. We note that $k_n H = n\pi - \omega^2 H/(n\pi) + \mathrm{O}(n^{-3})$ and $N_n = \mathrm{O}(1)$ as $n \to \infty$. These depth eigenfunctions form an orthonormal set since

$$\int_{-H}^{0} \phi_n(z)\phi_m(z)\,\mathrm{d}z = \delta_{mn}. \tag{2.10}$$

For the region $x > 0$ we use the following set of functions:

$$\psi_n(z) = M_n^{-1}\cos \kappa_n(z + H), \qquad M_n^2 = \tfrac{1}{2}(H\omega^2 - (5\kappa_n^4 + 1 - \delta)\sin^2\kappa_n H), \quad n \ge -2, \tag{2.11}$$

where κ_n are the solutions to the elastic-plate dispersion relation

$$\omega^2 + (\kappa_n^4 + 1 - \delta)\kappa_n \tan \kappa_n H = 0. \tag{2.12}$$

Here $\kappa_{-2} = s + \mathrm{i}t$ $(s > 0, t > 0)$, $\kappa_{-1} = s - \mathrm{i}t = \bar{\kappa}_{-2}$, $\kappa_0 = -\mathrm{i}\kappa$ $(\kappa > 0)$, and κ_n, $n \ge 1$ are real and positive. We note that $\kappa_n H = n\pi + \mathrm{O}(n^{-5})$ and $M_n = \mathrm{O}(1)$ as $n \to \infty$. These depth eigenfunctions are not orthogonal, but

$$\omega^2 \int_{-H}^{0} \psi_n(z)\psi_m(z)\,\mathrm{d}z = \delta_{mn} + (\kappa_n^2 + \kappa_m^2)\psi_n'(0)\psi_m'(0). \tag{2.13}$$

Which of k and κ is the greater depends on the values of δ and ω^2 in (2.9) and (2.12). For $\delta = 0$ we always have $\kappa < k$ and since $\delta \ll 1$, we will assume in what follows that $\kappa < k$. This puts a lower bound on the values of ω^2 that can be used, but will be true for any parameter values of practical importance.

For future reference we note that

$$\int_{-H}^{0} \phi_n(z)\psi_m(z)\,\mathrm{d}z = \frac{A_n B_m}{\kappa_m^2 - k_n^2}, \tag{2.14}$$

where

$$A_n = N_n^{-1}\cos k_n H, \qquad B_n = M_n^{-1}(\kappa_n \sin \kappa_n H + \omega^2 \cos \kappa_n H), \tag{2.15}$$

both of which are O(1) as $n \to \infty$ and we define

$$\alpha_n = (k_n^2 + l^2)^{1/2}, \qquad \beta_n = (\kappa_n^2 + l^2)^{1/2} \tag{2.16}$$

with the square root always chosen to have positive real part, or, if the real part is zero, negative imaginary part. We have, as $n \to \infty$,

$$\alpha_n = \frac{n\pi}{H}\left(1 + \frac{H^2}{2n^2\pi^2}\left(l^2 - \frac{2\omega^2}{H}\right)\right) + \mathrm{O}(n^{-3}), \qquad \beta_n = \frac{n\pi}{H}\left(1 + \frac{H^2 l^2}{2n^2\pi^2}\right) + \mathrm{O}(n^{-3}). \tag{2.17}$$

3. Wave incident from the ocean

We consider first the diffraction of an incident plane wave making an angle θ_I with the positive x-axis. Such a wave can be represented by the potential $\exp(-\alpha_0 x)\phi_0(z)$, where $l = k\sin\theta_\mathrm{I}$, $\alpha_0 = -\mathrm{i}k\cos\theta_\mathrm{I} = -\mathrm{i}(k^2 - l^2)^{1/2} = -\mathrm{i}\alpha$, say. Clearly we must have $l < k$. The appropriate radiation conditions are

$$\Phi \sim (\mathrm{e}^{-\alpha_0 x} + R\,\mathrm{e}^{\alpha_0 x})\phi_0(z) \quad \text{as } x \to -\infty, \tag{3.1}$$

$$\Phi \sim T\,\mathrm{e}^{-\beta_0 x}\psi_0(z) \quad \text{as } x \to \infty \text{ if } l < \kappa, \tag{3.2}$$

$$\Phi \to 0 \quad \text{as } x \to \infty \text{ if } l > \kappa, \tag{3.3}$$

where R and T are the reflection and transmission coefficients, respectively. Note that with this definition T is not the ratio of the amplitude of the transmitted to the incident wave, though this is easily recovered if we multiply T by $\psi_0'(0)/\phi_0'(0)$. If $l > \kappa$ the incident wave is totally reflected and the wave field in the elastic plate remains localised to the plate edge, whereas if $l < \kappa$ a plane wave is transmitted to infinity through the plate making an angle $\theta_\mathrm{T} = \sin^{-1}(l/\kappa)$ with the positive x-axis and we write $\beta_0 = -\mathrm{i}\kappa\cos\theta_\mathrm{T} = -\mathrm{i}(\kappa^2 - l^2)^{1/2} = -\mathrm{i}\beta$, say.

In $x < 0$ we can expand the potential as an eigenfunction series as follows:

$$\Phi = \mathrm{e}^{-\alpha_0 x}\phi_0(z) + \sum_{n=0}^{\infty} a_n\,\mathrm{e}^{\alpha_n x}\phi_n(z), \tag{3.4}$$

whereas in $x > 0$ we expand Φ as

$$\Phi = \sum_{n=-2}^{\infty} b_n\,\mathrm{e}^{-\beta_n x}\psi_n(z). \tag{3.5}$$

Here a_n, $n \geq 0$, and b_n, $n \geq -2$, are unknown complex coefficients, and $R = a_0$, $T = b_0$.

The continuity of Φ and Φ_x across $x = 0$, the orthogonality of the functions $\phi_n(z)$, Eq. (2.14), and the fact that $\beta_n^2 - \alpha_m^2 = \kappa_n^2 - k_m^2$, can be used to show that

$$\delta_{0m} + a_m = A_m \sum_{n=-2}^{\infty} \frac{b_n B_n}{\beta_n^2 - \alpha_m^2}, \quad m \geq 0, \tag{3.6}$$

$$-\alpha_0\delta_{0m} + \alpha_m a_m = -A_m \sum_{n=-2}^{\infty} \frac{\beta_n b_n B_n}{\beta_n^2 - \alpha_m^2}, \quad m \geq 0, \tag{3.7}$$

from which

$$2\alpha_0 A_0^{-1}\delta_{0m} = \sum_{n=-2}^{\infty} \frac{b_n B_n}{\beta_n - \alpha_m}, \quad m \geq 0, \tag{3.8}$$

$$2\alpha_m a_m A_m^{-1} = -\sum_{n=-2}^{\infty} \frac{b_n B_n}{\beta_n + \alpha_m}, \quad m \geq 0. \tag{3.9}$$

The first of these is an infinite system of equations for the b_n's, the solution of which will involve two arbitrary constants. These constants will be determined by the application of the edge conditions (2.6) and then the a_n's can be found from (3.9).

Consider the function

$$g(z) = \frac{G(z^2 + \gamma_1 z + \gamma_2)}{(z-\beta_{-2})(z-\beta_{-1})(z-\beta_0)} \prod_{n=1}^{\infty} \frac{1 - z/\alpha_n}{1 - z/\beta_n}, \tag{3.10}$$

where G, γ_1, and γ_2 are constants to be determined. The infinite product is uniformly convergent on compact sets excluding the points β_n and it is possible to show that $g(z) = \mathrm{O}(z^{-1})$ as $|z| \to \infty$ provided we avoid these points (see [15, Section 5.2.1], for example). We then consider the numbers

$$I_m = \lim_{N\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{C_N} \frac{g(z)}{z - \alpha_m}\,\mathrm{d}z, \qquad J_m = \lim_{N\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{C_N} \frac{g(z)}{z + \alpha_m}\,\mathrm{d}z, \quad m \geq 0, \tag{3.11}$$

where C_N are contours chosen to avoid the discrete set of points mentioned above and on which $|z| \to \infty$ as $N \to \infty$. The behaviour of g for large z implies that $I_m = J_m = 0$ and then Cauchy's residue theorem gives

$$\delta_{m0} g(\alpha_0) + \sum_{n=-2}^{\infty} \frac{\mathrm{R}(g:\beta_n)}{\beta_n - \alpha_m} = 0, \quad m \geq 0, \tag{3.12}$$

$$g(-\alpha_m) + \sum_{n=-2}^{\infty} \frac{\mathrm{R}(g:\beta_n)}{\beta_n + \alpha_m} = 0, \quad m \geq 0, \tag{3.13}$$

where $\mathrm{R}(g:z_0)$ means the residue of $g(z)$ at $z = z_0$.

Comparison with (3.8) and (3.9) shows that

$$b_n = B_n^{-1}\mathrm{R}(g:\beta_n), \qquad a_n = A_n(2\alpha_n)^{-1} g(-\alpha_n) \tag{3.14}$$

provided G is chosen so that $g(\alpha_0) = -2\alpha_0 A_0^{-1}$, i.e. with

$$g(z) = -\frac{2\alpha_0(z^2 + \gamma_1 z + \gamma_2)}{A_0(\alpha_0^2 + \gamma_1\alpha_0 + \gamma_2)} \tilde{g}(z), \tag{3.15}$$

where

$$\tilde{g}(z) = \prod_{n=-2}^{0} \frac{\alpha_0 - \beta_n}{z - \beta_n} \prod_{n=1}^{\infty} \frac{(1 - z/\alpha_n)(1 - \alpha_0/\beta_n)}{(1 - z/\beta_n)(1 - \alpha_0/\alpha_n)}. \tag{3.16}$$

Both a_n and b_n are $\mathrm{O}(n^{-2})$ as $n \to \infty$. This completes the solution apart from the evaluation of the two constants γ_1 and γ_2 from the edge conditions.

In the region under the plate we now have

$$\Phi = \sum_{n=-2}^{\infty} B_n^{-1}\mathrm{R}(g:\beta_n)\,\mathrm{e}^{-\beta_n x}\psi_n(z), \tag{3.17}$$

and so the numbers c_i which appear in the edge conditions are given by

$$c_i = \sum_{n=-2}^{\infty} B_n^{-1}\mathrm{R}(g:\beta_n)(-\beta_n)^i \psi_n'(0) = \sum_{n=-2}^{\infty} \mathrm{R}(g:\beta_n)\frac{(-\beta_n)^i}{\delta - \kappa_n^4}, \tag{3.18}$$

where (2.11), (2.12) and (2.15) have been used. If we write

$$S_i = \sum_{n=-2}^{\infty} \mathrm{R}(\tilde{g}:\beta_n)\frac{\beta_n^i}{\delta-\kappa_n^4} \tag{3.19}$$

then the conditions (2.6) can be written as a 2×2 matrix equation for the unknowns γ_1 and γ_2:

$$\begin{pmatrix} S_3-\nu l^2 S_1 & S_2-\nu l^2 S_0 \\ S_4-\nu_1 l^2 S_2 & S_3-\nu_1 l^2 S_1 \end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix} \nu l^2 S_2 - S_4 \\ \nu_1 l^2 S_3 - S_5\end{pmatrix}. \tag{3.20}$$

If the quantity β_0 is real, which happens when $l > \kappa$, we can show that $\bar{S}_i = \exp(\mathrm{i}\zeta)S_i$, where ζ is independent of i, from which it follows that γ_1 and γ_2 must both be real.

The reflection coefficient is given, from (3.14), by

$$R = a_0 = -\frac{\alpha_0^2-\gamma_1\alpha_0+\gamma_2}{\alpha_0^2+\gamma_1\alpha_0+\gamma_2}\prod_{n=-2}^{0}\frac{\alpha_0-\beta_n}{-\alpha_0-\beta_n}\prod_{n=1}^{\infty}\frac{(1+\alpha_0/\alpha_n)(1-\alpha_0/\beta_n)}{(1+\alpha_0/\beta_n)(1-\alpha_0/\alpha_n)}, \tag{3.21}$$

$$= \frac{w_\gamma^-(\alpha)}{w_\gamma^+(\alpha)}\frac{(\alpha-\mathrm{i}\beta_0)}{(\alpha+\mathrm{i}\beta_0)}\exp[2\mathrm{i}\chi(\alpha)], \tag{3.22}$$

where

$$w_\gamma^\pm = x^2 \pm \mathrm{i}\gamma_1 x - \gamma_2, \tag{3.23}$$

and, with $\beta_{-2} = \bar{\beta}_{-1} = \sigma + \mathrm{i}\tau$,

$$\chi(x) = \frac{1}{2}\pi + \tan^{-1}\left(\frac{x+\tau}{\sigma}\right) + \tan^{-1}\left(\frac{x-\tau}{\sigma}\right) + \sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{x}{\beta_n}\right) - \tan^{-1}\left(\frac{x}{\alpha_n}\right)\right). \tag{3.24}$$

If $l > \kappa$, the quantity β_0 is real and, since in this case γ_1 are γ_2 are both real, we then have $|R| = 1$ as required.

If $l < \kappa$, the transmission coefficient is given by

$$T = b_0 = \frac{2\alpha(\alpha-\beta)w_\gamma^+(\beta)}{A_0 B_0 w_\gamma^+(\alpha)}P, \tag{3.25}$$

where

$$P = \frac{(\alpha-\mathrm{i}\beta_{-2})(\alpha-\mathrm{i}\beta_{-1})}{(\beta-\mathrm{i}\beta_{-2})(\beta-\mathrm{i}\beta_{-1})}\prod_{n=1}^{\infty}\frac{(1+\mathrm{i}\beta/\alpha_n)(1+\mathrm{i}\alpha/\beta_n)}{(1+\mathrm{i}\beta/\beta_n)(1+\mathrm{i}\alpha/\alpha_n)}. \tag{3.26}$$

The phase of T is then readily determined, since

$$\frac{P}{\bar{P}} = \exp[2\mathrm{i}(\chi(\alpha)-\chi(\beta))], \tag{3.27}$$

where $\chi(\cdot)$ is defined in (3.24). The modulus of T can of course be obtained from (3.25), but it is most easily calculated from (3.22), noting that

$$|R|^2 + \frac{\beta}{\alpha\omega^2}|T|^2 = 1. \tag{3.28}$$

This result, which represents the conservation of energy, can be found by applying Green's theorem to Φ and its complex conjugate which leads to the equation

$$\mathrm{Im}\int \Phi\frac{\partial\bar{\Phi}}{\partial n}\,\mathrm{d}s = 0, \tag{3.29}$$

the integral being taken around the boundary of the fluid domain, including vertical lines at $x = \pm\infty$. The derivation is lengthy, but standard. Note that the relation (3.28) has a much simpler form than the equivalent expression in [1] due to the particular form of the normalisation factors N_n and M_n.

For normal incidence ($l = 0$) there is clearly considerable simplification, and in the case $l = \delta = 0$ it turns out that we can solve the system for γ_1 and γ_2 explicitly. In this case we have

$$S_i = -\sum_{n=-2}^{\infty} \mathrm{R}(\tilde{g} : \kappa_n)\kappa_n^{i-4}. \tag{3.30}$$

For $i = 4, 5$ we can write the sum as an integral

$$S_i = -\int_{\Gamma} \tilde{g}(z) z^{i-4}\,\mathrm{d}z, \tag{3.31}$$

where Γ is a contour from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ indented to pass to the left of the pole at $z = \kappa_0 = -\mathrm{i}\kappa$. The equivalence of the two expressions follows by closing the contour in the right-half plane and noting that $\tilde{g}(z) = \mathrm{O}(z^{-3})$ as $|z| \to \infty$. On the other hand, closing the contour in the left-half plane shows that $S_4 = S_5 = 0$. Thus for normal incidence and $\delta = 0$, we have $\gamma_1 = \gamma_2 = 0$. A similar simplification was noted in [8] for this special case when the Wiener–Hopf technique is used. It then follows that the reflection and transmission coefficients take the forms

$$R = \frac{k-\kappa}{k+\kappa}\exp[2\mathrm{i}\chi(k)], \tag{3.32}$$

$$T = \frac{2k\omega}{k+\kappa}\exp[2\mathrm{i}(\chi(k) - \chi(\kappa))], \tag{3.33}$$

where $\chi(\cdot)$ is defined in (3.24), but with σ, τ, α_n, β_n replaced by s, t, k_n, κ_n, respectively.

4. Wave incident from the ice

We consider next the diffraction of an incident plane wave from the ice region making an angle θ_I with the x-axis. Such a wave can be represented by the potential $\exp(\beta_0 x)\psi_0(z)$, where $l = \kappa \sin\theta_\mathrm{I}$, $\beta_0 = -\mathrm{i}\beta = -\mathrm{i}\kappa\cos\theta_\mathrm{I} = -\mathrm{i}(\kappa^2 - l^2)^{1/2}$. Clearly we must have $l < \kappa$. The potential for this problem will be labelled Ψ and the appropriate radiation conditions are now

$$\Psi \sim \mathcal{T}\mathrm{e}^{\alpha_0 x}\phi_0(z) \quad \text{as} \quad x \to -\infty, \tag{4.1}$$

$$\Psi \sim (\mathrm{e}^{\beta_0 x} + \mathcal{R}\,\mathrm{e}^{-\beta_0 x})\psi_0(z) \quad \text{as} \quad x \to \infty. \tag{4.2}$$

Since $\kappa < k$ there is always a transmitted wave in this case, making an angle $\theta_\mathrm{T} = \sin^{-1}(l/k)$ with the x-axis and we write $\alpha_0 = -\mathrm{i}\alpha = -\mathrm{i}k\cos\theta_\mathrm{T} = -\mathrm{i}(k - l^2)^{1/2}$.

In $x < 0$ we can expand the potential as an eigenfunction series as follows:

$$\Psi = \sum_{n=0}^{\infty} a_n\,\mathrm{e}^{\alpha_n x}\phi_n(z) \tag{4.3}$$

with $\mathcal{T} = a_0$, whereas in $x > 0$ we expand Ψ as

$$\Psi = \mathrm{e}^{\beta_0 x}\psi_0(z) + \sum_{n=-2}^{\infty} b_n\,\mathrm{e}^{-\beta_n x}\psi_n(z) \tag{4.4}$$

with $\mathcal{R} = b_0$. The equations which result from matching across $x = 0$ are

$$0 = \frac{B_0}{\beta_0 + \alpha_m} - \sum_{n=-2}^{\infty} \frac{b_n B_n}{\beta_n - \alpha_m}, \quad m \geq 0, \tag{4.5}$$

$$2\alpha_m a_m A_m^{-1} = \frac{B_0}{\beta_0 - \alpha_m} - \sum_{n=-2}^{\infty} \frac{b_n B_n}{\beta_n + \alpha_m}, \quad m \geq 0. \tag{4.6}$$

To solve (4.5) consider the function

$$f(z) = \frac{F(z^2 + \mu_1 z + \mu_2)}{(z - \beta_{-2})(z - \beta_{-1})(z + \beta_0)} \prod_{n=0}^{\infty} \frac{1 - z/\alpha_n}{1 - z/\beta_n}, \tag{4.7}$$

where F, μ_1, and μ_2 are constants to be determined. This function is of the same form as $g(z)$ defined in (3.10) except that an extra zero (at α_0) and an extra pole (at $-\beta_0$) are included. Applying Cauchy's residue theorem as before, we find

$$\frac{\mathrm{R}(f : -\beta_0)}{-\beta_0 - \alpha_m} + \sum_{n=-2}^{\infty} \frac{\mathrm{R}(f : \beta_n)}{\beta_n - \alpha_m} = 0, \quad m \geq 0, \tag{4.8}$$

$$f(-\alpha_m) + \frac{\mathrm{R}(f : -\beta_0)}{-\beta_0 + \alpha_m} + \sum_{n=-2}^{\infty} \frac{\mathrm{R}(f : \beta_n)}{\beta_n + \alpha_m} = 0, \quad m \geq 0. \tag{4.9}$$

Hence

$$b_n = B_n^{-1}\mathrm{R}(f : \beta_n), \qquad a_n = A_n(2\alpha_n)^{-1} f(-\alpha_n) \tag{4.10}$$

provided F is chosen so that $\mathrm{R}(f : -\beta_0) = B_0$, i.e. with

$$f(z) = \frac{B_0(z^2 + \mu_1 z + \mu_2)}{\beta_0^2 - \mu_1\beta_0 + \mu_2} \tilde{f}(z), \tag{4.11}$$

where

$$\tilde{f}(z) = \frac{(\beta_0 + \beta_{-2})(\beta_0 + \beta_{-1})}{(z - \beta_{-2})(z - \beta_{-1})(z + \beta_0)} \prod_{n=0}^{\infty} \frac{(1 - z/\alpha_n)(1 + \beta_0/\beta_n)}{(1 - z/\beta_n)(1 + \beta_0/\alpha_n)}. \tag{4.12}$$

The matrix equation for the unknowns μ_1 and μ_2 is (3.20) exactly as before, though because of the presence of the incident wave in the ice-covered region we must now define S_i by

$$S_i = \frac{(-\beta_0)^i}{\delta - \kappa_0^4} + \sum_{n=-2}^{\infty} \mathrm{R}(\tilde{f} : \beta_n) \frac{\beta_n^i}{\delta - \kappa_n^4}. \tag{4.13}$$

In the case $l = \delta = 0$ we again have $\mu_1 = \mu_2 = 0$.

The reflection coefficient is now

$$\mathcal{R} = b_0 = \frac{w_\mu^+(\beta)}{w_\mu^-(\beta)} \frac{(\alpha - \beta)}{(\alpha + \beta)} \exp[-2\mathrm{i}\chi(\beta)], \tag{4.14}$$

and the transmission coefficient is given by

$$\mathcal{T} = a_0 = \frac{2A_0 B_0 \beta}{(\alpha - \beta)(\alpha + \beta)^2} \frac{w_\mu^-(\alpha)}{w_\mu^-(\beta)} \bar{P}^{-1}, \tag{4.15}$$

where P is given by (3.26). The modulus is $\mathcal{T}$ is best determined from the conservation of energy equation which is now

$$|\mathcal{R}|^2 + \frac{\alpha\omega^2}{\beta}|\mathcal{T}|^2 = 1. \tag{4.16}$$

For normal incidence and $\delta = 0$ the results are particularly simple:

$$\mathcal{R} = \frac{k-\kappa}{k+\kappa}\exp[-2\mathrm{i}\chi(\kappa)], \tag{4.17}$$

$$\mathcal{T} = \frac{2\kappa}{\omega(k+\kappa)}\exp[2\mathrm{i}(\chi(k)-\chi(\kappa))], \tag{4.18}$$

where $\chi(\cdot)$ is as in (3.32) and (3.33).

5. Discussion

The reflection and transmission coefficients for the two problems solved above are related. If we apply Green's theorem to the two potentials Φ and Ψ we obtain

$$|R| = |\mathcal{R}|, \qquad \beta T = \omega^2\alpha\mathcal{T}, \tag{5.1}$$

and if we use Φ and $\bar{\Psi}$ we find that

$$\Theta_1 + \Theta_2 = 2\Theta \pm \pi, \tag{5.2}$$

where $\arg R = \Theta_1$, $\arg\mathcal{R} = \Theta_2$, and $\arg T = \arg\mathcal{T} = \Theta$. These relations (which are equivalent to those derived using a time-reversal argument in [17]) can be used as checks on the numerical results.

The computation of reflection and transmission coefficients from the formulas given in the preceding sections involves three main steps. These are the evaluation of the roots of the dispersion relations (2.9) and (2.12), the evaluation of the function χ defined in (3.24) and various infinite products, and the evaluation of γ_1, γ_2 and μ_1, μ_2 from (3.20). None of these steps presents any great difficulty.

As far as the dispersion relations are concerned the only possible difficulty is the computation of the complex root κ_{-2} (κ_{-1} is just its complex conjugate) which lies in the positive quadrant. However, a simple application of Newton's method beginning with the naive choice of $1+\mathrm{i}$ appears to work perfectly well.

The terms in the summation in the definition of χ are $\mathrm{O}(n^{-3})$ as $n \to \infty$. This is computationally acceptable, but the series is easily accelerated if we subtract off the leading order asymptotics of the summand. Thus using (2.17) we obtain

$$\sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{x}{\beta_n}\right) - \tan^{-1}\left(\frac{x}{\alpha_n}\right)\right) = -\frac{xh^2\omega^2}{\pi^3}\zeta(3) + \sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{x}{\beta_n}\right) - \tan^{-1}\left(\frac{x}{\alpha_n}\right) + \frac{xh^2\omega^2}{n^3\pi^3}\right), \tag{5.3}$$

in which ζ is the Riemann zeta function and the terms are now $\mathrm{O}(n^{-5})$ as $n \to \infty$. All the infinite products can be accelerated in the same way after first taking their logarithms.

In view of the fact that the qualitative nature of the results for this problem are evident from the many results previously presented, one set of accurate results will suffice. These are presented in Table 1 for easy comparison. The table shows the values of $|R|$, Θ_1, and Θ_2 (from which $|T|$, $|\mathcal{T}|$, and Θ are easily evaluated) for the case when $\delta = 0, \theta_{\mathrm{I}} = 20^{\circ}$, and $h = 0.2\pi$, for values of the non-dimensional frequency between zero and the critical frequency above which there is total reflection.

The analysis presented in this paper shows how the classic problem of wave scattering by a semi-infinite ice sheet in an ocean of finite depth can be solved exactly without having to resort to the Wiener–Hopf technique. Moreover the solution is presented in a form which makes its computation straightforward.

Table 1
Modulus and phase of the reflection coefficients $R = |R|\,e^{i\Theta_1}$ and $\mathcal{R} = |R|\,e^{i\Theta_2}$ when $\delta = 0$, $\theta_I = 20^\circ$, and $h = 0.2\pi$

ω	$\|R\|$	Θ_1	Θ_2
0.2	0.0008	−2.3160	2.3175
0.4	0.0124	−1.3698	1.4164
0.6	0.0463	−0.4354	0.6674
0.8	0.0939	0.3203	0.1670
1.0	0.1452	0.9067	−0.1986
1.2	0.1979	1.3736	−0.5204
1.4	0.2531	1.7591	−0.8494
1.6	0.3154	2.1000	−1.2353
1.8	0.4097	2.4718	−1.7973
1.9	0.5316	2.7290	−2.2884
1.95	0.7321	2.8954	−2.7534
1.96	0.8580	2.9326	−2.9545
1.963	0.9830	2.9439	−3.1208

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