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Exact asymmetric slope distributions in stochastic Gauss–Lagrange ocean waves

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ABSTRACT

The stochastic Lagrange wave model is a realistic alternative to the Gaussian linear wave model, which has been successfully used in ocean engineering for more than half a century. This paper presents exact slope distributions and other characteristic distributions at level crossings for symmetric and asymmetric Lagrange space and time waves. These distributions are given as expectations in a multivariate normal distribution, and they have to be evaluated by simulation or numerical integration. Interesting characteristic variables are: slopes obtained by asynchronous sampling in space or time, slopes in space or time, and horizontal particle velocity, when waves are observed when the water level crosses a predetermined level.

1. Introduction

In the safety analysis of marine vessels and structures the statistical asymmetry properties of irregular ocean waves have drawn considerable interest. The linear Gaussian wave model does not provide any solution, and descriptions based on nonlinear wave equations are hard to calibrate and generalize. However, more or less continuously sampled data from offshore platforms are available, giving empirical distributions of various asymmetry characteristics. An example is Stansell et al. [1], who analyzed 5 Hz sampled sea surface data from a North Sea platform at 130 m water depth, giving front–back steepness distributions for various individual wave heights, showing considerable front–back asymmetry.

Neither the linear nor the second order random wave theories, based on Gaussian elementary waves, allow such asymmetry. The alternative stochastic Lagrange model is a promising model, that agrees with observations and allows theoretical analysis of many of its statistical properties, some of them to be shown in this paper.

A thorough study of irregular, stochastic, Lagrange models was made by Gjøsund [2], and the theory was further developed by Socquet-Juglard et al. [3] and Fouques [4]. Also Woltering and Daemrich [5] studied empirical properties of the stochastic model, based on the previous studies of regular waves.

Systematic theoretical studies of the statistical properties of Lagrange ocean has just begun. In a series of recent papers, [6–10], Lindgren and Åberg have derived expressions for the exact statistical distributions of many individual wave characteristics, including steepness/slope, both for the space formulation (with time frozen), and the time formulation (at fixed location).

In this paper we extend and systematize the theoretical results concerning asymmetry properties of different first order Lagrange model. Emphasis is on front–back asymmetry. We restrict the analysis to unidirectional 2D waves, with one time and one space co-ordinate. The main message in the presentation is that the statistical correlations between the vertical and horizontal water particle movements, uniquely determine the exact skewness and asymmetry distributions in the first order stochastic Lagrange model.

For engineering purposes there are many variables that are of interest, related to upcrossing or downcrossing in space or time. Quantities to be dealt with in this paper are slopes in asynchronous sampling (AS, AT), space slopes in space waves (SS), time slopes (TT), space slopes (ST), and horizontal particle speed (VT), when waves are observed at level crossings in space or time, respectively. All these are quantities important for calculation of wave impact, for example on offshore structures: the TT-case shows how fast the water level will rise, once it has reached a high level, case ST deals with the geometry of the wave when it reaches the high level, and case VT deals with forces that a wave can exert on a marine or offshore structure.

The original results for slope distributions at crossings in Sections 5 and 6 were derived in [6,9,10]. The asynchronous distributions in Section 4 are new.

2. Stochastic Lagrange models

2.1. The Gauss–Lagrange model

A stochastic Lagrange wave is a stochastic version of the Miche waves, the depth dependent modification of the Gerstner waves;

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[11,12]. The Gerstner–Miche model describes the vertical and horizontal movements of individual water particle as functions of time t and original horizontal location, also called the *reference co-ordinate*. In the first order model, elementary components with different frequencies and wave numbers act independently of each other, and their effects are added. We consider here only particles on the free water surface.

The *stochastic 2D Lagrange model* is obtained by letting the vertical and horizontal displacements be correlated random processes with a time parameter t and a space parameter u. The vertical process, which describes the sea surface elevation above the still water level and is denoted $W(t,u)$, is taken to be a Gaussian process with mean zero, as in the linear Gaussian wave model, and so is the horizontal displacement process, denoted $X(t,u)$. Hence, a more complete name is *Gauss–Lagrange model*.

Expressed verbally, in the stochastic 2D Lagrange wave model, a water particle with still water location $(u,0)$ is, at time t, located at position

$$(X(t,u), W(t,u)),$$

and the height of the water surface at location $x = X(t,u)$ is equal to $W(t,u)$. Due to the randomness in horizontal displacement, it may happen that there are many u-values that satisfy $X(t,u) = x$; then the surface height is not uniquely defined. The probability of this *folding* is negligible, for all but very shallow waters.

The Gauss–Lagrange model is completely defined by the auto- and cross-covariance functions

$$\begin{aligned} r^{ww}(t,u) &= \mathsf{Cov}(W(s,x), W(s+t,x+u)) \\ &= \int_0^\infty \cos(\kappa u - \omega t)\, S(\omega)\mathrm{d}\omega, \end{aligned} \tag{1}$$

$$r^{xx}(t,u) = \mathsf{Cov}(X(s,x), X(s+t,x+u)),$$

$$r^{wx}(t,u) = \mathsf{Cov}(W(s,x), X(s+t,x+u)).$$

Here, in (1), $S(\omega)$ is the *orbital spectrum* and wave number $\kappa > 0$ and wave frequency $\omega > 0$ satisfy the depth dependent dispersion relation, $\omega^2 = g\kappa \tanh \kappa h$, with water depth h, and g denoting the gravitational constant.

The general form of the Gauss–Lagrange model is very flexible, and it can be used for many types of random deformation of a well-defined space–time process. In the following section we shall describe some physically motivated models for its covariance structure.

2.2. The free stochastic Lagrange model

The *free Gauss–Lagrange wave model* is the stochastic version of the Miche waves, that was introduced by Gjøsund [2]. In this model, the horizontal displacement process $X_M(t,u)$ is obtained as a linear filtration of the vertical Gaussian process $W(t,u)$ with depth and frequency dependent amplitude and phase response function

$$H_M(\omega) = \mathrm{i}\,\frac{\cosh \kappa h}{\sinh \kappa h}, \tag{2}$$

where the subscript M stands for *Miche filtration*; [4].

Expressed as stochastic Fourier integrals, the relation between the vertical and horizontal processes is

$$W(t,u) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\kappa u - \omega t)}\, \mathrm{d}\zeta(\omega),$$

$$X_M(t,u) = u + \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\kappa u - \omega t)}\, H_M(\omega)\mathrm{d}\zeta(\omega), \tag{3}$$

where the complex Gaussian process $\zeta(\omega)$ has uncorrelated increments such that

$$\mathsf{E}(\mathrm{d}\zeta(\omega)\cdot \overline{\mathrm{d}\zeta(\omega')}) = \begin{cases} 0, & \text{if } \omega \neq \omega', \\ \frac{1}{2} S(|\omega|), & \text{if } \omega = \omega'. \end{cases}$$

The cross-covariance function between the processes is

$$\begin{aligned} r_M^{wx}(t,u) &= \mathsf{Cov}(W(s,x), X_M(s+t,x+u)) \\ &= \int_0^\infty \cos(\kappa u - \omega t + \pi/2)\, \frac{\cosh \kappa h}{\sinh \kappa h}\, S(\omega)\mathrm{d}\omega. \end{aligned}$$

At this point we note that the components X_M and W are dependent Gaussian processes, but that, for fixed t and u, the random variables $X_M(t,u)$ and $W(t,u)$ are independent.

2.3. The Lagrange model with linked components

In the free Lagrange model the individual water particles move unaffected by outer forces. The dependence between vertical and horizontal movements is taken care of by the Miche filtration as described only by the simplified hydrodynamical laws. For wind driven waves this is unrealistic, and one would like to include some external influence in the interaction. However, the interaction between wind and waves is complex, and a stochastic description seems out of reach, at least at present.

A very flexible approach, still within the Gaussian realm, is to replace the response function (2) by a general complex response function,

$$H(\omega) = \rho(\omega)\,\mathrm{e}^{\mathrm{i}\theta(\omega)},$$

leading to the cross-covariance function

$$\begin{aligned} r^{wx}(t,u) &= \mathsf{Cov}(W(s,x), X_M(s+t,x+u)) \\ &= \int_0^\infty \cos(\kappa u - \omega t + \theta(\omega))\, \rho(\omega)\, S(\omega)\mathrm{d}\omega, \end{aligned} \tag{4}$$

and the spectral representation

$$X(t,u) = u + \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\kappa u - \omega t + \theta(\omega))}\, \rho(\omega)\mathrm{d}\zeta(\omega). \tag{5}$$

Comparing (5) with (3) we see that the free Lagrange model represents a phase shift between vertical and horizontal movement of $\theta = \pi/2 = 90°$, while the general model has a frequency dependent phase shift.

One way to formulate a more general relation between the vertical and horizontal processes is to let the horizontal acceleration of the water particles depend linearly on the vertical process, e.g. to take $X(t,u)$ as the solution to an equation

$$\frac{\partial^2 X(t,u)}{\partial t^2} = \frac{\partial^2 X_M(t,u)}{\partial t^2} - \alpha\, W(t,u) - \beta\, \frac{\partial W(t,u)}{\partial u},$$

with $\alpha > 0$, $\beta \geq 0$. With $G(\omega) = -\frac{\alpha + \beta \mathrm{i}\kappa(\omega)}{(-\mathrm{i}\omega)^2}$, the response function will be

$$\begin{aligned} H(\omega) &= H_M(\omega) + G(\omega) \\ &= \mathrm{i}\frac{\cosh \kappa h}{\sinh \kappa h} - \frac{\alpha + \beta \mathrm{i}\kappa}{(-\mathrm{i}\omega)^2} = \rho(\omega)\,\mathrm{e}^{\mathrm{i}\theta(\omega)}. \end{aligned} \tag{6}$$

By adjusting the values for the α- and β-parameters, one can obtain a covariance function (4) close to an observed covariance function. However, the analysis in this paper is not restricted to any particular mode to generate vertical and horizontal correlation; for example, any rational function in κ or ω would have a physical interpretation.

2.4. Time and space waves

The stochastic 2D Lagrange model consists of two dependent Gaussian processes, depending on a time and space parameter (t,u), which describe the vertical and horizontal movements of sea surface water particles. From these two processes, one can then build observable processes, recorded in time or recorded in space.

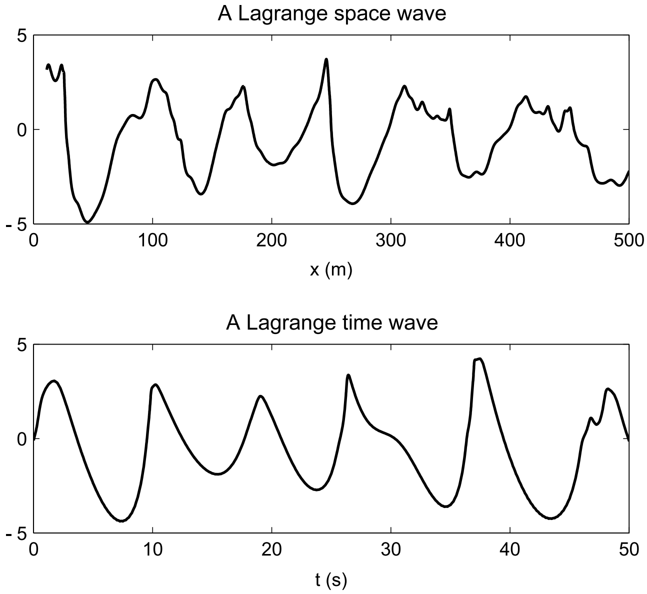

Fig. 1. Exaggerated skewed Lagrange space and time waves from model (6); Jonswap orbital spectrum, water depth $= 8$ m, $\alpha = 0.8$, $\beta = 0$. Note, that in the space waves, the downcrossings at the steep (right) wave front correspond to steep upcrossings in the time wave.

The *space wave* is obtained as the parametric curve

$$u \mapsto (X(t_0, u), W(t_0, u)),$$

by keeping time $t = t_0$ fixed. It resembles the free water surface profile $L(t_0, x)$ along the studied direction. A complication in the model is that there may well occur double points, where $X(t_0, u_1) = X(t_0, u_2)$ with $u_1 \neq u_2$ and $W(t_0, u_1) \neq W(t_0, u_2)$. The space wave is defined implicitly through the relation

$$L(t_0, X(t_0, u)) = W(t_0, u),$$

and explicitly, if there is only one $u = X^{-1}(t_0, x)$ satisfying $X(t_0, u) = x$, as

$$L(t_0, x) = W(t_0, X^{-1}(t_0, x)).$$

The space wave distributions are relevant for the analysis of photos or satellite observations of the sea.

The *time wave* is obtained as measurements of the free water level $L(t, x_0)$ at a fixed location in space with co-ordinate x_0, viz. the curve

$$t \mapsto W(t, X^{-1}(t, x_0)),$$

provided that the inverse $X^{-1}(t, x_0) = \{u; X(t, u) = x_0\}$ is uniquely defined at time t. Then there is only one water particle located at position x_0 at that time. Otherwise the Lagrangian time wave takes multiple values.

Fig. 1 shows exaggerated examples of Lagrange space and time waves for shallow water, $h = 8$ m, and moderately strong coupling in model (6), $\alpha = 0.8$, $\beta = 0$. The orbital spectrum is a Pierson–Moskowitz spectrum. Observe that the space waves, moving from left to right, have a steep right front and a less steep left back, which corresponds to the reversed skewness in the time wave.

The time and space partial derivatives $L_t(t, x) = \partial L(t, x)/\partial t$ and $L_u(t, x) = \partial L(t, x)/\partial u$ will be derived in Section 3.1.

2.5. The example model

We will illustrate the theory on a model with Pierson–Moskowitz (PM) orbital spectrum, with spectral density

$$S(\omega) = \frac{5H_s^2}{\omega_p(\omega/\omega_p)^5}\, \mathrm{e}^{-\frac{5}{4}(\omega/\omega_p)^{-4}}, \quad 0 < \omega < \omega_c,$$

where $H_s = 4\sqrt{\mathrm{Var}(W(t, u))}$ is the *significant wave height* in the linear model, and ω_p is the *peak frequency*, at which the spectral density has its maximum. The *peak period* is defined as $T_p = 2\pi/\omega_p$. We use a fixed value $H_s = 7$ m for the significant wave height, and assume a finite cut off frequency ω_c to obtain finite spectral moments and avoid small but high frequency wave components. The *steepness* parameter, H_s/T_p^2, is important for the degree of front–back asymmetry, and to see the effects we illustrate the distributions for very steep waves with $T_p = 12$ s. The cut off frequency is $\omega_c = 32/T_p$. We do the calculations for four different depths, $h = 8, 32, 64, \infty$ m and three different degrees of linkage, $\alpha = 0, 0.4, 0.8$, where $\alpha = 0$ means "no linkage".

For the examples we simulated samples of the vertical and horizontal displacement fields, $W(t, u), X(t, u)$, and constructed the space and time Lagrange waves. The simulations were made by the Fourier simulation routines in the MATLAB toolbox WAFO, freely available for download at reference [13].

3. Wave characteristics

3.1. Observable characteristics

The Lagrange model gives a stochastic description of the 2D sea surface in time and space. It is completely determined by the orbital spectrum and the transfer function between the vertical and horizontal displacement, and hence by their correlations. There are many wave characteristic distributions, which are important in ocean engineering practice, and they can all be derived exactly from these correlation properties.

The statistical distribution of wave characteristics, such as crest height and wave period, are to be interpreted in a frequentistic way as what one can empirically observe in an infinitely extended, statistically stationary recording of the sea level, either in space or in time. In different applications one can identify many different characteristic quantities related to the wave profiles. Some of them are described by their statistical distribution when the waves are sampled at a constant sampling rate. Others are coupled only to level crossings and the wave profile near instances when the wave reaches some specified level above or below the mean water level. We list six variables of special interest, which we will analyze in the examples:

(AS) Asynchronous slopes in space: This is the distribution of the space slope $L_x(t_0, x)$ observed at equidistant sampling of the space wave, corresponding to slope distribution in a (2D) satellite image.

(AT) Asynchronous slopes in time: This is the distribution of the time slope $L_t(t, x_0)$ observed at constant rate sampling of the time wave, i.e *asynchronous sampling* at a wave station.

(SS) Slope in space at level crossings in space: This is the distribution of the slope $L_x(t_0, x)$ observed only at the upcrossing or downcrossing of a fixed level v by the space wave $L(t_0, x)$, (*synchronous sampling in space*). This describes how the space wave front–back asymmetry depends on the water level.

(TT) Slope in time at level crossings in time: This is the distribution of the slope $L_t(t, x_0)$ observed only at the upcrossing or downcrossing of a fixed level v by the time wave $L(t, x_0)$, (*synchronous sampling in time*). Note that this is the rate of increase of the water level at the times when it has already reached level v at a fixed point.

(ST) Slope in space at level crossings in time: This is the distribution of the space slope $L_x(t, x_0)$ observed at the instances when the time wave reaches level v. Note that this is the slope of the moving wave front that may hit an upper deck of an offshore construction when a high wave reaches the deck.

(VT) Profile velocity at level crossings in time: This is the distribution of the horizontal velocity of the water particles at the times when the wave reaches level v. Note that this is a quantity of interest for the impact on a structure in case of an extreme wave.

3.2. Observable distributions

The observable statistical distributions in the different cases are defined as follows; we take the case (ST) as an example. Consider the Lagrange time wave, $L(t, x_0)$ observed at a location x_0. For a fixed level v, identify all time instances $t_k > 0$ where $L(t, x_0)$ upcrosses the level v, and define the upcrossing counter

$$N(T) = \#\{t_k, 0 < t_k < T; L(t_k, x_0) = v, \text{ upcrossing}\}.$$

Suppose further, that one also observes the slope of the spatial wave profile just at the crossing instance, and define the restricted upcrossing counter

$$N(y; T) = \#\left\{t_k, 0 < t_k < T; \left.\frac{\partial L(t_k, x)}{\partial x}\right|_{x=x_0} \le y\right\}.$$

Taken as a function of y, the ratio $N(y; T)/N(T)$ is the empirical distribution of space slopes at time wave upcrossings of the level v. For an ergodic process this ratio will converge, as $T \to \infty$, to the ratio between the corresponding expectations, $E(N(y; 1))/E(N(1))$, which thus gives the observable statistical distribution.

3.3. Model characteristics

To find the theoretical statistical model distribution of the observable characteristics in cases (AS)–(VT) we need to express the corresponding quantities in terms of the vertical and horizontal processes, $W(t, u)$ and $X(t, u)$. We derive the model variables in terms of the partial time and space derivatives, which we denote as $X_t(t, u) = \partial X(t, u)/\partial t$, $X_u(t, u) = \partial X(t, u)/\partial u$, $W_t(t, u) = \partial W(t, u)/\partial t$, $W_u(t, u) = \partial W(t, u)/\partial u$, etc.

Consider first the time wave. The Lagrange time wave satisfies, by definition, the relation $L(t, X(t, u)) = W(t, u)$. By differentiating with respect to t, we obtain

$$\begin{aligned}\frac{\partial L(t, X(t, u))}{\partial t} &= W_t(t, u) \\ &= L_t(t, X(t, u)) + L_u(t, X(t, u))\, X_t(t, u).\end{aligned}$$

By differentiating with respect to u, we obtain further,

$$W_u(t, u) = L_u(t, X(t, u))\, X_u(t, u),$$

giving the fundamental definition of the time wave slope at location $X(t, u)$,

$$L_t(t, X(t, u)) = W_t(t, u) - W_u(t, u)\,\frac{X_t(t, u)}{X_u(t, u)}. \tag{7}$$

Eq. (7) is a mathematical identity, and if $X^{-1}(t, x_0)$ is uniquely defined it also gives the unique slope of the Lagrange time wave $L(t, x_0)$ at location x_0. If there are multiple u-values such that $X(t, u) = x_0$, then we define $L_t(t, x_0)$ by (7) for each of these u-values.

We also need the slope of the space wave observed at a time t_0. The space wave is implicitly defined by $L(t_0, X(t_0, u)) = W(t_0, u)$, or explicitly, if there is only one $u = X^{-1}(t_0, x)$ satisfying $X(t_0, u) = x$, by $L(t_0, x) = W(t_0, X^{-1}(t_0, x))$. Its slope is,

$$L_x(t_0, x) = \frac{W_u(t_0, u)}{X_u(t_0, u)}, \tag{8}$$

with the understanding that (8) defines the slope for each of the solutions, if there are many.

We will use L_T and L_S as generic notations for the right-hand sides in (7) and (8), respectively.

4. Asynchronous slopes, cases (AS) and (AT)

The asynchronous slopes are obtained by *asynchronous sampling*, for example equidistant in time or in space, and their model representatives are defined by (7) and (8).

To find the slope distributions, in time or space, at a location x_0 at time t_0 one has to identify the reference co-ordinate for the particle that occupies position x_0 at time t_0, i.e. to find the solution, rather solutions, since there may be more than one, to the equation $X(t_0, u) = x_0$. Due to the stationarity in time and space we can take $t_0 = 0, x_0 = 0$.

Now, $X(0, u)$ is a Gaussian process with parameter u, with non-constant mean u, and stationary covariance function $r^{xx}(0, u) = \mathsf{Cov}(X(0, 0), X(t, u))$. It is correlated with the four derivatives in L_T and L_S. Thus, the slope distributions in time and space are equal to the distribution of the representations (7) and (8), respectively, under the condition that u is a point of crossing of the level 0 by the process $X(0, u)$. To reduce the effect of multiple solutions as much as possible, we consider only upcrossings.

The number of solutions to $X(0, u) = 0$ is random, with a small probability of there being more than one, and so are the reference co-ordinates of the solutions. Following the idea outlined in Section 3.2, we define the counters

$$N^+ = \#\{u; X(0, u) = 0, \text{ upcrossing}\}, \tag{9}$$

$$\begin{aligned}N^T(a, b) = \#\Big\{&u; X(0, u) = 0, \text{ upcrossing}, \\ &a < W_t(0, u) - W_u(0, u)\frac{X_t(0, u)}{X_u(0, u)} \le b\Big\},\end{aligned} \tag{10}$$

$$N^S(a, b) = \#\left\{u; X(0, u) = 0, \text{ upcrossing}, a < \frac{W_u(0, u)}{X_u(0, u)} \le b\right\}. \tag{11}$$

Thus, the distribution functions of the asynchronous slopes, L_T, L_S in time and space, are

$$F(L_T \le x) = \frac{\mathsf{E}(N^T(-\infty, x))}{\mathsf{E}(N^+)}, \tag{12}$$

$$F(L_S \le x) = \frac{\mathsf{E}(N^S(-\infty, x))}{\mathsf{E}(N^+)}. \tag{13}$$

This interpretation of the ratio between the expectations has the consequences that an outcome with multiple solutions will produce more than one outcome of the slope. For realistic water depths the model probability of multiple solutions is negligible.

Define the indicator functions

$$I^{(T)}(a, b) = I\left\{a < W_t(0, u) - W_u(0, u)\frac{X_t(0, u)}{X_u(0, u)} \le b\right\}, \tag{14}$$

$$I^{(S)}(a, b) = I\left\{a < \frac{W_u(0, u)}{X_u(0, u)} \le b\right\}, \tag{15}$$

equal to one if the event occurs and zero otherwise. Further, write

Table 1
Estimated skewness and kurtosis excess with standard errors of asynchronous slope distributions, PM orbital spectrum with $H_s = 7$ m, $T_p = 11$ s.

Depth		$\alpha = 0$	$\alpha = 0.4$	$\alpha = 0.8$
$h = \infty$	$A_S =$	0.0 (0.02)	−0.3 (0.02)	−0.6 (0.03)
	$A_T =$	0.0 (0.01)	0.5 (0.01)	0.9 (0.02)
	$B_S =$	0.2 (0.03)	0.4 (0.05)	1.2 (0.14)
	$B_T =$	0.1 (0.02)	0.4 (0.05)	1.4 (0.07)
$h = 64$	$A_S =$	0.0 (0.02)	−0.3 (0.02)	−0.7 (0.03)
	$A_T =$	0.0 (0.01)	0.5 (0.01)	0.9 (0.02)
	$B_S =$	0.1 (0.04)	0.3 (0.05)	1.3 (0.16)
	$B_T =$	0.1 (0.02)	0.5 (0.03)	1.4 (0.09)
$h = 8$	$A_S =$	0.0 (0.02)	−1.1 (0.08)	−2.6 (0.12)
	$A_T =$	0.0 (0.03)	0.8 (0.05)	1.5 (0.08)
	$B_S =$	0.8 (0.08)	5.2 (0.66)	17.2 (1.41)
	$B_T =$	1.1 (0.08)	2.3 (0.26)	5.0 (0.29)

$p_u(0, z)$ for the bi-variate normal density of

$$\mathbf{U}(0, u) = (X(0, u), X_u(0, u)),$$

evaluated at $(0, z)$. For exact expressions for the mean and covariance matrix of $\mathbf{U}(0, u)$, see [9]. Write $\mathsf{E}(X \mid Y = y)$ for the conditional expectation of the random variable X given that another random variable takes the value $Y = y$.

4.1. Distributions of the asynchronous time and space slopes

The observable distributions of the asynchronous time and space slopes in the Gauss–Lagrange model are given by the generalized Rice's formula for marked crossings (see, Leadbetter et al. [14, Ch. 7.5]), as

$$F(L_S \le y) = \frac{1}{\mathsf{E}(N^+)} \int_{u=-\infty}^{\infty} \int_{z=0}^{\infty} z\, p_u(0, z) \times \mathsf{E}\left(I^S(-\infty, y) \mid \mathbf{U}(0, u) = (0, z)\right) \mathrm{d}z\mathrm{d}u, \quad (16)$$

$$F(L_T \le y) = \frac{1}{\mathsf{E}(N^+)} \int_{u=-\infty}^{\infty} \int_{z=0}^{\infty} z\, p_u(0, z) \times \mathsf{E}\left(I^T(-\infty, y) \mid \mathbf{U}(0, u) = (0, z)\right) \mathrm{d}z\mathrm{d}u, \quad (17)$$

with

$$\mathsf{E}(N^+) = \int_{u=-\infty}^{\infty} \int_{z=0}^{\infty} z\, p_u(0, z)\mathrm{d}z\mathrm{d}u.$$

The expectations in (16)–(17) are easily found by conditional simulation of the 3D Gaussian vector $(W_t(t, u), W_u(t, u), X_t(t, u))$ given that $\mathbf{U}(0, u) = (0, z)$, and then one can perform numerical integration over u, z.

Remark 1. In the integrals (16)–(17), the integral over u comes from the random reference co-ordinate, the z-integral takes care of the slope bias that affects the location of the upcrossing, and the conditional expectation restricts the counter to the relevant slope values. The deviation of $\mathsf{E}(N^+)$ from 1 is a measure of the relative frequency of multiple solutions.

Example 1 (*Asynchronous sampling, (AS), (AT)*). We use the Pierson–Moskowitz (PM) orbital spectrum described in Section 2.5 and simulate the vertical and horizontal components. From these we construct space and time waves and observe the slopes. We also compute, by Monte Carlo simulation of the expectations, the exact cumulative distribution functions.

We simulated 50 replicates of space and time waves, with length 2048 m and 2048 s, respectively. Table 1 shows the estimated skewness (with standard error) for three water depths, and three degrees of linking dependence. The skewness (A) and kurtosis excess (B) measures in the observed slopes are used as estimates of the theoretical skewness of the slopes and deviation from normality, (with subscript indicating space or time waves),

$$A = \mathsf{E}(L_{\text{slope}}^3)/\mathsf{Var}(L_{\text{slope}})^{3/2},$$

$$B = \mathsf{E}(L_{\text{slope}}^4)/\mathsf{Var}(L_{\text{slope}})^2 - 3.$$

Figs. 2 and 3 show cumulative space and time slope distribution functions (CDF) for nine combinations of depth and linking. As seen, the skewness is much more severe in space than in time, but both cases show significantly non-Gaussian slope distributions.

5. Asymmetry in space waves, (SS)

From the definition, the distribution of the space wave slope at a crossing of the level v is simply the distribution of the ratio between the two derivatives in (8), conditioned on the event that u_k is a v-crossing point in the vertical Gaussian process $W(t_0, u)$.

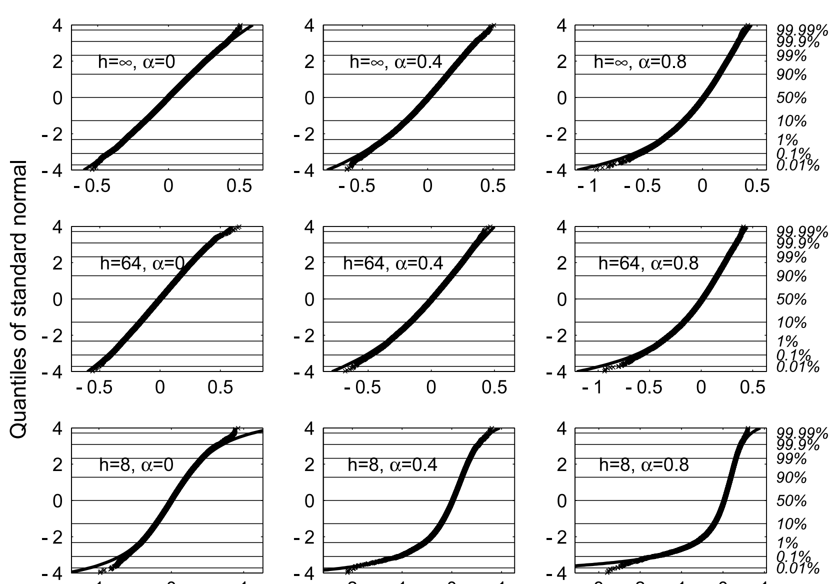

Fig. 2. CDF on normal probability paper for asynchronous slopes in space waves (AS). Crosses = simulated data, solid line = theoretical CDF. The Gaussian model would produce a straight line.

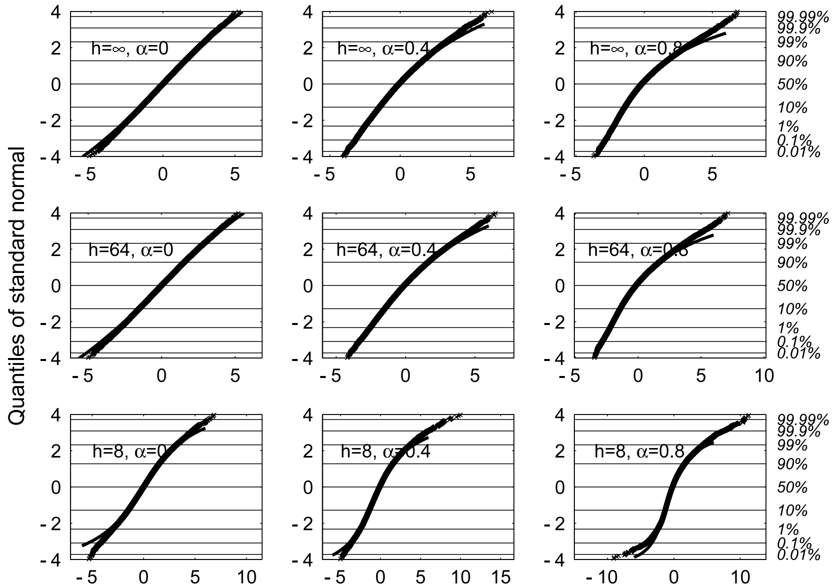

Fig. 3. Similar to Fig. 2 but for time waves (AT).

Now, it is well known that, if the derivative of a Gaussian process is observed only at points of crossing of a fixed level, then the observed derivative values have a two-sided Rayleigh distribution, in contrast to the normal distribution obtained when the derivative is observed without reference to any crossing; cf. Remark 1. Observed only at upcrossings or downcrossings, the derivative has a positive or negative Rayleigh distribution, respectively.

In the following representation of the space derivatives, R and U are two independent random variables, with probability densities, respectively, $f_R(r) = \frac{|r|}{2}\,\mathrm{e}^{-r^2/2}$, and $f_U(u) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-u^2/2}$, i.e. R has a two-sided Rayleigh distribution and U is standard normal. The notation $X \stackrel{\mathcal{L}}{=} Y$ means that the random variables X and Y have the same distribution. Further, $r^{ww} = \mathtt{Var}(W(t,u))$, $r^{wx}_{0u} = \mathtt{Cov}(W(t,u), X_u(t,u))$, etc. The following representation was derived in [6,10].

5.1. Representation of the space slope at crossings

The distribution of the space derivatives in (8), under the condition that $u = u_k$ is a v-crossing point in $W(t_0, u)$, can be expressed as,

$$W_u(t_0, u_k) \stackrel{\mathcal{L}}{=} R\sqrt{r^{ww}_{uu}},$$

$$X_u(t_0, u_k) \stackrel{\mathcal{L}}{=} 1 + v\,\frac{r^{wx}_{0u}}{r^{ww}} + R\frac{r^{wx}_{uu}}{\sqrt{r^{ww}_{uu}}} + U\sqrt{r^{xx}_{uu} - \frac{(r^{wx}_{0u})^2}{r^{ww}} - \frac{(r^{wx}_{uu})^2}{r^{ww}_{uu}}},$$

and hence the slope of the Gauss–Lagrange space wave at a crossing of level v at $x_k = X(t_0, u_k)$, has the representation

$$L_x(t_0, x_k) \stackrel{\mathcal{L}}{=} \frac{R\sqrt{r^{ww}_{uu}}}{1 + v\,\frac{r^{wx}_{0u}}{r^{ww}} + R\frac{r^{wx}_{uu}}{\sqrt{r^{ww}_{uu}}} + U\sqrt{r^{xx}_{uu} - \frac{(r^{wx}_{0u})^2}{r^{ww}} - \frac{(r^{wx}_{uu})^2}{r^{ww}_{uu}}}}. \quad (18)$$

The distributions at an upcrossing or at a downcrossing of the level v is obtained by replacing the two-sided Rayleigh variable by a one-sided, positive and negative, respectively. An explicit formula for the probability density can be found in [10].

As seen in the denominator in (18), the front–back asymmetry depends on the covariance r^{wx}_{uu} between the spatial derivatives of the vertical and horizontal processes. If it is zero the Lagrange slope distribution at an upcrossing is just the mirror of that at a downcrossing. If it is non-zero the Rayleigh distributed slope that appears in the nominator also influences the denominator and makes the Lagrange slope distribution asymmetric.

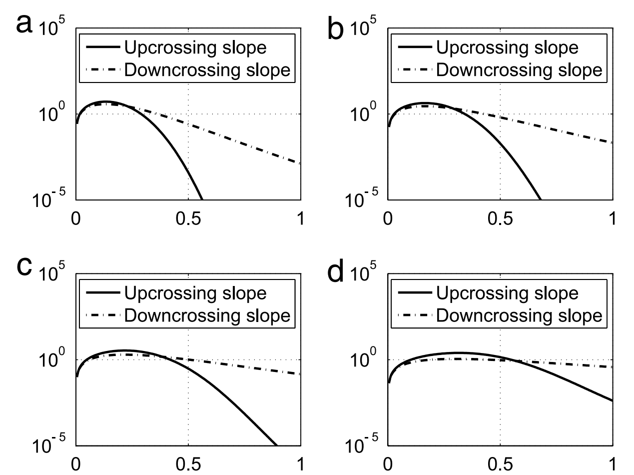

Fig. 4. PDF for space slopes in space waves (SS) at level up- (solid) and down-crossings (dash-dotted). Crossed levels = $[-1, 0, 1, 2] \times \sigma$, $4\sigma = H_s$ in (a)–(d). Infinite water depth, $\alpha = 0.4$.

Example 2 (*Space Slopes in Space Waves (SS)*). We use the orbital spectrum (PM) described in Section 2.5 and calculate the probability density function (PDF) of the space slopes of the space waves according to the representation (18). The results are shown in Figs. 4 and 5 for infinite and shallow water depth ($h = 32$ m). The linkage parameter is set to $\alpha = 0.4$.

6. Asymmetry in time waves

The time wave level crossings at a fixed position x_0 are more complicated than the space wave crossings, since one has to continuously follow the variations, both vertically and horizontally, of the random particle that happens to be located at x_0 as time changes. Thus, a crossing of the level v occurs at time t_k

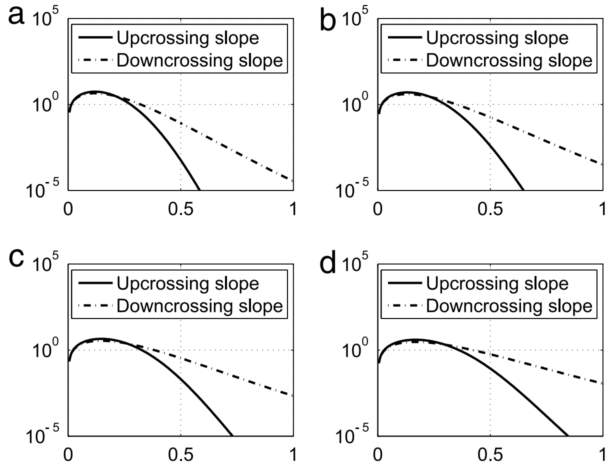

Fig. 5. Similar as Fig. 4, but water depth = 32 m.

if there is a particle with reference co-ordinate, which we denote by u_k, such that $W(t_k, u_k) = v$ and $X(t_k, u_k) = x_0$.

To solve problems (TT), (ST), (VT), we have to find the conditional distribution of the time slope, defined by (7), the space slope, defined by (8), and the horizontal velocity $X_t(t_k, u_k)$, conditioned on the crossing event, just defined.

By a remarkable generalization of Rice's formula for the number of level crossings, Mercardier [15] has given the tool for how to find conditional distributions like the ones we seek; see also [16, Theorem 6.4]. Mercardier's theorem was used, in the symmetric case by Åberg [6], and in the asymmetric case by Lindgren [9], to prove the following representations of the distributions in (TT), (ST), (VT).

To formulate the distributions, we define

$$D = |W_t(0,u)X_u(0,u) - W_u(0,u)X_t(0,u)|\,,$$

and, in addition to (14)–(15), the indicator function

$$I^{(V)}(a,b) = I\,\{a < X_t(0,u) \le b\}\,. \tag{19}$$

Further, write $q_u(v, x)$ for the bi-variate normal density of the variables

$\mathbf{V}(0,u) = (W(0,u), X(0,u))$,

evaluated at (v, x).

6.1. Representations of slopes at time wave crossings

(a) The cumulative long run observable distribution function for slopes at upcrossings of the level v in the Gauss–Lagrange time wave, is given by,

$$F_v^{TT+}(y) = \frac{1}{\mathsf{E}(N^+)} \int_{-\infty}^{\infty} g_y^{TT+}(u)\, q_u(v, x_0)\mathrm{d}u,$$

where

$$g_y^{TT+}(u) = \mathsf{E}(D \times I^{(T)}(0,y) \mid \mathbf{Z}(0,u) = (v, x_0)), \tag{20}$$

and

$$\mathsf{E}(N^+) = \int_{-\infty}^{\infty} g_{\infty}^{TT+}(u)\, q_{W(0,u),X(0,u)}(v, x_0)\mathrm{d}u. \tag{21}$$

(b) The cumulative distribution function of slopes at downcrossings is obtained by replacing the indicator function $I^{(T)}(0, y)$ by $I^{(T)}(-y, 0)$, properly adjusting the $\le$ sign.

(c) The distributions for space slope and velocity, (ST) and (VT), are found by replacing (14) by (15) and (19), respectively, in (20).

The conditional expectations in (20) and (21) have to be found by simulation of the 4D conditional normal variables, followed by straightforward numerical integration over u.

Example 3 (*Slopes in Time Waves (TT), (ST)*). Since the time waves are most easily registered, it is of some importance to see how one can obtain, not only time wave slopes, but also slopes in space observed or reconstructed from time wave recordings, in particular at level crossings. Both the time slope and the space slope distributions at time wave crossings can be found by the same technique, as well as that for many other variables.

A common technique to relate time registrations to space properties is the Fourier snapshot method, described in [17]. The

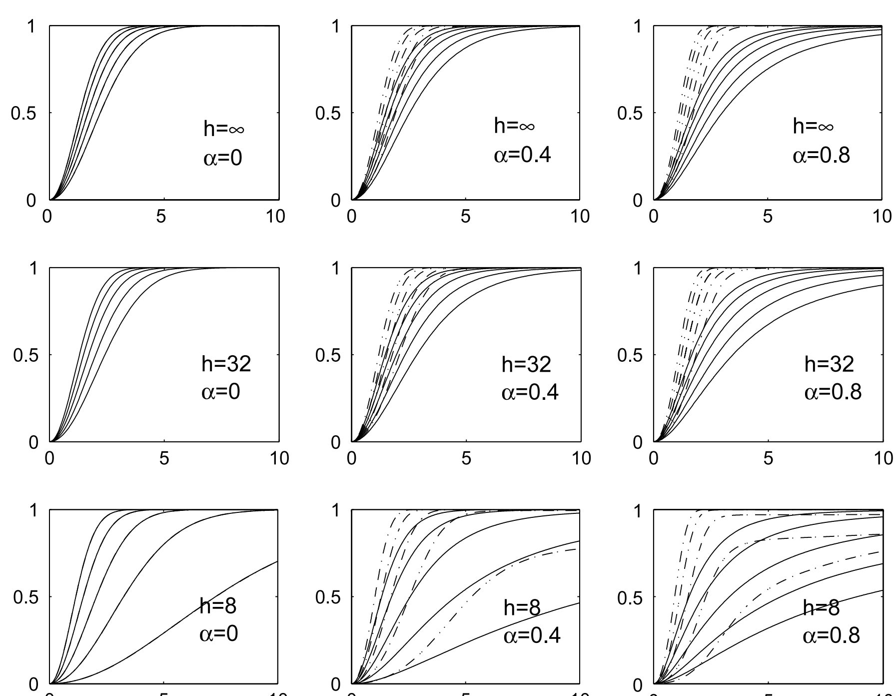

Fig. 6. CDF for time wave slopes (absolute values) at time wave crossings (TT) of different levels. Slope CDF at upcrossings (solid lines) and at downcrossings (dash-dotted lines). Levels v: $[-1, 0, 1, 2, 3] \times \sigma$, $4\sigma = H_s$. Largest absolute values correspond to highest level.

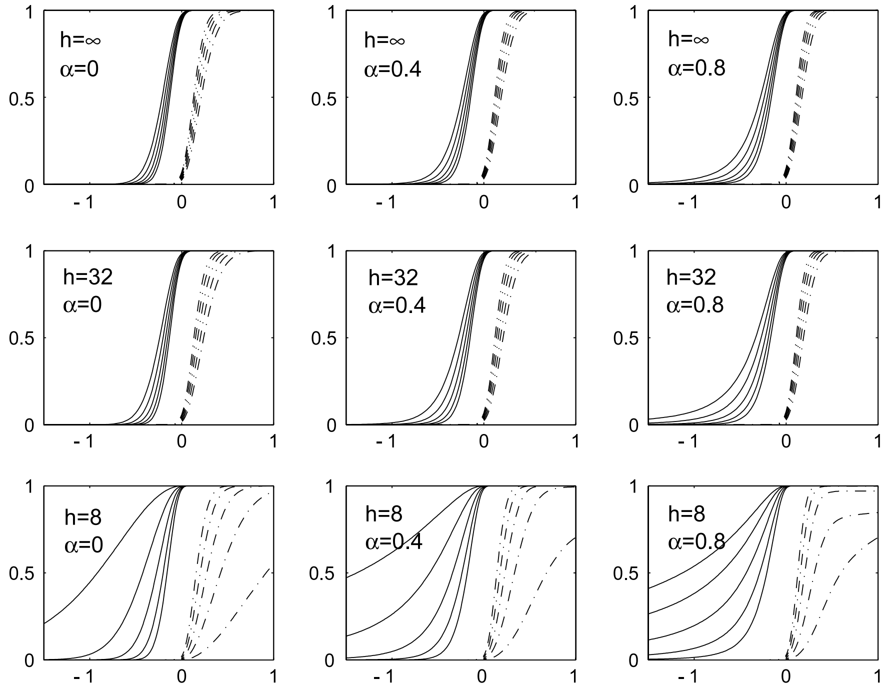

Fig. 7. CDF for space wave slopes at time wave crossings (ST) of different levels. Slope CDF at upcrossings (solid lines) and at downcrossings (dash-dotted lines). Levels v: $[-1, 0, 1, 2, 3] \times \sigma$, $4\sigma = H_s$. Most extreme values correspond to highest level.

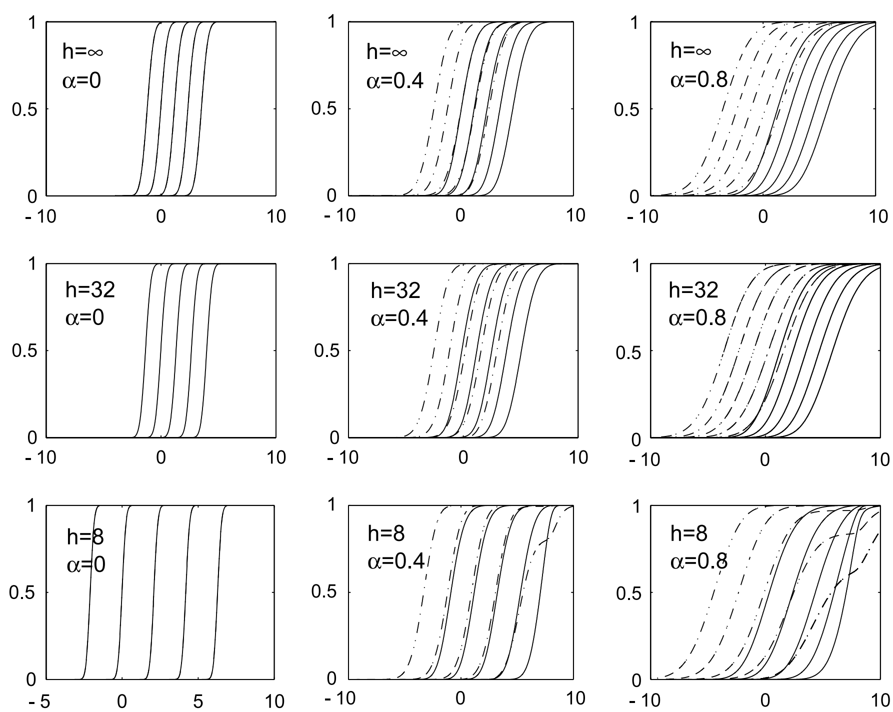

Fig. 8. CDF for horizontal velocity (m/s) at time wave crossings (VT) of different levels; at upcrossings (solid lines) and at downcrossings (dash-dotted lines). Levels v: $[-1, 0, 1, 2, 3] \times \sigma$, $4\sigma = H_s$. Largest positive velocities correspond to highest level.

space–time analysis presented in this paper might be an adequate method to handle this problem for genuinely asymmetric waves.

We illustrate the technique on the PM-example. Figs. 6 and 7 show the cumulative distribution functions for the slope distributions in time and space, observed at the level crossings in time.

Example 4 (*Velocity in Time Waves (VT)*). The final example shows the horizontal velocity distributions at occasions of time wave upcrossing and downcrossing of a level. The results for the PM-example are shown in Fig. 8. For realistic water depths the horizontal velocity distributions are obviously asymmetric, but have a rather normal distribution regardless of the level height.

7. Concluding remarks

The first order Lagrange model is basically a linear model in the sense that different frequency components are just superposed, and do not interact with each other in the (Gaussian) horizontal and vertical movements. It is the combination of the two that produces the non-Gaussian properties of the sea surface.

The first order Lagrange 2D wave model with linked components combines most of the advantages in the Gaussian wave model with some empirical realism. It allows easy simulation and exact calculation of many important wave characteristics, in particular those related to wave asymmetry, as presented in this paper. The input to the asymmetry distributions is the set of covariances between the synoptic horizontal and vertical displacements and their time and space derivatives, and since these can be measured in real or numerically obtained waves, one can calculate slope distributions without having access to the full energy spectrum. This will allow comparison both with real wave data, provided they are detailed enough to give reliable estimates of the covariance structure, and with numerically obtained data, either based on nonlinear wave equations or on higher order Fourier methods.

Slope analysis in the 3D model is more complicated than in the 2D model, since slope is directional dependent, even if the basic form of the first order model is the same. The generalization of the linking transfer function (6) may depend on direction, which makes comparison between the theory and real or numerical wave data more complicated. Thus, the next step in an analysis of the stochastic Gauss–Lagrange model, is the comparison of the 3D model with a numerical model, both without any linkage, and with a slightly restricted link equation.

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