A New Coastal Wave Model. Part III: Nonlinear Wave–Wave Interaction

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ABSTRACT

Resonant wave–wave interaction processes are studied with the nonlinear dispersion relationship for shallow water. The formulation was derived based on a Hamiltonian representation first reported by Zakharov. Results show that four waves are needed for resonant interactions at all depths. Furthermore, when the nonlinear dispersion relationship for waves in intermediate water is considered, two interaction modes can result depending on the water depth and the nonlinearity: (i) in deep water the classic Phillips interactions dominate, involving four waves of comparable wavelengths, whereas (ii) in shallow water the dominant interactions still consist of four waves, but with one component of vanishingly small wavenumber. As an approximate asymptotic limit, the latter become triadic shoaling wave interactions.

1. Introduction

The seminal paper by Phillips (1960) on weak nonlinear wave–wave interactions can rightfully be regarded as the beginning of modern water wave theory. In contrast to strong nonlinear waves where harmonic distortions dominate, weak wave–wave interactions involve slow, albeit persistent, energy exchanges among four interacting free wave components. Although the interaction process conserves energy, Phillips suggested correctly that the effects of these slow weak interactions, integrated through time, could drastically alter the sea state. Phillips's idea was confirmed and extended by Hasselmann (1962), who formulated the weak wave–wave interactions in terms of a wave spectrum. At first glance, it might seem paradoxical that a conservative wave–wave interaction process can play a dominating role in the wind wave generation and evolution process. The explanation, however, is simple: The key effect of the weak nonlinear wave–wave interactions is to cause the peak frequency of the spectrum to downshift. As the peak frequency downshifts, the waves become longer. The lengthening of the waves is a necessary condition for wave energy to grow. Otherwise, the waves will be too steep to be stable. Hasselmann's classic formulation correctly modeled this frequency downshift process. As a result, this formulation was the foundation for understanding wind wave generation and evolution processes for the last three decades.

With the introduction of weakly nonlinear wave–wave interactions, wave forecast methodologies have become much more sophisticated and mature. Some of the models (WAM, for example) have even been successfully implemented as operational tools. The success of this implementation can be attributed to the practical application of Hasselmann's formulation, as reviewed recently by Young and Van Vledder (1993) and Komen et al. (1994).

Successful as modern wave models are, there are some unsettling problems concerning the way in which weak nonlinear interactions are implemented. The difficulties are both theoretical and practical. On the theoretical side, we can list three difficulties. First, though the weak nonlinear wave–wave interactions are third-order events, the resonance condition is based on the linear dispersion relationship. The effects of third-order amplitude dispersion have been studied by McLean et al. (1981) and McLean (1982a, b) These studies show that there are, indeed, small but definite finite amplitude effects even in deep water waves. However, using the linear dispersion relation for a third-order wave–wave interaction study is an inconsistency. Even if the effects in deep water are small, whether they are also small in water of finite depth has never been explored thoroughly.

Second, though the weak nonlinear wave–wave interactions are shown to be among four free waves, Fre-

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lich and Guza (1984) found that the resonance conditions can also be satisfied by interactions of three free waves in shoaling waters where waves are no longer dispersive. If this were indeed true, what should be the crossover depth between the four-wave interactions and the three-wave interactions?

Finally, nonlinear wave–wave interactions as formulated by Hasselmann are based on the resonant interaction mechanism proposed by Phillips (1960). Although the Phillips-type resonant interactions are the most important ones for small amplitude waves, McLean et al. (1981) and McLean (1982a,b) have shown that as wave amplitude increases, there are two distinct types of interactions. The Phillips mechanism is predominately two-dimensional, while the other is predominately three-dimensional. Three-dimensional interactions lead to three-dimensional instabilities, which could cause energy spreading in directions other than the main wave propagation direction. Three-dimensional instability has indeed been observed by Su et al. (1982) and Su and Green (1984). However, no application has ever been made of this relatively recent discovery in wave modeling.

On the practical side, there are also many difficulties. Though weak nonlinear wave–wave interactions, as formulated by Hasselmann (1962), have been applied to wave modeling successfully, other alternatives have not been explored at all. The Hasselmann formulation was obtained through straightforward perturbation analysis up to the fifth order. Straightforward as the perturbation method is, there is no guarantee that the answer obtained will indeed converge accurately. Furthermore, it is well known (see, e.g., Kevorkian and Cole 1981) that for the study of weakly nonlinear oscillations, the two-time expansion method can offer a more accurate answer. The two-time expansion scheme and a Hamiltonian representation has been tried by Zakharov (1968, 1991) and Crawford et al. (1980) to obtain the nonlinear energy transfer. The result is much simpler than the Hasselmann formulation, for it is in terms of energy already. Theoretically, Hasselmann's and Zakharov's results are identical, as shown by Dyachenko and Lvov (1997) recently. However, being simpler algebraically, Zakharov's formulation should be computationally less time consuming. Furthermore, the Zakharov equation extends the results to a larger range of wave steepness, as shown by Crawford et al. (1981).

The computation time involved in evaluating the nonlinear wave–wave interactions is the real practical obstacle in any implementation. Although Hasselmann's formulation has been evaluated exactly by Hasselmann and Hasselmann (1985), the computer time required renders it impractical for anything other than pure research. Moreover, excluding the nonlinear source term will deprive the model of an important physical mechanism. Faced with this dilemma, various approximations have been introduced in wave models, as discussed by Young and Van Vledder (1993). However, on closer examination, we suggest that the present approximate solutions do not make physical sense; while the algorithms making physical sense are not practical. We will present a solution to alleviate this impasse in this study and the accompanying paper by Lin and Perrie (1997).

In this paper, we will concentrate on the theoretical side of the problem. Section 2 presents the derivation of the nonlinear dispersion relation for finite depth. Section 3 presents the nonlinear energy transfer rate for finite depth with nonlinear dispersion. Section 4 discusses the effect of nonlinear dispersion on the resonant trajectory and the energy transfer rate. The problem of evaluating the predominately three-dimensional interactions will be discussed in a separate paper. The practical aspect of implementing the nonlinear wave–wave interaction mechanism as a source function in wave modeling is presented in Lin and Perrie (1997).

2. Nonlinear dispersion

To solve the n-order resonant interaction problem consistently, we have to use the n-order dispersion relationship, in which the relationship between wave frequency and wavenumber should also be wave amplitude and water depth dependent (see, e.g., Whitham 1974).

a. Basic equations

The basic equation and boundary conditions for a potential incompressible flow are the following:

1) Continuity equation

$$\nabla_H^2\Phi + \frac{\partial^2\Phi}{\partial Z^2} = 0, \quad \text{for } -h \le Z \le \eta, \tag{1}$$

where

$$\mathbf{\nabla}_H = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}.$$

2) Boundary conditions

A. At the free surface: $Z = \eta$

a. Kinematics:

$$\frac{\partial \eta}{\partial t} + \mathbf{\nabla}_H\Phi \cdot \mathbf{\nabla}_H\eta = \frac{\partial \Phi}{\partial Z}, \tag{2}$$

b. Dynamics:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\mathbf{\nabla}_H\Phi)^2 + \frac{1}{2}\left(\frac{\partial\Phi}{\partial Z}\right)^2 + g\eta = 0. \tag{3}$$

B. At the bottom:

$$\frac{\partial \Phi}{\partial Z} + \nabla_H h \cdot \nabla_H \Phi = 0. \tag{4}$$

In the above equations t, H, h, g, η, and Φ are time, the horizontal coordinate, the depth of the ocean, the gravitational acceleration, the free surface elevation, and the potential function.

We define ω as the frequency, and expanding all functions as power series in ϵ, which is a small parameter, we obtain

$$\Phi = \epsilon\Phi_1 + \epsilon^2\Phi_2 + \cdots$$
$$\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots$$
$$\omega = \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \tag{5}$$

This leads to a series of systems of equations, at each order in the expansion.

b. Linear analysis: First order

Substituting the perturbation expansions in Eq. (5), the linear terms in Eqs. (2) and (3) imply

$$\frac{\partial^2 \Phi_1}{\partial t^2} + g\frac{\partial \Phi_1}{\partial Z} = 0. \tag{6a}$$

The solution to Eq. (6a) and corresponding linearizations of Eqs. (1)–(4) is

$$\Phi_1 = A_1 \exp\{i(\omega t + \mathbf{K}\cdot\mathbf{r})\}\cosh\left[|K|(Z + h)\right]$$
$$\eta_1 = a \exp\{i(\omega t + \mathbf{K}\cdot\mathbf{r})\}, \tag{6b}$$

where K is the wavenumber, and $K = |\mathbf{K}|$

$$A_1 = \frac{iag}{\omega \cosh Kh},$$

and a is the wave amplitude.

Substituting (6b) into (6a), we obtain the linear dispersion relationship

$$\omega_1^2 = gK \tanh(Kh). \tag{7}$$

c. Nonlinear analysis

1) SECOND ORDER

The second order in the perturbation expansions of Eqs. (2) and (3) at $z = \eta$ gives

$$\frac{\partial \eta_2}{\partial t} + \nabla_H\Phi_1\cdot\nabla_H\eta_1 = 0, \tag{8}$$

$$\frac{\partial \Phi_2}{\partial t} + \frac{1}{2}(\nabla_H\Phi_1)^2 + g\eta_2 = 0. \tag{9}$$

Combining Eqs. (8) and $\partial(9)/\partial t$ and substituting Eq. (6), we obtain

$$\frac{\partial^2 \Phi_2}{\partial t^2} = i2\omega K^2 A_1^2 \cosh^2 Kh \exp\{i2(\omega t + \mathbf{K}\cdot\mathbf{r})\}; \tag{10}$$

which implies that η_2 and Φ_2 assume the form

$$\eta_2 = a_2 \exp\{i2(\omega t + \mathbf{K}\cdot\mathbf{r})\}$$
$$\Phi_2 = A_2 \cosh 2Kh \exp\{i2(\omega t + \mathbf{K}\cdot\mathbf{r})\}, \tag{11}$$

and furthermore, we have

$$a_2 = a^2K\frac{3 + \tanh^2 Kh}{4\tanh^3 Kh}, \qquad A_2 = \frac{3aK}{8\sinh^3 Kh}A_1. \tag{12a}$$

The second-order nonlinear dispersion term ω_2 is equal to zero. Given Eq. (12a), the obvious choice for the expansion coefficient ϵ is the ratio of a_2 and a:

$$\epsilon = \frac{a_2}{a} = aK\frac{3 + \tanh^2 Kh}{4\tanh^3 Kh}. \tag{12b}$$

The small parameter ε has to be smaller than a certain critical value in order for small perturbative theory to be valid with reasonable accuracy. We call this critical value the upper limit of the nonlinearity, $\gamma = \varepsilon$. For deep water, γ is simply the critical wave slope. According to Crawford et al. (1981), an approximation to the nonlinear action transfer can still be considered reliable so long as its discrepancy with respect to the true nonlinear action transfer is less than 10%. As a measure of the accuracy of the approximation, we can examine the dimensionless perturbation wavenumber for the most unstable mode. Crawford et al. (1981) showed that for deep water, the discrepancy is 10% between the numerical results obtained from the exact water wave equations by Longuet-Higgins (1978) and (i) the approximation of Zakharov (1968) when γ reaches 0.3, whereas (ii) for the approximation of Hasselmann (1962), γ need only reach 0.06. Therefore, we can trust the results derived here up to $\gamma = 0.3$. Crawford et al. (1981) also showed the trend toward destabilization of the entire system for sufficiently large γ (at about $\gamma = 0.5$). This instability feature qualitatively agrees with the numerical results obtained by Longuet-Higgins (1978).

2) HIGH ORDER

Rearranging Eqs. (2) and (3) at $z = \eta$, we have

$$\frac{\partial \eta_n}{\partial t} + \nabla_H\Phi_{n-1}\cdot\nabla_H\eta_1 + \cdots + \nabla_H\Phi_1\cdot\nabla_H\eta_{n-1} = 0, \tag{13}$$

$$\frac{\partial \Phi_n}{\partial t} + \frac{1}{2}(\nabla_H\Phi_{n-1}\cdot\nabla_H\Phi_1 + \cdots + \nabla_H\Phi_1\cdot\nabla_H\Phi_{n-1}) + g\eta_n = 0. \tag{14}$$

where $n = 3,4,5, \cdots$. In order to obtain the nonlinear dispersion term, we may assume the free surface elevation has the following form:

$$\Phi_n = A_n \cosh nKh[\exp\{in(\omega t + \mathbf{K}\cdot\mathbf{r})\} - \exp\{-in(\omega t + \mathbf{K}\cdot\mathbf{r})\}]$$

$$\eta_n = a_n[\exp\{in(\omega t + \mathbf{K}\cdot\mathbf{r})\} - \exp\{-in(\omega t + \mathbf{K}\cdot\mathbf{r})\}], \qquad (15a)$$

and substitute Eqs. (6b), (7), (11), (12a), and (15a) into Eqs. (13) and (14) at $z = \eta$. After some algebra, we finally obtain the expression:

$$\omega_3 = \omega_1\left(\frac{9 - 10\tanh^2 Kh + 9\tanh^4 Kh}{8\tanh^4 Kh}\right)K^2a^2,$$

$$\omega_4 = 0. \qquad (15b)$$

A similar expression was obtained by Whitham (1974). When the water depth becomes large, Eq. (15b) reduces to the deep water Stokes expression exactly.

3. Nonlinear energy transfer rate

As mentioned in the introduction, Hasselmann (1962) obtained the nonlinear energy transfer rate for finite depth by using the perturbation method. Zakharov (1968, 1991) obtained the nonlinear energy transfer rate for deep water by using the Hamiltonian representation. Both methods only considered linear dispersion. Because the second method is simpler in algebra and therefore less time consuming (orders of magnitude less) in computation, we decided to adopt Zakharov's approach. We will extend the analysis to finite depth of water. In the derivation, we also include the nonlinear dispersion relationship.

Following Zakharov, we define

$$\psi_{(\mathbf{r},t)} = \Phi_{(\mathbf{r},z,t)}\big|_{z=\eta}$$

$$\frac{\partial\psi}{\partial t} = \frac{\partial\Phi}{\partial t} + \frac{\partial\eta}{\partial t}\frac{\partial\Phi}{\partial z}\bigg|_{z=\eta}. \qquad (16)$$

We assume Kh_o to be constant in one wavelength and $\varepsilon^2 h_o > \delta h$. Then from Eq. (7), we will have ω_1 as a function only of wavenumber K and water depth h. If we also limit the wave amplitude ah_o, as a constant in one wavelength, then from the Eq. (15b), we will also have ω_n as a function of K and h. Consequently, to third-order approximation, the frequency should also be truncated at third order. With these general assumptions we are able to make the Fourier representation:

$$\eta_{(\mathbf{r},t)} = \frac{1}{2\pi\sqrt{2}}\int_{-\infty}^{\infty}\frac{|K\tanh Kh|^{1/2}}{\omega_{(K)}^{1/2}}[b_{(\mathbf{K},t)}\exp\{i(\mathbf{K}\cdot\mathbf{r})\} + b^*_{(\mathbf{K},t)} \times \exp\{-i(\mathbf{K}\cdot\mathbf{r})\}]\,d\mathbf{K}; \qquad (17a)$$

$$\psi_{(\mathbf{r},t)} = -\frac{i}{2\pi\sqrt{2}}\int_{-\infty}^{\infty}\frac{\omega_{(K)}^{1/2}}{|K\tanh Kh|^{1/2}}[b_{(\mathbf{K},t)}\exp\{i(\mathbf{K}\cdot\mathbf{r})\} - b^*_{(\mathbf{K},t)} \times \exp\{-i(\mathbf{K}\cdot\mathbf{r})\}]\,d\mathbf{K}. \qquad (17b)$$

Following Zakharov (1968) and Yuen and Lake (1982), we define a complex variable,

$$b(\mathbf{K}, t) = \left(\frac{\omega}{2|\mathbf{K}|}\right)^{1/2}\hat{\eta}(\mathbf{K}, t) + i\left(\frac{|\mathbf{K}|}{2\omega}\right)^{1/2}\Phi(\mathbf{K}, t),$$

where $\hat{\eta}$ and $\hat{\Phi}$ are the Fourier coefficients of $\mathbf{K}$ for $\eta(\mathbf{r}, t)$ and $\hat{\Phi}$ ($\mathbf{r}$, $z = \eta$, t), We also define $b(\mathbf{K}, t) = [\varepsilon B\mathbf{K}, t) + \varepsilon^2 B'(\mathbf{K}, t)] \exp(—iwt)$. Instead of dividing the slow motion and fast motion, we use the solvability condition and adjoint operator, maping the n-dimensional system into an $n - 1$ dimensional system in order to obtain the resonant modes.

a. Second order

Recalling Eqs. (1), (4), (8), and (9),

$$\nabla_H^2\Phi_2 + \frac{\partial^2\Phi_2}{\partial Z^2} = 0, \quad \text{for } -h \le Z \le \eta$$

$$\frac{\partial\Phi_2}{\partial Z} + \boldsymbol{\nabla}_H h\cdot\boldsymbol{\nabla}_H\Phi_2 = 0, \quad \text{for } Z = -h$$

$$\frac{\partial\eta_2}{\partial t} + \boldsymbol{\nabla}_H\Phi_1\cdot\boldsymbol{\nabla}_H\eta_1 = 0, \quad \text{at } Z = \eta$$

$$\frac{\partial\Phi_2}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}_H\Phi_1)^2 + g\eta_2, \qquad \text{at } Z = \eta. \qquad (18)$$

We apply Fourier transform and introduce the two-time scales into Eq. (18):

$$\begin{aligned} i\frac{\partial B'_{(\mathbf{K},t)}}{\partial t} = \iint_{-\infty}^{\infty} [&V^{(-)}_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2)}B_{(\mathbf{K}_1)}B_{(\mathbf{K}_2)}\delta(\mathbf{K} - \mathbf{K}_1 - \mathbf{K}_2)\exp\{i(\omega_{(\mathbf{k})} - \omega_{(\mathbf{k}_1)} - \omega_{(\mathbf{k}_2)})t\} \\ &+ 2V^{(-)}_{(\mathbf{K}_1,\mathbf{K},\mathbf{K}_2)}B_{(\mathbf{K}_1)}B^*_{(\mathbf{K}_2)}\delta(\mathbf{K} - \mathbf{K}_1 + \mathbf{K}_2)\exp\{i(\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)})t\} \\ &+ V^{(+)}_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2)}B^*_{(\mathbf{K}_1)}B^*_{(\mathbf{K}_2)}\delta(\mathbf{K} + \mathbf{K}_1 + \mathbf{K}_2)\exp\{i(\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)})t\}]\,d\mathbf{K}_1\,d\mathbf{K}_2. \end{aligned} \qquad (19)$$

Since $B_{(\mathbf{K}_1)}$ and $B_{(\mathbf{K}_2)}$ are not functions of t', we can obtain $B'_{(\mathbf{K},t)}$ by integrating Eq. (19) with respect to t:

$$B'_{(\mathbf{K},t)} = \iint_{-\infty}^{\infty} \Bigg[V^{(-)}_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2)} B_{(\mathbf{K}_1)} B_{(\mathbf{K}_2)} \delta(\mathbf{K} - \mathbf{K}_1 - \mathbf{K}_2) \frac{\exp\{i(\omega_{(\mathbf{K})} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)})t\}}{\omega_{(\mathbf{K})} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}}$$
$$+ 2V^{(-)}_{(\mathbf{K}_1,\mathbf{K},\mathbf{K}_2)} B_{(\mathbf{K}_1)} B^*_{(\mathbf{K}_2)} \delta(\mathbf{K} - \mathbf{K}_1 + \mathbf{K}_2) \frac{\exp\{i(\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)})t\}}{\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)}}$$
$$+ V^{(+)}_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2)} B^*_{(\mathbf{K}_1)} B^*_{(\mathbf{K}_2)} \delta(\mathbf{K} + \mathbf{K}_1 + \mathbf{K}_2) \frac{\exp\{i(\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)})t\}}{\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)}} \Bigg] d\mathbf{K}_1\, d\mathbf{K}_2, \quad (20)$$

where $V^{(-)}$ and $V^{(+)}$ are functions of wave amplitude, wavenumber, and water depth. The detailed form of the coefficients $V^{(-)}$, $V^{(+)}$ is given in the appendix. These results are slightly different from those given by Zakharov (1968) and Crawford et al. (1980) because of the finite depth assumption and the nonlinear dispersion relationship.

b. *Third order*

We must apply the following solvability condition:

$$\left\langle \begin{pmatrix} \Phi^* \\ \eta^* \end{pmatrix} xL \begin{pmatrix} \Phi \\ \eta \end{pmatrix} - \begin{pmatrix} \Phi \\ \eta \end{pmatrix} xL^* \begin{pmatrix} \Phi^* \\ \eta^* \end{pmatrix} \right\rangle = 0, \quad (21)$$

to eliminate the divergent term, where

$$L = \begin{pmatrix} \nabla^2 & 0 \\ -\partial/\partial Z & \partial/\partial t \\ \partial/\partial t & g\eta \\ \partial/\partial Z & 0 \end{pmatrix} \quad (22a)$$

and

$$L^* = \begin{pmatrix} \nabla^2 & 0 \\ -\partial/\partial Z & -\partial/\partial t \\ -\partial/\partial t & g\eta \\ \partial/\partial Z & 0 \end{pmatrix}. \quad (22b)$$

The solvability condition in this case is therefore

$$\left\langle \Phi^*, \frac{\partial \Phi^{(1)}}{\partial t} \right\rangle + \left\langle \eta^*, \frac{\partial \eta^{(1)}}{\partial t} \right\rangle$$
$$= \left\langle \Phi^*, \left[-(\boldsymbol{\nabla}_H \Phi^{(2)}) \cdot (\boldsymbol{\nabla}_H \Phi^{(1)}) - \left(\frac{\partial \Phi^{(2)}}{\partial Z}\right)\left(\frac{\partial \Phi^{(1)}}{\partial Z}\right) \right] \right\rangle$$
$$+ \langle \eta^*, [-\boldsymbol{\nabla}_H \Phi^{(2)} \cdot \boldsymbol{\nabla}_H \eta^{(1)} - \boldsymbol{\nabla}_H \Phi^{(1)}) \cdot \boldsymbol{\nabla}_H \eta^{(2)}] \rangle. \quad (23)$$

Once more, we apply the Fourier transform of Eqs. (17a) and (17b), introduce the two-time scales into Eq. (23) and obtain

$$i\frac{\partial B_{(\mathbf{K},t)}}{\partial t} = -\iint_{-\infty}^{\infty} [V^{(-)}_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2)} (B_{(\mathbf{K}_1)} B'_{(\mathbf{K}_2)} + B_{(\mathbf{K}_2)} B'^*_{(\mathbf{K}_1)}) \delta(\mathbf{K} - \mathbf{K}_1 - \mathbf{K}_2) \exp\{i(\omega_{(\mathbf{K})} - \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)})t\}$$
$$+ 2V^{(-)}_{(\mathbf{K}_1,\mathbf{K},\mathbf{K}_2)} (B_{(\mathbf{K}_1)} B'^*_{(\mathbf{K}_2)} + B_{(\mathbf{K}_2)} B'^*_{(\mathbf{K}_1)}) \delta(\mathbf{K} - \mathbf{K}_1 + \mathbf{K}_2) \exp\{i(\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)})t\}$$
$$+ V^{(+)}_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2)} (B^*_{(\mathbf{K}_1)} B'^*_{(\mathbf{K}_2)} + B^*_{(\mathbf{K}_2)} B'^*_{(\mathbf{K}_1)}) \delta(\mathbf{K} + \mathbf{K}_1 + \mathbf{K}_2) \exp\{i(\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)})t\}]\, d\mathbf{K}_1\, d\mathbf{K}_2$$
$$+ \iiint_{-\infty}^{\infty} W_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} B^*_{(\mathbf{K}_1)} B_{(\mathbf{K}_2)} B_{(\mathbf{K}_3)} \delta(\mathbf{K} + \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3)$$
$$\times \exp\{i(\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)})t\}\, d\mathbf{K}_1\, d\mathbf{K}_2\, d\mathbf{K}_3, \quad (24)$$

where W is a function of amplitude, wavenumber, and water depth. The detailed form of the coefficients W is given in the appendix. Their forms are also slightly different from those given by Zakharov (1968) and Crawford et al. (1980) because of the finite-depth assumption and the nonlinear dispersion relationship, $\omega = \varepsilon\omega_1 + \varepsilon^3\omega_3 + \varepsilon^5\, \omega_5 + \cdots$.

By substituting Eq. (20) into Eq. (24), we will obtain the Zakharov equation to third order for B,

$$i\frac{\partial B}{\partial t}\int\int\int_{-\infty}^{\infty} T_{o,1,2,3}B_1^*B_2B_3\delta(\mathbf{K}+\mathbf{K}_1-\mathbf{K}_2-\mathbf{K}_3)\exp\{i[\omega_{(\mathbf{K})}+\omega_{(\mathbf{K}_1)}-\omega_{(\mathbf{K}_2)}-\omega_{(\mathbf{K}_3)}]t\}\, d\mathbf{K}_1\, d\mathbf{K}_2\, d\mathbf{K}_3. \quad (25)$$

The detailed form of the function T is given in the appendix. Following Crawford et al. (1980), the spectral action transfer rate for a homogeneous wave field is given by

$$\frac{\partial AC_{(\mathbf{K}_i)}}{\partial t} = 4\pi\int\int\int_{-\infty}^{\infty} T_{i,1,2,3}^2\delta(\mathbf{K}_i+\mathbf{K}_1-\mathbf{K}_2-\mathbf{K}_3)\delta[\omega_{(\mathbf{K})}+\omega_{(\mathbf{K}_1)}-\omega_{(\mathbf{K}_2)}-\omega_{(\mathbf{K}_3)}]$$
$$\times\{AC_{(\mathbf{K}_3)}AC_{(\mathbf{K}_2)}[AC_{(\mathbf{K}_1)}+AC_{(\mathbf{K}_i)}]-AC_{(\mathbf{K}_1)}AC_{(\mathbf{K}_i)}[AC_{(\mathbf{K}_3)}+AC_{(\mathbf{K}_2)}]\}\, d\mathbf{K}_1\, d\mathbf{K}_2\, d\mathbf{K}_3, \quad (26)$$

where AC is the action density, $AC = \langle|B_{\mathbf{K}}|^2\rangle + \langle|B'_{\mathbf{K}}|^2\rangle$ (Zakharov 1991). This is comparable to Crawford et al. (1980).

4. The effects of finite water depth and nonlinear dispersion

We now apply the theoretical results of the previous section to study the effects of finite water depth and nonlinear dispersion in determining the resonant trajectory and nonlinear energy transfer rate. We take the standard Joint North Sea Wave Project (JONSWAP) directional spectrum as our initial energy spectrum. This is shown in Fig. 1, in terms of frequency and direction, where lines A, B, . . . , G represent angles 0°, 30°, . . . , 180° 0° is pointing to the east, which is the main wave propagation direction.

Before proceeding further, we demonstrate that the expressions for the wave energy transfer rate derived for finite water depth really agree with those derived by Zakharov (1968, 1991) for deep water. Using the spectrum of Fig. 1, the nonlinear wave–wave interaction transfer rate was computed by using the expression of Zakharov (1968, 1991) and by Eq. (26) with depth set at 1000 m. To be consistent, only the linear dispersion relationship was used for both Zakharov's and our formulations. The results are identical, as presented in Fig. 2. The lines A, B, . . . , G represent angle 0°, 30°, . . . , 180° as in Fig. 1.

a. The effects on resonance conditions

Phillips (1960) pointed out that narrowband instability dominates in deep water. Our results basically support this conclusion even with the nonlinear dispersion relationship and the effects of finite water depth. There are, however, certain unexpected effects due to the nonlinearity in combination with the finite depth of the water.

Assuming the linear dispersion relation, the diagram by Phillips (1960) for the trajectory of the wavenumbers satisfying the resonance condition for third-order binary interactions in deep water, is shown in Fig. 3. Here, wavenumber one, K_1, and wavenumber two, K_2, are both set to 1, and both are propagating in the x direction. The horizontal and vertical coordinates represent the x and y components of wavenumber three (K_{x3}, K_{y3}), respectively. Figures 3b and 3c are the same as Fig. 3a except that they are for intermediate water ($Kh_o = 0.5$) and shallow water ($Kh_o = 0.1$), which are also described by Phillips (1960). Figures 3a–c show that the y component of the third interacting wavenumber decreases as the water depth decreases. Consequently, $\mathbf{K}_3$ tends to become more parallel to $\mathbf{K}_1$ and $\mathbf{K}_2$ when the water becomes shallower.

Figures 4 and 5 are the same as Fig. 3 except that the dispersion relation is nonlinear. As the effects of nonlinear dispersion are most pronounced in combination with a shallow depth, we concentrate our examination on very small nonlinearity. For the same depth range as in Fig. 3, Fig. 4 represents weak nonlinearity with $aK = 0.01$, whereas Fig. 5 represents strong nonlinearity with $aK = 0.03$.

The resonant trajectory in Fig. 4a is similar to that in Fig. 3a and assuming $Kh = 1.0$, $\gamma = 0.013$, and $\omega_3 = 0.86 \times 10^{-4}\ \omega_1$, where γ is the expansion coefficient defined in Eq. (12b) and ω_1 and ω_3 are related through Eq. (15b). Moreover, with $Kh = 0.5$, $\gamma = 0.022$, and $\omega_3 = 0.25 \times 10^{-3}\ \omega_1$, the resultant resonant trajectory is slightly narrower in the y direction than the linear dispersion case, as shown in Fig. 4b, as compared to Fig. 3b. In contrast to this, when we assume $Kh = 0.1$, $\gamma = 0.3$, and $\omega_3 = 0.5 \times 10^{-2}\ \omega_1$, the resonant trajectory is almost nonexistant except for a few solutions. This is shown in Fig. 4c and differs significantly from Fig. 3c.

In the strong nonlinearity case, assuming $Kh_o = 1.0$,

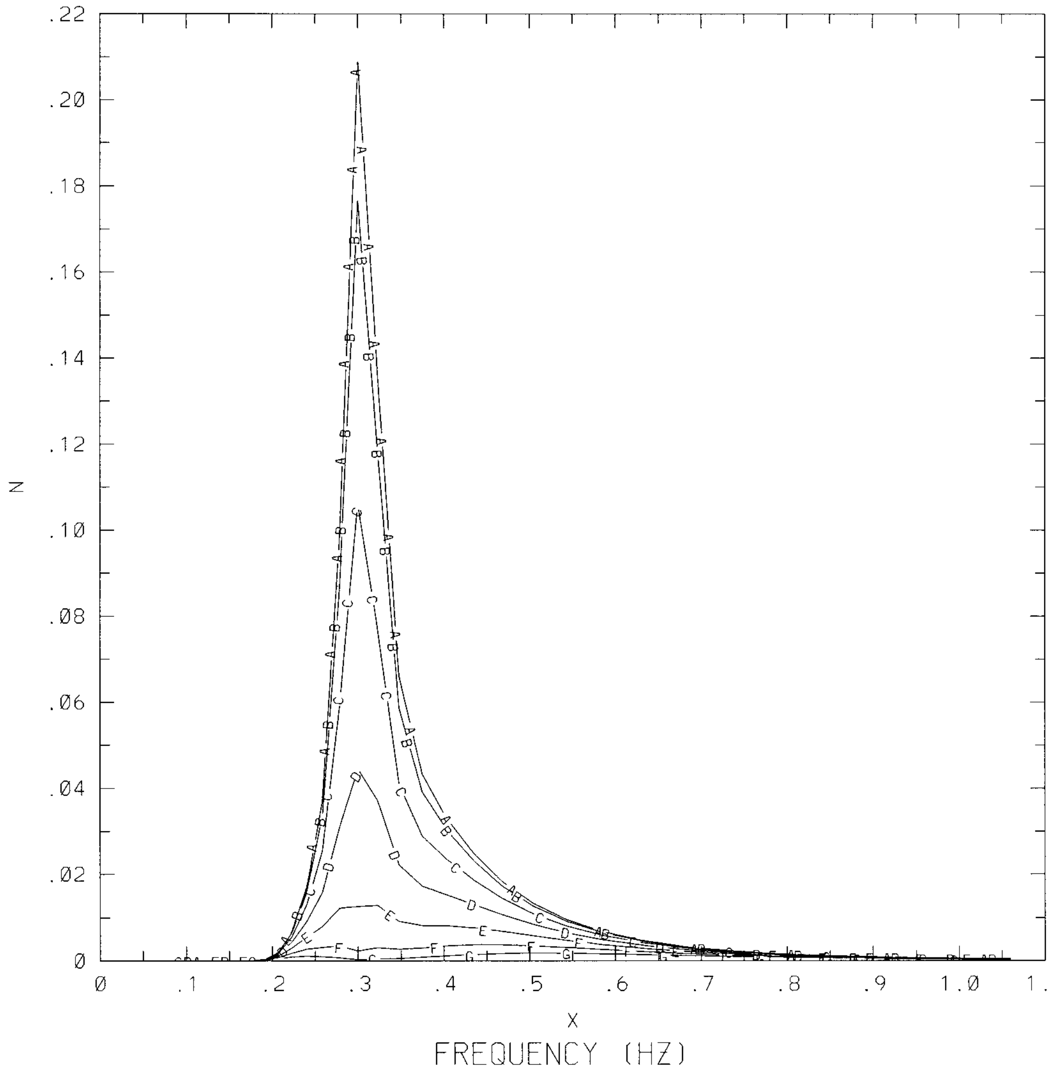

FIG. 1. The reference JONSWAP gravity wave spectrum with Hasselmann–Mistsuyasu directional speading, where lines A, B, C, . . . , G represent the angle 0°, 30°, 60°, . . . , 180°, with 0° toward the east.

$\gamma = 0.039$, and $\omega_3 = 0.78 \times 10^{-3}\ \omega_1$, the resultant trajectory, as shown in Fig. 5a, is similar to the linear dispersion case of Fig. 3a. Moreover, assuming $Kh_o = 0.5$, $\gamma = 0.065$, and $\omega_3 = 0.21 \times 10^{-2}\omega_1$, the trajectory in Fig. 5b appears only slightly different from that in Fig. 3b. However, in contrast to this, when $Kh_o = 0.1$, $\gamma = 0.3$, and $\omega_3 = 0.045\omega_1$ as in Fig. 5c, there are no solutions satisfying the resonance conditions of Fig. 3c.

Figure 6 shows the different domains of the nonlinear wave–wave interactions, as functions of the water depth and the nonlinearity, as measured by the wave slope. A solid line separates the figure into two parts. The Phillips mechanics (the nonlinear wave–wave resonant interaction involving four gravity waves) dominates in the upper area, where the water is deep and the wave slope is small. However, over the lower area, as the depth of water decreases and the wave slope increases, the classic tetrad interactions cease to operate. This indicates that the effects of nonlinear dispersion appear significant, especially for water of finite depth.

The lack of classic tetrad interactions is relevant to the results of Frelich and Guza (1984), who suggested that in shoaling water, the dominating instability mechanism is triadic. Their result was based on the nearly nondispersive properties of shoaling waves, that is,

$$\omega_1^2 = gh_oK^2, \tag{27}$$

which is the asymptotic form for vanishing depth for the linear expression given in Eq. (7). If we employ the full nonlinear dispersion relationship derived here, we find that the triadic interactions are not possible. It is an open question as to whether the *near* resonant triadic interactions at a lower order can overpower the *exact* tetradic interactions at a higher order. However, if the instability condition over shallow water is neither three-wave nor four-gravity-wave interactions, according to the precise dispersion relation-

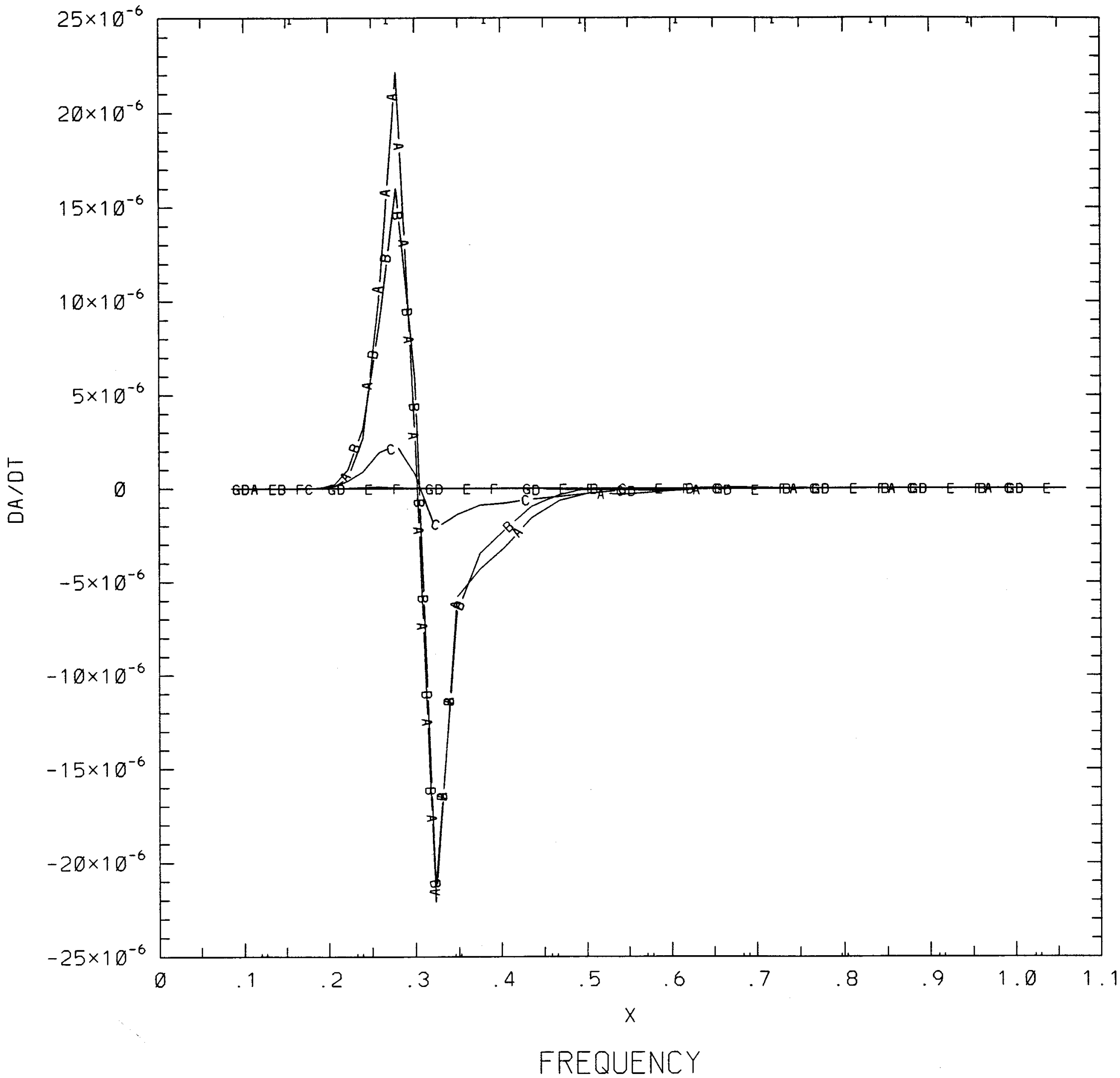

FIG. 2. The nonlinear action transfer rate in gravity wave spectrum in deep water, where lines A, B, . . . , G represent the angle 0°, 30° 60°, . . . , 180° and 0° is toward east. The results are identical using either Zakharov's formulation or our new model given in Eq. (26) and the appendix.

ship and resonance condition, then what are the actual interaction processes? One can always make the interacting components satisfy one of the resonance conditions (either in wavenumber or frequency space), and force the components also to satisfy the remaining conditions under special additional provisions. If this approach is adopted, we have three possibilities.

The first possibility is that one of the interacting components has a fixed wavenumber but zero frequency. A fixed topographic feature will satisfy this requirement as suggested by Phillips (1995). This would be a topographically trapped wave.

The second possibility, which is more plausible, is to introduce a different kind of wave with a different dispersion relationship as part of the interacting tetrad. Internal waves are among the possibilities, as shown by Ball (1964). However, internal waves might not be a prevailing phenomena in shallow water near the coast. Therefore, we have to look at other surface waves. Gallagher (1971) first proposed a long edge wave interacting with two surface waves. Guza and Davis (1974) suggested that a resonant triad could be formed among two edge waves and one surface gravity wave. Miles (1990, 1991) investigated the resonant excitation of two weakly nonlinear, oppositely traveling, edge waves on a gently sloping beach by a perfectly reflected obliquely incident gravity wave. However, based on kinematics, we find that the resonance condition still needs four-wave interactions over shallow water, consisting of three gravity waves and one edge wave. The three-wave interaction is only an asymptotic limit. Details will be presented in a separate paper.

The third possibility is for nonlinear waves to have the

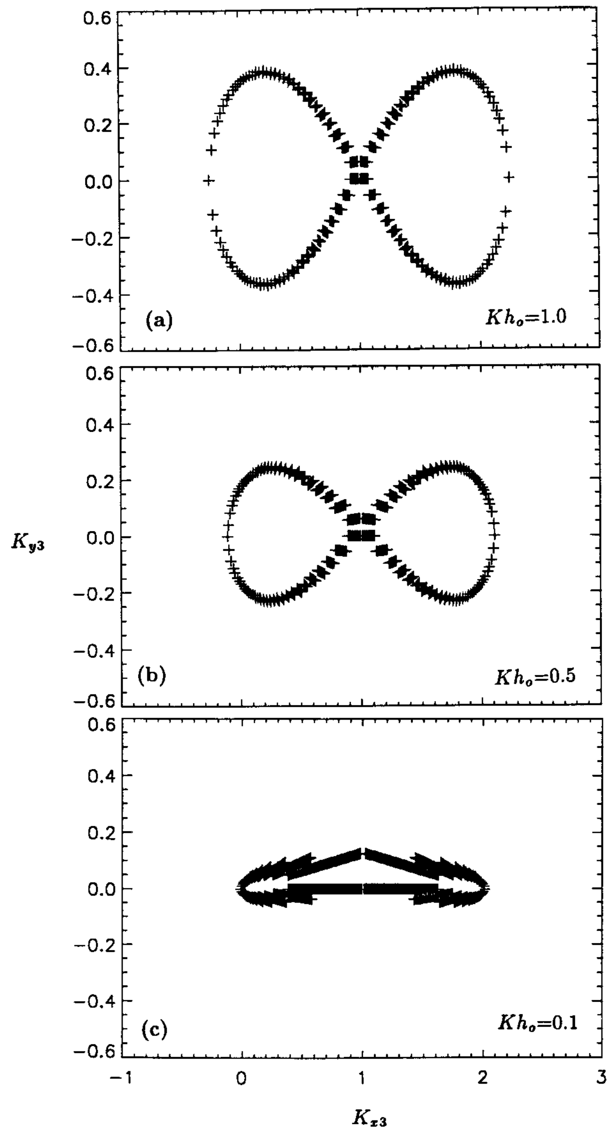

FIG. 3. The trajectory of the resonant wavenumber for third-order binary interaction of four gravity waves with a linear dispersion: (a) deep water ($Kh_o = 1.0$), (b) intermediate water ($Kh_o = 0.5$), and (c) shallow water ($Kh_o = 0.1$) for $K_1 = K_2 = 1$.

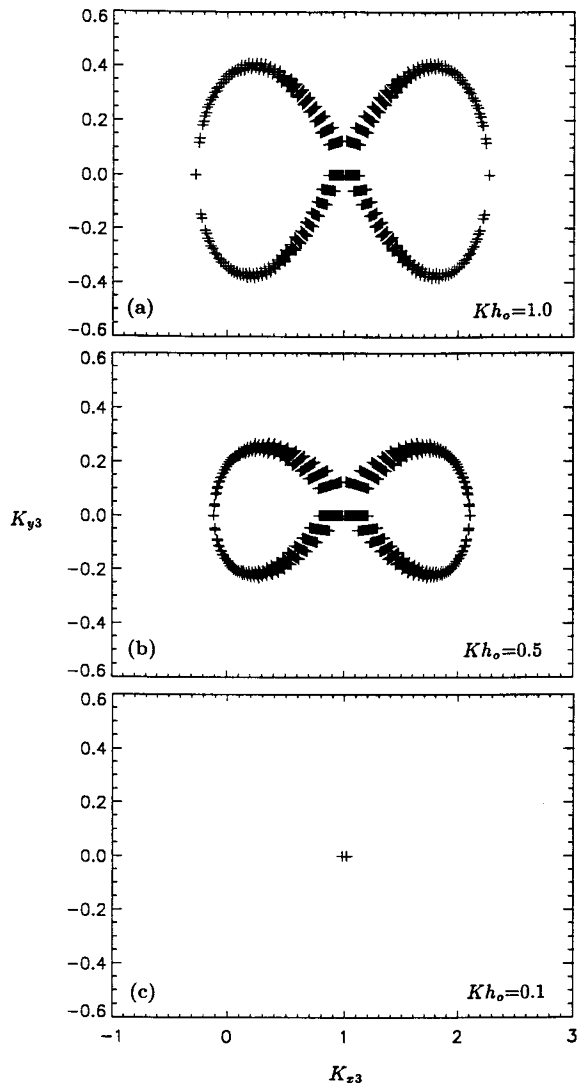

FIG. 4. As in Fig. 3 except the dispersion is nonlinear as represented in Eqs. (7) and (15b) for weak nonlinearity with ($aK = 0.01$): (a) deep water ($Kh_o = 1.0$), (b) intermediate water ($Kh_o = 0.5$), and (c) shallow water ($Kh_o = 0.1$).

classic tetrad interaction that occurs in wideband spectra. Figures 7 and 8 are the same as Fig. 3 except for the wave numbers: $K_1 = 1$, and $K_2 = 0.2$. Figure 7 compares linear and weakly nonlinear dispersion relationships, whereas Fig. 8 compares linear and strongly nonlinear dispersion relationships. In Fig. 7, we assume $Kh_o = 1.0$ and $\gamma = 0.3$, which implies $\omega_3 = 0.045\omega_1$ from Eqs. (12b) and (15b). In Fig. 8, we assume $Kh_o = 1.0$ and $\gamma = 0.5$, which implies $\omega_3 = 0.125\omega_1$. The trajectories for the linear and weakly nonlinear dispersion cases are both represented by two small circles in Fig. 7. The number of resonant solutions is therefore significantly decreased, in comparison with Fig. 3a. In contrast to this, the trajectory for strongly nonlinear dispersion in Fig. 8b is one large circle and the number of resonant solutions increases greatly in comparison with Fig. 7b. Thus, the possibility of wideband instability increases significantly, even in finite water depth, when nonlinear dispersion is considered. Full discussion of the wideband cases will be given separately in Lin and Perrie (1997). We now turn our attention to the influence of nonlinear dispersion and finite depth on the nonlinear energy transfer rate.

There is considerable uncertainty concerning the instability of Stokes waves in water of finite depth. According to Whitham's theory, Stokes waves should be stable when $Kh_o < 1.363$. Later, Hayes (1973) and Davey and Stewartson (1974) showed through more detailed analyses that Stokes waves should be unstable for all values of Kh_o, although for $Kh_o < 0.5$ the instability

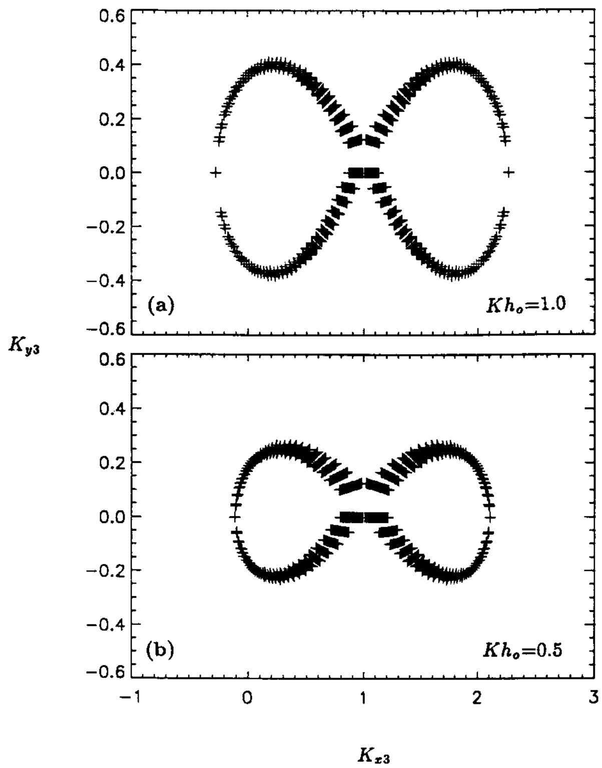

FIG. 5. As in Fig. 3 except the dispersion is nonlinear as represented in Eqs. (7) and (15b) for strong nonlinearity with ($aK = 0.03$): (a) $Kh_o = 1.0$ and (b) $Kh_o = 0.5$.

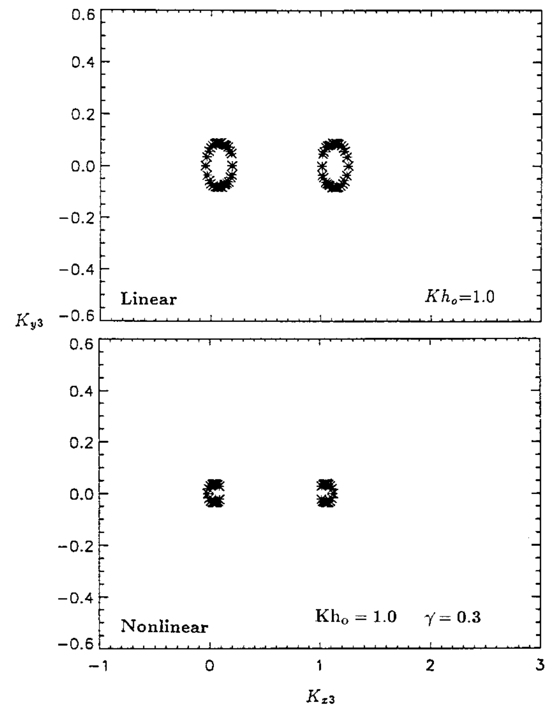

FIG. 7. The trajectories of the resonant wavenumbers for third-order binary interaction with $K_1 = 1$ and $K_2 = 0.2$: (a) linear dispersion relationship and (b) weak nonlinear dispersion with $aK(3 + \tanh^2 Kh)/4 \tanh^3 Kh = 0.3$ and $Kh_o = 1.0$.

would be hard to realize practically. As all analyses are subject to the restrictions of the model equations, the phenomena should be examined carefully with precise observations to sort out the true mechanism.

b. The effects in the action transfer rate

As shown in the last section, the effects of nonlinear dispersion are most pronounced when combined with

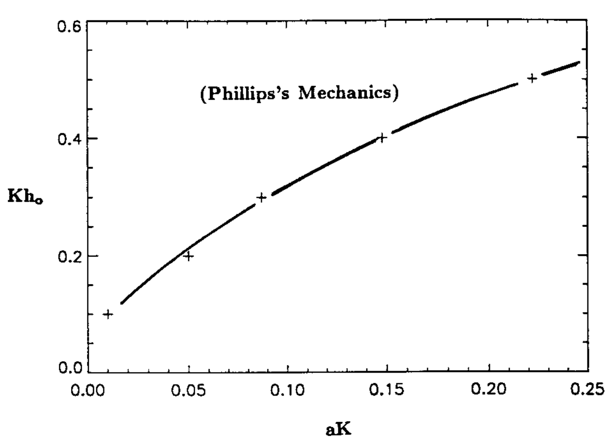

FIG. 6. The nonlinear instability domain of the Phillips mechanics as a function of wave slope, aK, representing the nonlinearity and the depth of the ocean, Kh_o.

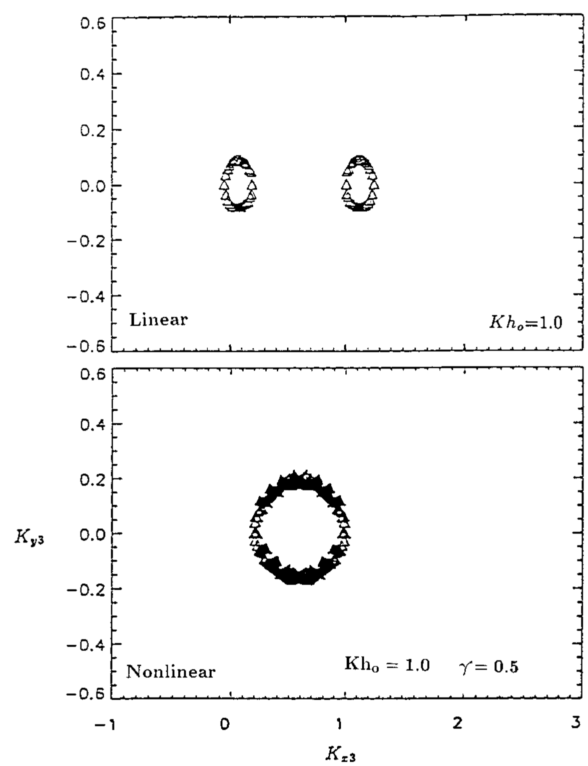

FIG. 8. As in Fig. 7 comparing (a) linear dispersion relationship and (b) strong nonlinear dispersion with $aK(3 + \tanh^3 Kh)/4 \tanh^3 Kh = 0.5$ and $Kh_o = 1.0$.

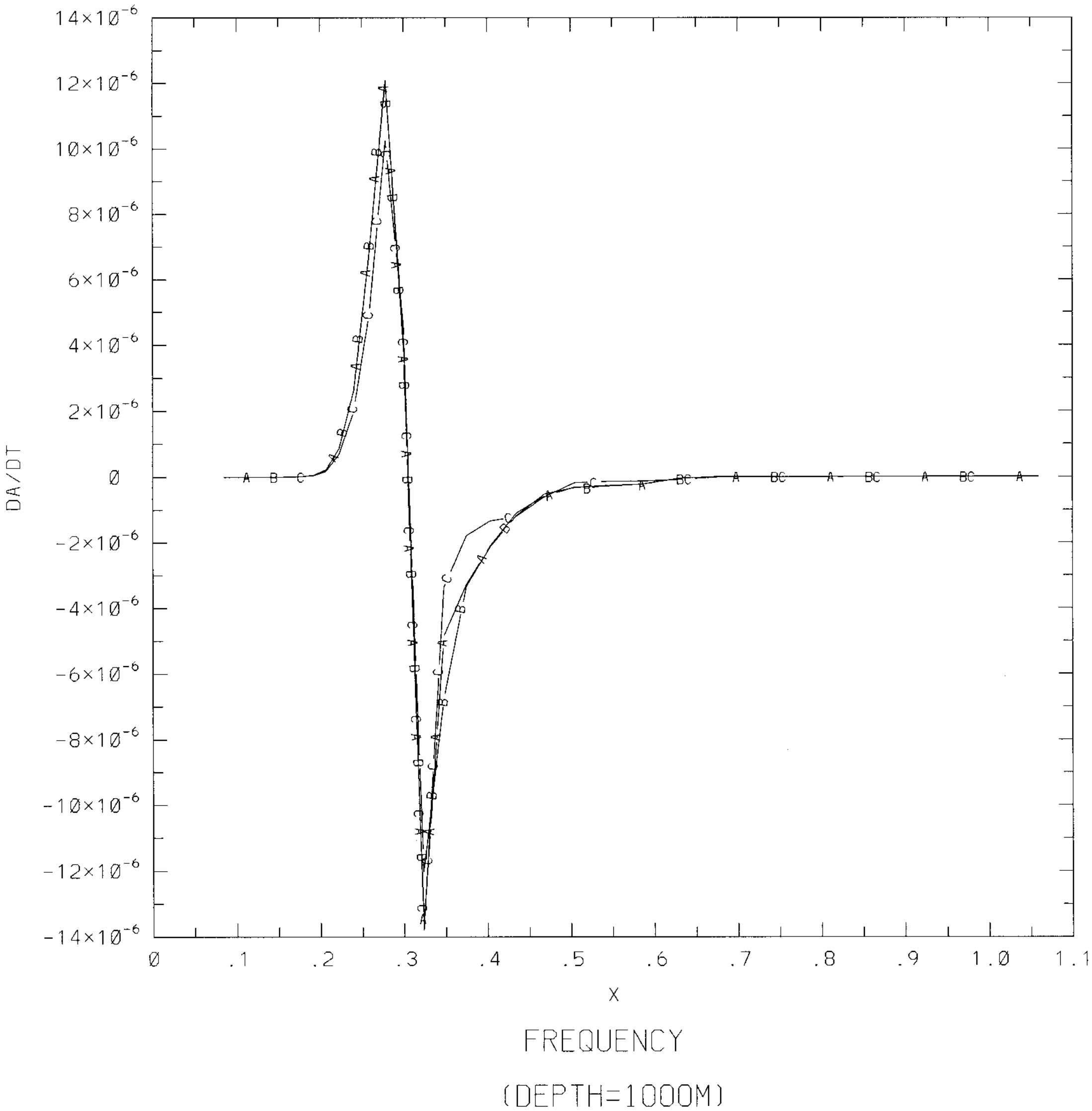

FIG. 9. The effects of the nonlinear dispersion relationship on the total nonlinear action transfer rate (summing over all the directions) in a gravity wave spectrum in deep water (h_o = 1000 m). Lines A, B, and C represent Zakharov's (1968, 1991) result using the linear dispersion relationship, the new formulation with $aK = 0.1$ (weak nonlinear dispersion), and the new formulation with $aK = 0.3$ (strong nonlinear dispersion), respectively.

finite water depth. To illustrate this further, we examine the effects of nonlinear dispersion on the energy transfer rate, starting with the deep water case and proceeding to shallow water.

Figure 9 shows the total nonlinear action transfer rate $\partial A/\partial t$ (summing all directions) in deep water ($h_o = 1000$ m). Line A represents the linear dispersion relation. Line B represents weak nonlinearity, with $\gamma = 0.1$. Line C represents strong nonlinearity, with $\gamma = 0.3$. This shows that the nonlinear effects are only discernible when the nonlinearity is strong. Even then, the difference is quantitative rather than qualitative. Figure 10 is the same as Fig. 9 except that it is for shallow water with $h_o = 10$ m. However, the action transfer rates of lines A and B in Fig. 10 are significantly smaller than the action transfer rates of lines A and B in Fig. 9. Moreover, in Fig. 10, the action transfer rate with strong nonlinear dispersion, line C, is 20% less than the cases with linear or weakly nonlinear dispersion in lines A and B. If one considers the peak frequency at 0.1 *Hz* instead of 0.3 *Hz*, the action transfer rates will be much smaller than the action transfer rates shown in Fig. 10. This is consistent with Figs. 3, 4, 5 and 6, in which the Phillips mechanism is shown to be less dominant when one considers strong nonlinear dispersion over shallow water. This trend is also consistent with the analyses by Hayes (1973) and Davey and Stewartson (1974). In fact, the interaction mechanism changes from classic tetradic four waves to three gravity waves plus a long wave, as discussed in the previous section.

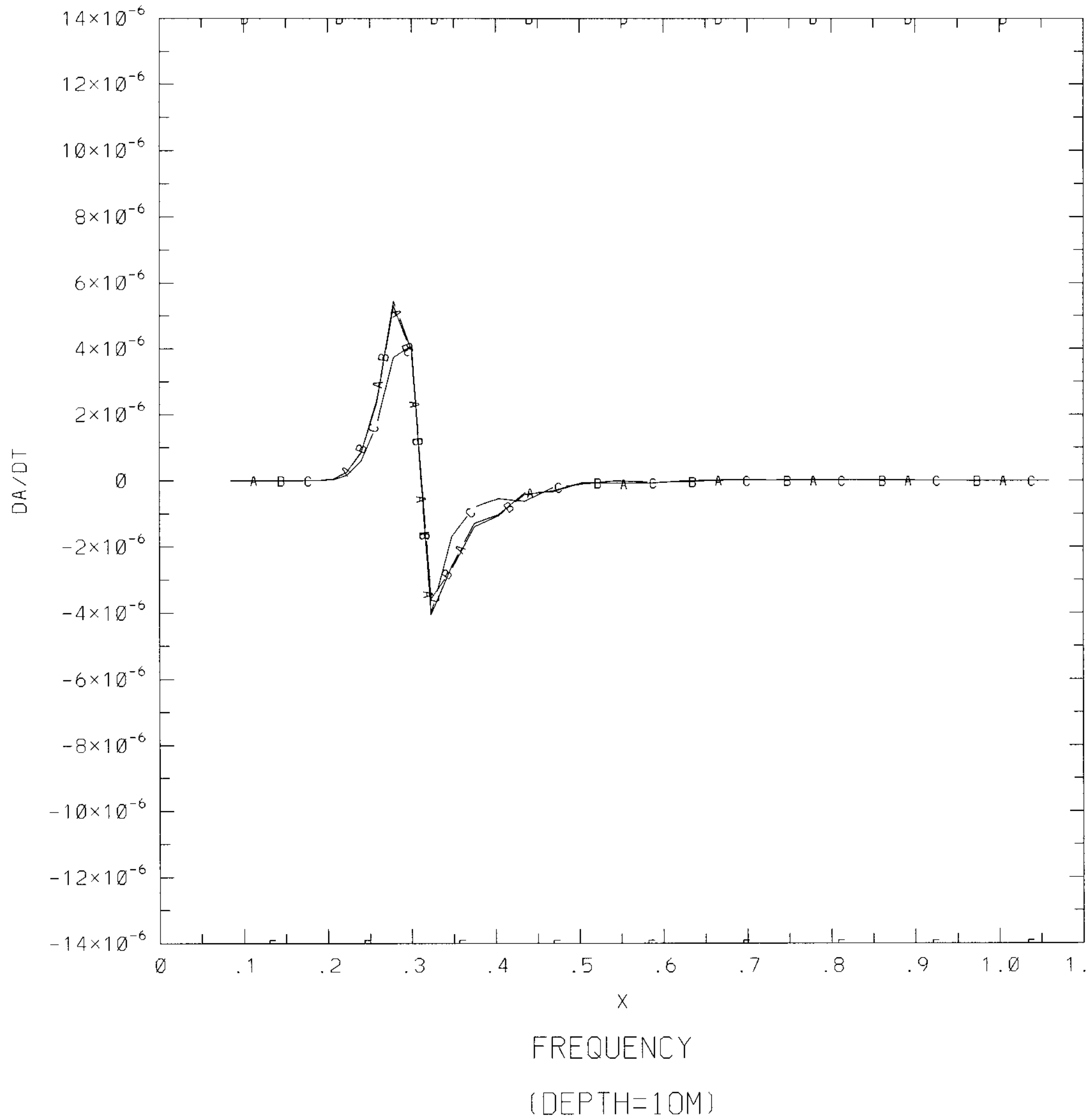

FIG. 10. As in Fig. 9 except in shallow water where h_o is 10 m: lines A, B, and C represent linear dispersion, weak nonlinear dispersion ($aK = 0.1$), and strong nonlinear dispersion ($aK = 0.3$), respectively. All the results were calculated by the new formulation.

5. Summary

We have derived the action transfer rate, based on a Hamiltonian representation, for waves of finite amplitude in water of finite depth. This is a generalization of the results obtained by Zakharov (1968) for deep water. In the limit of deep water and linear dispersion, our results become identical to those of Zakharov's. Using our new formulation, we have studied the effects of nonlinear dispersion and finite water depth.

The effects of nonlinear dispersion are not important in deep water, because

$$\gamma = aK(3 + \tanh^2 Kh)/4 \tanh^3 Kh$$

is very small compared with the linear term gk. However, the nonlinear effect is very important when the water becomes shallow. In shallow water, even the resonance condition has to be modified. In place of the classic tetradic interaction, the resonance condition involves three gravity waves and one low-frequency wave, such as an edge wave or a bottom topographic wave. We also showed that triadic interactions of gravity waves are only asymptotic approximations. Whether or not these asymptotic cases are really important approximations in the coastal ocean remains to be demonstrated.

Based on our analysis, we believe that for an accurate evaluation of the energy transfer rate in water of finite depth, the effects of nonlinear dispersion should not be neglected. For shallow water wave modeling, the new results derived here should be used as suggested by Lin and Perrie (1997).

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APPENDIX

The Interaction Coefficient

The third-order interaction coefficient $V(\pm)$ and W are given as follows:

$$V^{(\pm)}_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2)} = \frac{1}{8\pi\sqrt{2}}\Bigg\{[\mathbf{K}_i\cdot\mathbf{K}_1 \pm K_iK_1\tanh(K_ih)\tanh(K_1h)]\left[\frac{\omega_{(K_i)}\omega_{(K_1)}K_2\tanh K_2h}{\omega_{(K_2)}K_iK_1\tanh K_ih\tanh K_1h}\right]^{1/2}$$
$$+ [\mathbf{K}_i\cdot\mathbf{K}_2 \pm K_iK_2\tanh(K_ih)\tanh(K_2h)]\left[\frac{\omega_{(K_i)}\omega_{(K_2)}K_1\tanh K_1h}{\omega_{(K_1)}K_iK_2\tanh K_ih\tanh K_2h}\right]^{1/2}$$
$$+ [\mathbf{K}_1\cdot\mathbf{K}_2 + K_1K_2\tanh(K_1h)\tanh(K_2h)]\left[\frac{\omega_{(K_1)}\omega_{(K_2)}K_i\tanh K_ih}{\omega_{(K_i)}K_1K_2\tanh K_1h\tanh K_2h}\right]^{1/2}\Bigg\}, \qquad \text{(A1)}$$

and

$$W_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} = \overline{W}_{(-\mathbf{K},-\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} + \overline{W}_{(\mathbf{K}_2,\mathbf{K}_3,-\mathbf{K},-\mathbf{K}_1)} - \overline{W}_{(\mathbf{K}_2,-\mathbf{K}_1,-\mathbf{K},\mathbf{K}_3)} - \overline{W}_{(-\mathbf{K},\mathbf{K}_2,-\mathbf{K}_1,\mathbf{K}_3)} - \overline{W}_{(-\mathbf{K},\mathbf{K}_3,\mathbf{K}_2,-\mathbf{K}_1)}$$
$$- \overline{W}_{(\mathbf{K}_3,-\mathbf{K}_1,\mathbf{K}_2,-\mathbf{K})}, \qquad \text{(A2)}$$

$$W_{(\mathbf{K}_i,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)} = \frac{1}{64\pi^2}\left[\frac{\omega_{(K_i)}\omega_{(K_1)}}{\omega_{(K_2)}\omega_{(K_3)}}K_iK_1K_2K_3\tanh K_ih\tanh K_1h\tanh K_2h\tanh K_3h\right]^{1/2}$$
$$\times\Bigg[2\left(\frac{K_i}{\tanh K_ih} + \frac{K_1}{\tanh K_1h}\right) - |\mathbf{K}_1 + \mathbf{K}_3|\tanh\big(|\mathbf{K}_1 + \mathbf{K}_3|h\big) - |\mathbf{K}_1 + \mathbf{K}_3|\tanh\big(|\mathbf{K}_1 + \mathbf{K}_2|h\big)$$
$$- |\mathbf{K}_i + \mathbf{K}_3|\tanh\big(|\mathbf{K}_i + \mathbf{K}_3|h\big) - |\mathbf{K}_i + \mathbf{K}_2|\tanh\big(|\mathbf{K}_i + \mathbf{K}_2|h\big)\Bigg]. \qquad \text{(A3)}$$

$$T_{0,1,2,3} = -V^{(-)}_{(\mathbf{K}_3,\mathbf{K}_3-\mathbf{K}_1,\mathbf{K}_1)}V^{(-)}_{(\mathbf{K},\mathbf{K}_2,\mathbf{K}-\mathbf{K}_2)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} - \omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}-\mathbf{K}_2)} - \omega_{(\mathbf{K})}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}_2,\mathbf{K},\mathbf{K}_2-\mathbf{K})}V^{(-)}_{(\mathbf{K}_1,\mathbf{K}_1-\mathbf{K}_3,\mathbf{K}_3)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_3)} - \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_2-\mathbf{K})} - \omega_{(\mathbf{K}_2)}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}_2,\mathbf{K}_2-\mathbf{K}_1,\mathbf{K}_1)}V^{(-)}_{(\mathbf{K},\mathbf{K}_3,\mathbf{K}-\mathbf{K}_3)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_2)} - \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(\mathbf{K}_3)} + \omega_{(\mathbf{K}_3-\mathbf{K})} - \omega_{(\mathbf{K})}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}_3,\mathbf{K},\mathbf{K}_3-\mathbf{K})}V^{(-)}_{(\mathbf{K}_1,\mathbf{K}_1-\mathbf{K}_2,\mathbf{K}_2)}\left\{\frac{1}{\omega_{(\mathbf{K}_1-\mathbf{K}_2)} - \omega_{(\mathbf{K}_1)} + \omega_{(\mathbf{K}_2)}} + \frac{1}{\omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_3-\mathbf{K})} - \omega_{(\mathbf{K}_3)}}\right\}$$
$$- V^{(-)}_{(\mathbf{K}+\mathbf{K}_1,\mathbf{K},\mathbf{K}_1)}V^{(-)}_{(\mathbf{K}_2+\mathbf{K}_3,\mathbf{K}_2,\mathbf{K}_3)}\left\{\frac{1}{\omega_{(\mathbf{K}+\mathbf{K}_1)} - \omega_{(\mathbf{K})} - \omega_{(\mathbf{K}_1)}} + \frac{1}{\omega_{(\mathbf{K}_2+\mathbf{K}_3)} - \omega_{(\mathbf{K}_2)} - \omega_{(\mathbf{K}_3)}}\right\}$$
$$- V^{(+)}_{(-\mathbf{K}_2-\mathbf{K}_3,\mathbf{K}_2,\mathbf{K}_3)}V^{(+)}_{(\mathbf{K},\mathbf{K}_1,-\mathbf{K}-\mathbf{K}_1)}\left\{\frac{1}{\omega_{(\mathbf{K}_2+\mathbf{K}_3)} + \omega_{(\mathbf{K}_2)} + \omega_{(\mathbf{K}_3)}} + \frac{1}{\omega_{(\mathbf{K}+\mathbf{K}_1)} + \omega_{(\mathbf{K})} + \omega_{(\mathbf{K}_1)}}\right\} + W_{(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)}. \qquad \text{(A4)}$$

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