Harmonic Generation of Shallow Water Waves Over Topography[1]

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ABSTRACT

We investigate the reflection and nonlinear interaction between the first and second harmonics of a two-dimensional Boussinesq wavetrain. Effects of topography are included, the depth departing from a constant in a finite region. It is found that topography can speed up or retard energy transfer between first and second harmonics. The reflection coefficient in the present context is significantly different from the one obtained by using linear theory. This is partly due to partitioning of energy between harmonics.

1. Introduction

We propose to investigate theoretically the effects of bottom topography on moderate-amplitude shallow water waves. This problem is relevant to the understanding of the distribution of wave energy in near-shore processes.

A distinctive character of these waves as they enter shallow water is the generation of secondary crests. Mason and Keulegan (1944) observed that multiple secondary crests are generated when long waves move over reefs modeled by step-like topography; the criteria for their formation is

$$\alpha/\beta>4, \tag{1.1}$$

where α and β are, respectively, defined as a/H and H/λ, where a is the wave amplitude, λ the wavelength and H the depth over the reef. This relation was verified by Horikawa and Wiegel (1959) who also observed that the formation of secondary waves is a characteristic of shallow water waves rather than the shape of the bottom. Experiments of Goda (1967) further strengthened this observation. Recently Mei and Ünlüata (1972) studied the harmonic generation of shallow water waves over flat bottom and showed that the appropriate criteria should be α/β^2 greater than some constant.

Williams (1964) investigated the passage of a deep-water wavetrain onto a thin submerged horizontal plate. He observed the formation of higher harmonics as the wave moved over that obstacle. In accord with the above criteria, the second and third harmonic content increases as the depth over the plate is decreased, while the wave amplitude of the fundamental is held fixed. For relatively shallow depths, the first and second harmonics were of comparable magnitude. Furthermore, reflections were significant. Byrne's (1969) field observations of induced multiple gravity waves generated by submerged offshore bars lying parallel to the shore show similar characteristics: as the tide increased the depth over the crest of the bar, he noticed that only the large-amplitude waves would generate secondary crests. From Byrne's oscillograph traces, there was strong evidence of reflection. Fourier decomposition of his surface traces indicates that the energy content of the first and second harmonics are dominant, though higher harmonics are present.

From these observations it appears that reflection is important when topography is present. We shall investigate the reflection and nonlinear interaction between the first and second harmonics of a moderate-amplitude, two-dimensional shallow water wavetrain over a scattering region $0<\bar{x}<\bar{x}_0$ in which the depth varies with $\bar{x}$. For simplicity we will neglect third and higher harmonics. Therefore, our results are valid only in cases for which higher harmonics are negligible.

2. Formulation

We assume the bottom topography to have horizontal characteristic scales comparable to or smaller than those associated with the waves. The presence of this topography could induce reflection.

Consider a two-dimensional wavetrain and let $(\bar{x},\bar{z})$ be the spatial coordinates where the undisturbed free surface lies at $\bar{z}=0$, and $\bar{z}=-\bar{h}(\bar{x})$ denotes the bottom. Let $\bar{x}=0$ be the shoreline and measure $\bar{x}$ positive seaward. In the Boussinesq approximation, the moderate-amplitude long-wave equations are

$$\bar{q}_{\bar{t}}+g\bar{\zeta}_{\bar{x}}+\bar{q}\bar{q}_{\bar{x}}=\frac{\bar{h}^2}{3}\bar{q}_{\bar{x}\bar{x}\bar{t}}+\bar{h}\bar{h}_{\bar{x}}\bar{q}_{\bar{x}\bar{t}}+\frac{\bar{h}\bar{h}_{\bar{x}\bar{x}}}{2}\bar{q}_{\bar{t}}, \tag{2.1}$$

$$\bar{\zeta}_{\bar{t}}+[(\bar{\zeta}+\bar{h})\bar{q}]_{\bar{x}}=0, \tag{2.2}$$

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where

$$\bar{q}=\frac{1}{(\bar{h}+\bar{\zeta})}\int_{-\bar{h}}^{\bar{\zeta}}\bar{u}(\bar{x},\bar{z},\bar{t})d\bar{z} \tag{2.3}$$

is the depth-averaged horizontal velocity and $\bar{\zeta}$ is the free surface displacement. This set of equations can be obtained if one averages over depth the equations of shallow water waves as derived by Mei and Méhanté [1966; Eqs (30), (31)]; the approximations behind these equations are discussed in that paper. Some general discussion about these equations can also be found in Hoogstraten (1968).

Let us scale the dimensional quantities as

$$\left.\begin{aligned} &x=\bar{x}/H, \quad h=\bar{h}/H, \quad \zeta=\bar{\zeta}/H \\ &t=\bar{t}/(H/g)^{\frac{1}{2}}, \quad q=\bar{q}/a(g/H)^{\frac{1}{2}}, \end{aligned}\right\}, \tag{2.4}$$

where H is a characteristic depth. The nondimensional equations read

$$q_t+\zeta_x+\alpha q q_x=\frac{h^2}{3}q_{xxt}+hh_xq_{xt}+\frac{hh_{xx}}{2}q_t, \tag{2.5}$$

$$\zeta_t+[(\alpha\zeta+h)q]_x=0, \tag{2.6}$$

where

$$\alpha=a/H\ll 1. \tag{2.7}$$

It can be shown [see Mei and Ünlüata (1972) for the special case of $h=\text{constant}$] that solution of these equations obtained by a straightforward expansion in α is inadequate because the second order terms grow linearly and soon become of the same magnitude as the first order terms. Since we have not yet used the wavelength λ in our scaling an additional horizontal variable must be introduced; thus, we write

$$x=\bar{x}/\lambda=\beta x \quad \text{where} \quad \beta=H/\lambda. \tag{2.8}$$

The problem then involves α and β; for simplicity, we will only look at the class of waves for which

$$O(\alpha)=O(\beta). \tag{2.9}$$

From Mason and Keulegan's (1944) criterion for the "disintegration" of a wavetrain we deduce that the above range of parameters puts us in the regime in which growth of second harmonics is possible. We will further assume that the bottom topography could induce effects comparable in size to the nonlinearities, i.e., we write

$$h(x,X)=1+\alpha f(x,X), \tag{2.10}$$

where the departures of h from a flat surface are $O(\alpha)$; $f(x,X)$ is an $O(1)$ function. In order to further simplify the mathematics we will take

$$f(x,X)=0 \quad \text{for} \quad x<0, \quad x>x_0. \tag{2.11}$$

Now we introduce a two-scale expansion, *viz.*,

$$\left.\begin{aligned} &\frac{\partial}{\partial x}=\frac{\partial}{\partial x}+\alpha\frac{\partial}{\partial X} \\ &q=q^{(1)}(x,X,t)+\alpha q^{(2)}(x,X,t)+\cdots \\ &\zeta=\zeta^{(1)}(x,X,t)+\alpha\zeta^{(2)}(x,X,t)+\cdots \end{aligned}\right\}, \tag{2.12}$$

and substitute the above into (2.5) and (2.6). Equating like powers of α, we obtain the equations for the first two orders as

$$\left.\begin{aligned} q_t^{(1)}+\zeta_x^{(1)}-\tfrac{1}{3}q_{xxt}^{(1)}&=0 \\ \zeta_t^{(1)}+q_x^{(1)}&=0 \end{aligned}\right\}, \tag{2.13}$$

$$\left.\begin{aligned} q_t^{(2)}+\zeta_x^{(2)}-\tfrac{1}{3}q_{xxt}^{(2)}=&-\zeta_X^{(1)}-q^{(1)}q_x^{(1)}+\tfrac{2}{3}q_{Xxt}^{(1)} \\ &+\tfrac{2}{3}fq_{xxt}^{(1)}+\tfrac{1}{2}f_{xx}q_t^{(1)}+f_xq_{xt}^{(1)} \\ \zeta_t^{(2)}+q_x^{(2)}=&-q_X^{(1)}-\zeta_x^{(1)}q^{(1)}-\zeta^{(1)}q_x^{(1)} \\ &-f_xq^{(1)}-fq_x^{(1)} \end{aligned}\right\}. \tag{2.14}$$

Upon elimination of the free-surface displacements $\zeta^{(1)}$ and $\zeta^{(2)}$, we find

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{1}{3}\frac{\partial^4}{\partial x^2\partial t^2}\right)q^{(1)}=0 \tag{2.15}$$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{1}{3}\frac{\partial^4}{\partial x^2\partial t^2}\right)q^{(2)}=G(x,X,t), \tag{2.16}$$

where

$$G(x,X,t)=-\zeta_{tX}^{(1)}+\tfrac{2}{3}q_{ttxX}^{(1)}+q_{xX}^{(1)}-(q^{(1)}q_x^{(1)})_t+(\zeta^{(1)}q^{(1)})_{xx} +\tfrac{2}{3}fq_{xxtt}^{(1)}+f_xq_{xtt}^{(1)}+\tfrac{1}{2}f_{xx}q_{tt}^{(1)}+(fq^{(1)})_{xx}. \tag{2.17}$$

Because (2.15) and (2.16) have the same linear differential operator, terms found in G may produce resonance.

Field and experimental evidence suggest that the leading order fields contain first and second harmonics. Thus, we write

$$(\zeta^{(1)},q^{(1)})=\sum_{j=1}^{2}e^{i\omega_jt}[Y_j^{(1)}(x,X);\, Q_j^{(1)}(x,X)]+*, \tag{2.18}$$

where the asterisk denotes complex conjugation, ω_1 is the fundamental frequency, and $\omega_2=2\omega_1$. $Q_j^{(1)}$ satisfies

$$\left[-\omega_j^2-\left(1-\frac{\omega_j^2}{3}\right)\frac{\partial^2}{\partial x^2}\right]Q_j^{(1)}=0; \tag{2.19}$$

and if we write

$$Y_j^{(1)}(x,X)=a_j(X)e^{ik_jx}+b_j(X)e^{-ik_jx}, \tag{2.20}$$

where a_j, b_j are the amplitudes of the incident and reflected waves, we find from (2.13) that

$$Q_j^{(1)}(x,X) = -(\omega_j/k_j)[a_j(X)e^{ik_jx} - b_j(X)e^{-ik_jx}]. \quad (2.21)$$

Substituting (2.21) into (2.19), we get the dispersion relation

$$D(\omega_j,k_j) = \omega_j^2(1+k_j^2/3) - k_j^2 = 0. \quad (2.22)$$

In the present normalization, $O(\omega_j) \sim O(k_j) \ll 1$ for long waves. When k is small, a plot of $D(\omega,k)=0$ in the (ω,k) plane departs slightly from the straight line $\omega = k$. As a result, when $\omega_2 = 2\omega_1$,

$$k_2 = 2k_1 + \Delta k, \quad (2.23)$$

where Δk measures the deviation of the curve $D(\omega,k)=0$ from the line $\omega=k$ at $\omega = 2\omega_1 = \omega_2$; Δk is a detuning or mismatching factor. For small mismatch, near resonance conditions occur; energy from the first harmonic is readily transferred to the second harmonic. For the higher harmonics, $\omega_j = j\omega_1$ $(j \geqslant 3)$, we deduce from (2.22) that the mismatching factor $|k_j - jk_1| \approx j(j^2-1)\Delta k/6$. Thus, they are more strongly detuned than the second harmonic and we are justified in neglecting harmonics higher than the second. The $O(\alpha^0)$ equations do not provide the equations for the amplitudes $a_j(X)$, $b_j(X)$. These are found by removal of secularities at the $O(\alpha)$. The forcing $G(x,X,t)$ contains exponentials with exponents $\pm i(\omega_j t \pm k_j x)$; $\pm i[(\omega_j \pm \omega_n)t \pm (k_j \pm k_n)x]$ with j, $n = 1$, 2. Several tones are produced by nonlinear mixing. The "dangerous" ones are those that are in near resonance with the linear operator. We expand $G(x,X,t)$ as

$$G(x,X,t) = \sum_{j=1}^{2} e^{i\omega_j t} G_j(x,X) + {}^* + \cdots, \quad (2.24)$$

where the dots denote remaining tones that do not produce resonance. The $G_j(x,X)$ written in terms of $a_j(X)$ and $b_j(X)$ are found in the Appendix. If we assume that the bottom profile can be Fourier-decomposed as

$$f(x,X) = \sum_{-\infty}^{\infty} F_n(X) e^{in|k_1|x}, \quad (2.25)$$

with $F_n = F_{-n}^*$, the solvability conditions for (2.16) are that G_1 and G_2 must be orthogonal to $\exp(\pm ik_1x)$ and $\exp(\pm i2k_1x)$, respectively, i.e.,

$$\frac{k_1}{2\pi}\int_0^{2\pi/k_1} e^{\pm ik_1x} G_1(x,X)dx = 0, \quad (2.26)$$

$$\frac{k_1}{\pi}\int_0^{\pi/k_1} e^{\pm i2k_1x} G_2(x,X)dx = 0. \quad (2.27)$$

We observe that Δkx is an additional long scale corresponding to distances over which near-resonance conditions have significant influence. This is treated on equal footing with X as an independent variable. From (2.26) and (2.27) we obtain four coupled nonlinear differential equations for the complex amplitudes a_j and $b_j (j=1,2)$. To $O(\Delta k/k_1)$ these are:

$$a_{1X} + i\gamma(\omega_1,k_1;X)a_1 + i\delta(\omega_1,-k_1;X)b_1 + iS_1a_1^*a_2e^{i\Delta kx} = 0, \quad (2.28)$$

$$b_{1X} + i\gamma(\omega_1,-k_1;X)b_1 + i\delta(\omega_1,k_1;X)a_1 - iS_1b_1^*b_2e^{-i\Delta kx} = 0, \quad (2.29)$$

$$a_{2X} + i\gamma(\omega_2,k_2;X)a_2 + i\delta(\omega_2,-k_2;X)b_2e^{-i2\Delta kx} + iS_2a_1^2e^{-i\Delta kx} = 0, \quad (2.30)$$

$$b_{2X} + i\gamma(\omega_2,-k_2;X)b_2 + i\delta(\omega_2,k_2;X)a_2e^{i2\Delta kx} - iS_2b_1^2e^{i\Delta kx} = 0, \quad (2.31)$$

where

$$\gamma(\omega_j,k_j;X) = \frac{|jk_1|}{2\pi}\int_{X_0}^{(2\pi/|jk_1|)+X_0} \mathcal{A}(\omega_j,k_j;x,X)dx, \quad (2.32)$$

$$\delta(\omega_j,k_j;X) = \frac{|jk_1|}{2\pi}\int_{X_0}^{(2\pi/|jk_1|)+X_0} \mathcal{A}(\omega_j,k_j;x,X) \times \exp[i(\operatorname{sgn}k_j)j2k_1x]dx, \quad (2.33)$$

for $j=1,2$. The quantities $\mathcal{A}$, S_1 and S_2 are defined in the Appendix. The meaning of the various terms and the behavior of these equations will be discussed in the next section by means of simple examples.

3. Examples

The effects of bottom topography are embedded in the complex coefficients $\gamma(\omega_j,k_j;X)$ and $\delta(\omega_j,k_j;X)$. The former represents the influence of the bottom on an unidirectional wavetrain whereas the latter provides the coupling and interaction between waves traveling in opposite directions. In the absence of $\delta(\omega_j,k_j;X)$ the incident and reflected wavetrains are uncoupled and may be analyzed independently. The coefficients $S_j(j=1,2)$ provide the nonlinear interactions between harmonics and are responsible for the transfer of energy between the first and second harmonics. Therefore, we must retain these terms if we are to model the growth of second harmonics.

a. Example I

Here, the topography scale is assumed long compared to the wave scale, i.e., take $F_0(X)$ as the only non-zero term in (2.25). Then

$$\gamma(\omega_j,k_j;X) = \frac{k_j(3-2\omega_j^2)}{2(3-\omega_j^2)}F_0(X) \quad (3.1)$$

is real and

$$\delta(\omega_j,k_j;X) = 0, \quad (3.2)$$

so that incident and reflected waves are uncoupled. Since reflection is not important in this case we need only consider an incident wavetrain. The governing equations are

$$a_{1x}^*-i\alpha\gamma(\omega_1,k_1;x)a_1^*-i\alpha S_1e^{-i\Delta kx}a_1a_2^*=0, \quad (3.3)$$

$$a_{2x}+i\alpha\gamma(\omega_2,k_2;x)a_2+i\alpha S_2e^{-i\Delta kx}a_2^2=0, \quad (3.4)$$

where we reverted to x instead of X. We can write these equations in a more convenient form if we define

$$\left.\begin{aligned} a_j&=\rho_j\exp(i\phi_j),\quad j=1,2,\\ \theta&=-\Delta kx-\phi_2+2\phi_1,\\ (u,v)&=\frac{1}{(2\alpha E)^{\frac{1}{2}}}\left(\frac{\rho_1}{(S_1)^{\frac{1}{2}}},\frac{\rho_2}{(S_2)^{\frac{1}{2}}}\right)\\ \zeta&=x\alpha S_1(2\alpha S_2E)^{\frac{1}{2}}\end{aligned}\right\}, \quad (3.5)$$

where E is a constant of the motion:

$$E=\left(\frac{\rho_1^2}{2\alpha S_1}+\frac{\rho_2^2}{2\alpha S_2}\right)_{\zeta=\zeta_0}. \quad (3.6)$$

The real and imaginary part of (3.3), (3.4) become

$$\frac{du}{d\zeta}=-uv\sin\theta, \quad (3.7)$$

$$\frac{dv}{d\zeta}=u^2\sin\theta, \quad (3.8)$$

$$\frac{d\theta}{d\zeta}=\Delta S+\Delta\gamma F_0(\zeta)+(u^2/v-2v)\cos\theta, \quad (3.9)$$

where

$$\left.\begin{aligned} l&=[\alpha S_1(2\alpha S_2E)^{\frac{1}{2}}]^{-1}\\ \Delta S&=-l\Delta k\\ \Delta\gamma&=\alpha l\left[\frac{k_2}{2}\left(\frac{3-2\omega_2^2}{3-\omega_2^2}\right)-k_1\left(\frac{3-2\omega_1^2}{3-\omega_1^2}\right)\right]\end{aligned}\right\}. \quad (3.10)$$

Eqs. (3.7)–(3.9) with $\Delta\gamma=0$ have been discussed by Armstrong *et al.* (1962) in the context of nonlinear optics and by Mei and Ünlüata (1972) in the context of harmonic generation of shallow water waves in the absence of topography. The quantity l is a dimensionless length, called the "interaction length" by Armstrong. For $\Delta S=0$, about 75% of the energy of the first harmonic is transferred to the second harmonic in a distance l.

For $\Delta\gamma=0$, the salient features of the solution are:

(i) There exists a first integral of the motion expressing the conservation of energy between the two harmonics,

$$u^2+v^2=1.$$

(ii) ΔS is a measure of the phase mismatch. If $|\Delta S|\neq 0$ the energy will be transferred back and forth between first and second harmonics and this oscillating transfer of energy occurs over a characteristic length called "beat length." For $|\Delta S|=0$, the beat length is infinite. As $|\Delta S|$ increases the beat length as well as the maximum energy transferred to the second harmonic decrease.

When topography is present we could rewrite (3.9) as

$$\frac{d\theta}{d\zeta}=\Delta S_{\rm eff}(\zeta)+(u^2/v-2v)\cos\theta, \quad (3.11)$$

where the effective ΔS is defined as

$$\Delta S_{\rm eff}=\Delta S+\Delta\gamma F_0(\zeta)\approx-\Delta kl\left[1-\frac{\alpha}{2}F_0(\zeta)\right], \quad (3.12)$$

and depends upon the local depth. For a plane sloping beach, $|\Delta S_{eff}|$ decreases for decreasing depth. Therefore we expect the maximum amplitude of the second harmonic, $V_{\rm max}$, and the beat length to increase monotonically so that at some location significant energy is transferred to the second harmonic. If higher harmonics were included in $\zeta^{(1)}$ and $q^{(1)}$, energy may also be transferred to these harmonics. We have assumed these transfers to be small. The numerical integration of (3.7)–(3.9) is in agreement with the above qualitative description: for a beach of 0.15% slope in the region $0<\zeta<\zeta_0$, $\Delta S=0.5$ and $\Delta\gamma=0.25$[2], one finds that the second harmonic intensity $|V^2_{\rm max}/U^2(\zeta_0)|$ at $(\zeta-\zeta_0)=6$ (i.e., six interaction lengths shoreward of the foot of the slope) to be about 8% greater that for the flat bottom case. This increase is even higher for steeper slopes. A plot of $|V^2_{\rm max}/U^2(\zeta_0)|$ for waves entering shallow and deep water is shown in Fig. 1.

b. Example II

Reflection becomes important when wave-scale and subwave-scale variations in the bottom topography are present. Barcilon *et al.* (1972) found for a related linear problem that reflection is at a maximum when the correlation coefficient between the second harmonic and the topography is large; other harmonics did not affect the passage of these linear waves. In the case of nonlinear, moderate-amplitude long waves it can be shown by substituting (2.25) into (2.33) that all the odd harmonics as well as those multiplied by F_2 and F_4 reflect waves. For simplicity we will consider the bottom topography with F_2 as the only nonzero

[2] These values of ΔS and $\Delta\gamma$ correspond to waves of 2 m amplitude and 18-sec period propagating in water of averaged depth of 6 m. These are approximately the conditions observed by Byrne (1969).

coefficient:

$$F_2=\begin{cases}\text{constant}, & 0<\zeta<\zeta_0\\ 0, & \zeta<0,\quad \zeta_0<\zeta\end{cases} \tag{3.13}$$

Then, in the scattering region, $0<\zeta<\zeta_0$

$$\left.\begin{aligned}&\gamma(\omega_j,k_j;\zeta)=0\\ &\delta(\omega_j,k_j;\zeta)=\delta_{j1}\frac{k^2}{2}\left(\frac{15-2\omega_1^2}{3-\omega_1^2}\right)F_2\end{aligned}\right\}, \tag{3.14}$$

(where δ_{j1} is the Kronecker delta) and

$$\left.\begin{aligned}&a_{1x}^*+i\alpha\delta b_1^*-i\alpha S_1e^{-i\Delta kx}a_1a_2^*=0\\ &b_{1x}+i\alpha\delta a_1-i\alpha S_1e^{-i\Delta kx}b_1^*b_2=0\\ &a_{2x}+i\alpha S_2e^{-i\Delta kx}a_1^2=0\\ &b_{2x}^*+i\alpha S_2e^{-i\Delta kx}b_1^{*2}=0\end{aligned}\right\}. \tag{3.15}$$

Again writing

$$\left.\begin{aligned}&a_j=\rho_j\exp(i\phi_j),\quad j=1,\,2\\ &b_j=\sigma_j\exp(i\psi_j),\quad j=1,\,2\\ &\theta=-\Delta kx-\phi_2+2\phi_1\\ &\mu=-\Delta kx+\psi_2-2\psi_1\\ &\nu=\phi_1-\psi_1\end{aligned}\right\}, \tag{3.16}$$

$$\left.\begin{aligned}&(u,p)=(\rho_1,\sigma_1)/(2\alpha S_1E)^{\frac{1}{2}}\\ &(v,q)=(\rho_2,\sigma_2)/(2\alpha S_2E)^{\frac{1}{2}}\end{aligned}\right\}, \tag{3.17}$$

where E is given by (3.6) we can write the real and imaginary parts of (3.15) as

$$\frac{du}{d\zeta}=-uv\sin\theta+\Delta\delta p\sin\nu, \tag{3.18}$$

$$\frac{dv}{d\zeta}=u^2\sin\theta, \tag{3.19}$$

$$\frac{dp}{d\zeta}=-pq\sin\mu+\Delta\delta u\sin\nu, \tag{3.20}$$

$$\frac{dq}{d\zeta}=-p^2\sin\mu, \tag{3.21}$$

$$\frac{d\theta}{d\zeta}=\Delta S+(u^2/v-2v)\cos\theta+2\Delta\delta(p/u)\cos\nu, \tag{3.22}$$

$$\frac{d\mu}{d\zeta}=-\Delta S+(p^2/q-2q)\cos\mu+2\Delta\delta(u/p)\cos\nu, \tag{3.23}$$

$$\frac{d\nu}{d\zeta}=-v\cos\theta-q\cos\mu+\Delta\delta[(p/u)+(u/p)]\cos\nu. \tag{3.24}$$

ΔS is given by (3.9) and

$$\Delta\delta=\alpha\delta(\omega_1,k_1)l. \tag{3.25}$$

Note $O(\Delta\delta)=O(1)$. As boundary conditions we specify u, v; ν, μ, θ at the leading edge of the scattering region $\zeta=\zeta_0$, and demand that p, q vanish at the "trailing edge," $\zeta=0$. The amount of reflection at ζ_0 depends upon ΔS, $\Delta\delta$ and the length of the scattering region ζ_0. Thus, even though a first integral can be obtained from (3.18)–(3.21) as

$$(u^2+v^2)-(p^2+q^2)=C, \tag{3.26}$$

the constant of integration C is unknown until the problem is solved. The results for $\Delta S=0.3$, $\Delta\delta=1.0$ and $\zeta_0=1$ are shown in Fig. 2. The general features are:

(i) The second harmonic is not reflected, i.e., $q^2\approx 0$ everywhere. This is anticipated because in the present case, the bottom profile is "tuned" to reflect the first harmonics only (cf. Barcilon *et al.*, 1972).
(ii) Nonlinear interaction between harmonics of the incident wave is still present. The rate of energy transfer from the first to the second harmonic is slower than that for the flat bottom case because the energy of the first harmonic is partially reflected.
(iii) The total amount of reflection is measured by the reflection coefficient

$$R=\left(\frac{p^2+q^2}{u^2+v^2}\right)^{\frac{1}{2}}_{\zeta=\zeta_0}. \tag{3.27}$$

For the present case $R=0.46$.

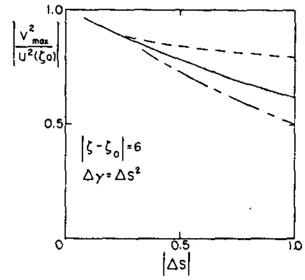

FIG. 1. Maximum second harmonic intensity $|V^2_{max}/U^2(\zeta_0)|$ vs mismatch parameter ΔS: waves propagating over flat bottom ——, waves climbing plane beach of 0.15% slope — — —, waves propagating into deep water over beach of 0.15 slope — — —.

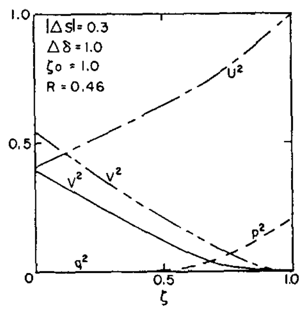

FIG. 2. Distribution of harmonic intensities vs distance along the scattering region with bottom profile given by (3.13): incident first harmonic intensity U^2 — — —, incident second harmonic intensity V^2 ——, reflected first harmonic intensity p^2 — — —, incident second harmonic intensity for the flat bottom case without reflection V^2 — — — —.

If the nonlinear terms in (3.15) are neglected, we have

$$\left.\begin{aligned} a_{1x}^*+i\alpha\delta b_1^*=0 \\ b_{1x}+i\alpha\delta a_1=0 \end{aligned}\right\}, \qquad (3.28)$$

and $a_2=b_2=0$. These are the equations one obtains when Airy's linear shallow water wave equations are used to describe the present problem. The reflection coefficient computed by using these linear equations is

$$|b_1/a_1|_{\zeta=\zeta_0}=\tanh|\Delta\delta\zeta_0|=0.76, \qquad (3.29)$$

which is about twice the one obtained from nonlinear analysis. The strong reflection is due to the strong correlation of the present bottom profile with the incident wave. The substantial difference between the linear and the nonlinear cases occurs because when harmonic interaction is present, the energy content of the incident first harmonic is partially reflected and partially transferred to the second harmonic, and the latter suffers no reflection.

4. Summary

We have only studied two simple examples in an effort to gain an understanding of the physical mechanisms involved in this problem. The results show that for long-scale topography the rate of harmonic generation depends upon the local depth which affects the detuning and thus the energy transfer between the first and second harmonics. When wave-scale and subwave-scale variations are found, reflection plays an important part. There is a competition for the partitioning of energy between reflection and harmonic generation. In Example II, the topography reflects most strongly the first harmonic and thus depletes some of its energy content so that there is less energy for transfer to the second harmonic. On the other hand, if we were to consider topography scales which would most strongly reflect the second harmonic, we would find that the first harmonic would undergo very little reflection. In that case, the growth of the second harmonic by harmonic generation will be attenuated due to reflection. The detail dependence of the reflection coefficient on ΔS, $\Delta\delta$, a more realistic bottom topography, as well as the inclusion of higher harmonics in (2.18), would be worthwhile investigations in the future.

APPENDIX

Definition of G_j

For $j=1, 2$:

$$G_1(x,X)=\mathcal{B}(\omega_1)\{[-ia_{1X}+\mathcal{A}(\omega_1,k_1;x,X)a_1+S_1a_1^*a_2e^{i\Delta kx}]e^{ik_1x}+[-ib_{1X}+\mathcal{A}(\omega_1,-k_1;x,X)b_1-S_1b_1^*b_2e^{-i\Delta kx}]e^{-ik_1x}\},$$

$$G_2(x,X)=\mathcal{B}(\omega_2)\{[-ia_{2X}+\mathcal{A}(\omega_2,k_2;x,X)a_2+S_2a_1^2e^{-i\Delta kx}]e^{i(2k_1+\Delta k)x}+[-ib_{2X}+\mathcal{A}(\omega_2,-k_2;x,X)b_2-S_2b_1^2e^{i\Delta kx}]e^{-i(2k_1+\Delta k)x}\},$$

where

$$\mathcal{B}(\omega_j)=2\omega_j(1-\omega_j^2/3),$$

$$\mathcal{A}(\omega_j,k_j;x,X)=\left[\omega_jk_j(1-\tfrac{2}{3}\omega_j^2)f+i\omega_j^3f_x-\frac{\omega_j}{k_j}\left(1-\frac{\omega_j^2}{2}\right)f_{xx}\right]\Big/\mathcal{B}(\omega_j),$$

$$S_1=\left[\omega_1^2\omega_2\left(\frac{1}{k_1}-\frac{1}{k_2}\right)+\left(\frac{\omega_2}{k_2}+\frac{\omega_1}{k_1}\right)(k_2-k_1)^2\right]\Big/\mathcal{B}(\omega_1),$$

$$S_2=(\omega_1/k_1)(2\omega_1^2+4k_1^2)/\mathcal{B}(\omega_2).$$

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