The Effects of Finite Depth on the Propagation of Nonlinear Wave Packets

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23 January 1978 and 11 May 1978

ABSTRACT

It is shown that soliton wave packets have some unexpected properties when they move on waters of finite depth and, in particular, that the wave packet forces a long gravity wave. Furthermore, the long gravity wave affects the shape of the wave packet in factors that decay linearly with increasing depth. Thus, although the carrier wave may be effectively short compared with the depth, the wave packet is not.

1. Introduction

An envelope of deep water waves is known to obey the nonlinear Schrödinger equation (Whitham, 1974). Experiments by Yuen and Lake (1975, 1977) have shown that pulses of nonlinear waves break up into solitons and that the observed properties of the solitons are in qualitative agreement with predictions obtained from the nonlinear Schrödinger equation.

Here we note that soliton wave packets have some unexpected properties when water depth is included in the analysis. The modulation envelope is shown to force a long gravity wave whose amplitude is inversely proportional to the difference between the square of the group velocity of the carrier wave and the square of the speed of a long gravity wave. Because of the forced gravity wave the depth is important even for carrier waves with values of kh as large as 10. When kh approaches 1.363 the wavelength of the packet becomes infinite and for values of $kh < 1.363$ the wave packet cannot exist.

2. Mathematical development

We consider the propagation of a group of waves of nearly identical frequencies. We introduce two time scales and two space scales, allowing a pair of rapidly varying scales to measure the local wave dynamics and a pair of space and time scales to follow the modulation envelope. The rapidly varying space and time scales are x and t, and the slowly varying modulation scales are X and T. Benney and Roskes (1969) and Davey and Stewartson (1974) have derived the equations governing such a modulation envelope. Previous interest has focused on the stability of wave packets and in three-dimensional effects. Here we focus on the depth dependence of coherent wave packets.

The inviscid equations and free surface conditions for irrotational fluid motion are applied in the domain $y > -h$, where h is the water depth and $y = \epsilon\eta$, where η is the free surface elevation.[1] The velocity potential and free surface elevation are expanded in the series

$$\phi = A_0(y,X,T) + A_1(y,X,T)e^{i\chi} + \epsilon A_2(y,X,T)e^{2i\chi} + \text{conjugate} + O(\epsilon^2), \quad (2.1)$$

$$\eta = b_1(X,T)e^{i\chi} + \epsilon b_0(X,T) + \epsilon b_2(X,T)e^{2i\chi} + \text{conjugate} + O(\epsilon^2), \quad (2.2)$$

where X and T are slow space and time scales $X = \epsilon x, T = \epsilon t$. The basic wavetrain of wavenumber k has a central frequency $\omega(k)$ and the phase function for the wavetrain is $\chi = kx - \omega(k)t$. Since the potential must satisfy Laplace's equation,[2] we find

$$A_1 = a_1 \frac{\cosh k(y+h)}{\cosh kh} + \epsilon\left[-i\frac{\partial a_1}{\partial X}(y+h)\frac{\sinh k(y+h)}{\cosh kh}\right] + \epsilon^2\left[-\tfrac{1}{2}\frac{\partial^2 a_1}{\partial X^2}(y+h)^2 \times \frac{\cosh k(y+h)}{\cosh kh}\right] + \ldots, \quad (2.3)$$

$$A_0 = a_0 + \epsilon^2\left[-\tfrac{1}{2}(y+h)^2\frac{\partial^2 a_0}{\partial X^2}\right] + \ldots, \quad (2.4)$$

[1] As is customary ϵ is the small amplitude of the wave.

[2] Because Laplace's equation is solved subject to the restriction that $\epsilon kh \ll 1$, the equations are not valid for large values of kh. A solution for $kh \gg 1$ was derived by Larsen (1978).

0022-3670/78/0923-0925$05.00
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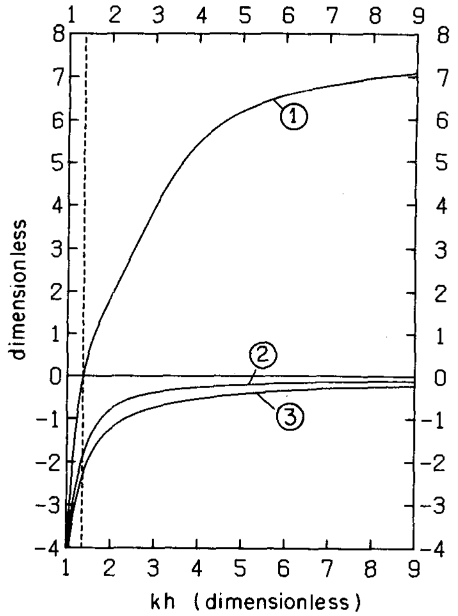

FIG. 1. The left-hand sides of Eqs. (2.18) [curve 1], (2.20) [curve 2] and (2.21) [curve 3]. Curve 1 is the inverse of the square of the width of the wave packet. It vanishes when $kh = 1.363$. Curve 2 is the amplitude of the long gravity wave and curve 3 the amplitude of the velocity potential for the long gravity wave. The dashed line is inserted at $kh = 1.363$.

$$A_2 = a_2 \frac{\cosh 2k(y+h)}{\cosh 2kh}, \tag{2.5}$$

where each $a_i = a_i(X,T)$ and $a_1 = -(ig/\omega)b_1$.

Benney and Roskes (1969) show that the modulation envelope is determined by a set of three coupled equations, which in a coordinate system (X',T') moving with the group velocity of the wave packet are

$$\frac{\partial b_1}{\partial T'} - \frac{i\epsilon}{2}\frac{\omega_{11}g^2}{\omega^3} b_{1_{X'X'}} + i\epsilon b_1\left[k\frac{\partial a_0}{\partial X'} + \frac{\omega k}{2}\frac{\operatorname{sech}^2 kh}{\tanh kh} b_0\right] + i\omega\epsilon k^2\beta b_1^2 b_1^* = 0, \tag{2.6}$$

$$\frac{\partial a_0}{\partial T'} - c_g\frac{\partial a_0}{\partial X'} + gb_0 + gk\frac{\operatorname{sech}^2 kh}{\tanh kh} b_1 b_1^* = 0. \tag{2.7}$$

$$\frac{\partial b_0}{\partial T'} - c_g\frac{\partial b_0}{\partial X'} + ha_{0_{X'X'}} + \frac{2gk}{\omega}\frac{\partial}{\partial X'} b_1 b_1^* = 0, \tag{2.8}$$

where

$$\beta = \tfrac{1}{2}\left[4 + \lambda\frac{\operatorname{sech}^2 kh}{\tanh kh} + \mu(2\tanh 2kh \tanh kh - 4)\right], \tag{2.9}$$

$$\omega_{11} = \tfrac{1}{4}[-(kh)^2 \operatorname{sech}^2 kh(1 + 3\tanh^2 kh) + 2kh \tanh kh \operatorname{sech}^2 kh - \tanh^2 kh], \tag{2.10}$$

$$\lambda + \mu \tanh 2kh = \frac{1}{\tanh kh}, \tag{2.11}$$

$$\lambda + 2\mu \tanh kh = \tfrac{1}{2}\left(3\tanh kh - \frac{1}{\tanh kh}\right), \tag{2.12}$$

and c_g is the group velocity of the carrier wave. Eqs. (2.7) and (2.8) together represent a long gravity wave equation driven by forcing proportional to the modulation energy. Because the forcing travels at the group velocity the long gravity wave has an amplitude increasing as the group velocity nears the long wave velocity. Eqs. (2.6)–(2.8) admit solutions in terms of the Jacobian elliptic functions. Rather than include the full set of solutions in the analysis we investigate the limiting case of a solitary wave packet.

We seek solutions to (2.6)–(2.8) of the form

$$b_1 = \alpha_0 e^{i(krX' - s\omega T')} \operatorname{sech} kB(X' - VT'), \tag{2.13}$$

$$a_0 = \alpha_1 \tanh kB(X' - VT'), \tag{2.14}$$

$$b_0 = \alpha_2 \operatorname{sech}^2 kB(X' - VT'). \tag{2.15}$$

Direct substitution in the modulation equations yields the following relations:

$$\frac{V\omega}{g} = \frac{\epsilon r\omega_{11}}{\tanh kh}, \tag{2.16}$$

$$s = \frac{\epsilon r^2\omega_{11}}{2\tanh^2 kh} - \alpha, \tag{2.17}$$

$$B^2 = \frac{2\alpha\tanh^2 kh}{\omega_{11}}, \tag{2.18}$$

$$\alpha = -\frac{\beta\alpha_0^2 k^2}{2}\left[1 + \frac{1}{c_\Delta^2\beta\tanh kh} \times \left(2 + \frac{2(V+c_g)}{c}\operatorname{sech}^2 kh + \frac{kh\operatorname{sech}^4 kh}{2\tanh kh}\right)\right], \tag{2.19}$$

$$\frac{\alpha_2}{\alpha_0(\alpha_0 k)} = \frac{1}{c_\Delta^2}\left[\frac{2(V+c_g)}{c} + \frac{kh}{\sinh kh\cosh kh}\right], \tag{2.20}$$

$$\frac{\omega}{gk}\frac{\alpha_1}{\alpha_0} = \frac{1}{c_\Delta^2}\left[2 + \frac{(V+c_g)}{c}\frac{\operatorname{sech}^2 kh}{\tanh kh}\right], \tag{2.21}$$

where

$$c_\Delta^2 = \frac{k}{g}[(V+c_g)^2 - gh] \tag{2.22}$$

and c is the phase speed $(=\omega/k)$.

Since r is of order unity (or less), V is a small number when compared with c_g. Thus, the essential features of the solution may be seen by examining the equations with $V = 0$, that is, the wave packet moves at the group velocity. In Fig. 1 we illustrate the parameters for this limiting case. First, we note that ω_{11} which is a dimensionless form for the curvature of the group velocity is always negative. Thus, a wave packet may exist only when α is negative (2.18).

Illustrated in Fig. 1 are the left-hand sides of Eqs. (2.18), (2.20) and (2.21). The width of the wave packet, described by B, becomes infinite as B approaches zero. This occurs for $kh = 1.363$, the dashed line in Fig. 1, and represents the shallowest water in which a wave packet may propagate. With kh near 1.363 the wave packet is long compared to ϵx and the expansion used in this paper is not valid. Johnson (1977) discusses soliton behavior near this critical value of kh. The two other curves in Fig. 1 are the amplitude and velocity potential of the forced gravity wave. The amplitude increases with diminishing values of kh. Because the long gravity wave modifies the modulation envelope (2.6) the width of the soliton depends on the depth.

The slow rate at which B^2 approaches its asymptotic value of 8 is surprising. For kh values on the order of 10, B^2 is still 10% removed from its asymptotic value. The reason for this is that the speed of long waves increases as the square root of the depth. Eqs. (2.7) and (2.8) for a_0 and b_0 are the nonhomogeneous shallow water wave equations driven by the modulation envelope.

3. The bottom pressure

The pressure fluctuation P is determined from the Bernoulli equation, which at the bottom, $y = -h$, is

$$\frac{P}{\rho} - gh = -\phi_t - \epsilon\phi_T - 2(\phi_x^2 + 2\epsilon\phi_x\phi_X + \epsilon^2\phi_X^2). \quad (3.1)$$

The fluid density is ρ. Note that (3.1) is in a stationary coordinate system (X,T). Since the pressure field associated with the carrier wave exhibits no unusual behavior we examine only the low-frequency component of the pressure. Inserting the expansion for the velocity potential (2.1) into (3.1), we find

$$\frac{P}{\rho} - gh = gk\alpha_0^2\left\{\frac{-(1+\alpha_0 k)}{\tanh kh}\operatorname{sech}^2 kh + \frac{\alpha_0 k}{c_\Delta^2} \times \left[\frac{2(V+c_g)}{c} + \frac{kh}{\sinh^2 kh}\right]\right\} \times \operatorname{sech}^2 kB[X-(c_g+V)T]. \quad (3.2)$$

We have replaced the small parameter ϵ with α_0 in order to present the solution in physical terms. Because c_Δ^2 is linear in kh the pressure fluctuations are inversely proportional to depth.

In deep water the term involving c_Δ^2 dominates the bottom pressure fluctuations. Although small in magnitude near the surface the signature of the forced wave may dominate at depth. Low sea level is predicted from (3.2) beneath the highest waves in the wave packet. This agrees with observations of surf beats (Longuet-Higgins and Stewart, 1963).

Acknowledgment. This study was supported by the National Science Foundation under Grant OCE77-17660 and represents Contribution No. 1027 of the Department of Oceanography, University of Washington.

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