NOTES AND CORRESPONDENCE

Dispersion of a Passive Contaminant in Oscillatory Fluid Flows[1]

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ABSTRACT

We consider dispersion in an oscillatory boundary layer in an effort to ascertain the relative importance of velocity shear and phase. Although salt is not strictly a passive contaminant, the model considered is relevant to the induced transport of salt in a tidal estuary. The results indicate a transport directed toward higher concentrations near a boundary whenever the velocity phase near that boundary leads the interior flow.

1. Introduction

Here we examine dispersion in an oscillatory boundary layer in order to ascertain the relative importance of velocity shear and phase. Dispersion in an oscillatory flow has application to both the flow of blood in the body and to the intrusion of salt into a tidal estuary. Because the prime motivation for this study was the estuary problem, we shall refer to the dispersant as salt. However, the conclusions are sufficiently general that they have application outside of the estuary problem.

In the middle reaches of many partially mixed estuaries the tidally averaged horizontal gradient of salinity is observed to be independent of depth and to vary nearly linearly with distance into the estuary (Pritchard, 1954). As a consequence, much of the observed tidal variation in salinity results from advection of the mean horizontal gradient of salinity. Longitudinal transport of salt will result if there is a nontrivial correlation between the tidal velocity u and the tidal portion s of the total salinity field. The purpose of this note is to establish that the correlation may be negative and consequently there are regions within the fluid that exhibit anomalous dispersion. The total transport of salt, which is the integral of the velocity salinity correlation, is always directed toward decreasing salinity.

Anomalous dispersion or dispersion in the direction of increasing salinity results from a phase dependence in the velocity profile which may be either in the lateral or vertical planes. In the vertical plane salt is transported seaward near the bottom and in the lateral plane it is transported seaward near the shore. Although no proof has been found for arbitrary velocity profiles, the results indicate that the requirement for anomalous dispersion is that the phase of the tide lead the mid-channel tide near the boundary whether it be the bottom or the shore.

Although the negative transport could be viewed as just diminishing the effective dispersion, it can also represent a real factor in the dynamics of a partially mixed estuary. In a partially mixed estuary the basic balance is between the fresh water runoff, the gravitational circulation and dispersion of salt into the estuary. Vertical mixing of salt is required to maintain the balance. If, however, the negative flux of salt near the bottom is used to balance the gravitational circulation a dynamic balance could be achieved with less vertical mixing.

In the lateral direction anomalous dispersion near the shore will enhance the cross-channel density gradient, thereby increasing the cross-channel circulation. Fisher (1976) and Smith (1976) discuss the consequences of the cross-channel density variations. We omit a discussion in order to restrict the topic to the prime conclusions.

2. The transport equations

If the horizontal gradient of the fluctuating salinity field is small compared with the mean horizontal gradient and cross-channel variations are neglected, then the fluctuating salinity s obeys

$$\frac{\partial s}{\partial t}+u\frac{dS_0}{dx}=\frac{\partial}{\partial z}K\frac{\partial s}{\partial z}. \tag{1}$$

In the above, u is the periodic tide velocity, dS_0/dx

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the mean gradient of salinity and K the vertical diffusivity. It is assumed that the estuary is of constant depth h and of uniform cross section. The coordinate system is x, z, t, with x increasing on moving away from the sea, z increasing with distance above the bottom and t increasing with time. Eq. (1) is identical to that used by Bowden (1965) in his study of induced salt flux. His study was restricted to a shallow estuary where the tide velocity was assumed to have a constant phase.

For our model of the tide velocity we consider a horizontally uniform flow driven by a given oscillatory pressure gradient, i.e.,

$$\frac{\partial u}{\partial t}=\frac{\partial U}{\partial t}+\frac{\partial}{\partial z}N\frac{\partial u}{\partial z}. \tag{2}$$

In (2), $\partial U/\partial t$ is used to represent the oscillatory forcing of the tide. Because the objective of this study is to examine the interdependence of shear and phase in the tidal flow on dispersion we omit the complications of advection in the horizontal momentum equations. In the analysis we take z to be the vertical coordinate. However, z may also represent a lateral coordinate in a rectangular estuary of sidewall friction only.

Appropriate boundary conditions for (1) are a vanishing of the vertical salt flux at the bottom and surface of the estuary. For (2) we assume the velocity shear to vanish at the surface and the bottom velocity to vanish. In a turbulent flow the eddy viscosity is large in the interior and small near the bottom. Solutions to (2) can be found which match a log layer near the bottom of an interior flow of constant viscosity. If in the bottom layer the viscosity varies linearly with distance above the bottom the solutions are given by the modified Bessel functions. The major consequence of the matching is that the apparent shear seen by the interior flow is smaller than that experienced by a laminar fluid. Since the velocity is small in the log layer the contribution to the salt flux from this layer is small. In order to avoid complications involved with the matching, we seek solutions to (2.2) with a constant eddy coefficient subject to arbitrary shear conditions at the bottom.

If we scale time on the tidal frequency σ, velocities on the amplitude of the forcing U_0, the vertical coordinate on the boundary layer thickness $\delta = (2N/\sigma)^{\frac{1}{2}}$, and salinity on the magnitude of the tidal advection of the mean salinity gradient $(-\sigma/U_0)dS_0/dx$, the problem reduces to three parameters—the imposed bottom stress, the Schmidt number $p = N/K$ and a nondimensional depth $\beta = h/\delta$. The negative sign is included in the scaling of salinity in order that positive values S correspond to increases in salinity. Thus letting $U = e^{it}$ and introducing a reduced velocity F such that

$$u = [1 - F(z)]e^{it}, \tag{3}$$

Eqs. (1) and (2) become

$$2is=2(1-F)+\frac{1}{p}\frac{\partial^2 s}{\partial z^2}, \tag{4}$$

$$2iF=\frac{\partial^2 F}{\partial z^2}. \tag{5}$$

The salt flux is the expected value over a tide cycle of the product of velocity and salinity. Using carets to denote dimensional quantities and asterisks for the conjugate variable, the salt flux is

$$q=\frac{-\hat{q}\sigma}{U_0^2 dS_0/dx}=\tfrac{1}{4}[(1-F)s^*+(1-F)^*s]. \tag{6}$$

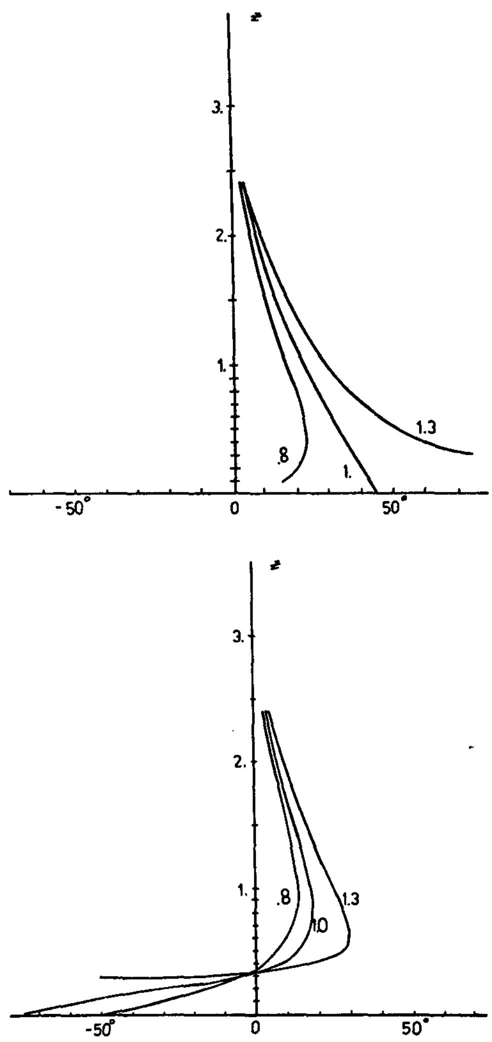

FIG. 1. Phase of the tide velocity (deg) as a function of depth. The curves are for relative magnitudes of the shear of 0.8, 1.0 and 1.3. In (a) the phase of the shear is identical to that in a laminar flow and in (b) θ is increased to 20°.

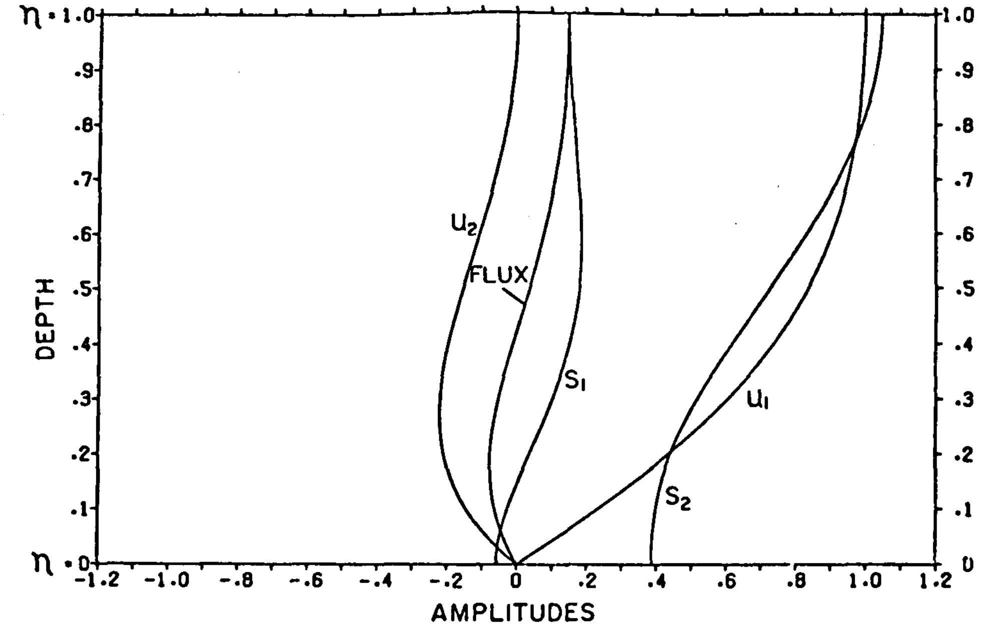

FIG. 2. Vertical profiles of velocity, salinity anomaly and salt flux. The scale for the velocity profiles is fractions of the surface velocity. Salinity is measured according to units of $-(U_{sfc}/\sigma)(dS_0/dx)$ and flux in units of $-(U^2_{sfc}/\sigma)(dS_0/dx)$. This graph is for an estuary of depth 2.5δ and of Schmidt number $p=1.5$.

The transport per unit width of estuary is

$$T=\frac{-\hat{T}\sigma}{\delta U_0^2 dS_0/dx}=-\int_0^\beta q dz. \qquad (7)$$

Letting χ denote homogeneous solution of (2.4), we find

$$q=[i/4(p-1)][(F-p\chi)+p(\chi F^*)-\text{conjugate}], \qquad (8)$$

$$T=-\int_0^\beta q dz=[-1/8(p-1)]\Big[F^*(0)\tau+F(0)\tau^* + \int_0^\beta \Big(\frac{\partial\chi}{\partial z}\frac{\partial F^*}{\partial z}+\frac{\partial\chi^*}{\partial z}\frac{\partial F}{\partial z}\Big)dz\Big], \qquad (9)$$

where $\tau = (\partial F/\partial z)_{z=0}$ is a complex number whose value determines the velocity shear at the bottom of the estuary. In arriving at (8) we note that the solution of (4) is

$$s = -i + [ip/(p-1)](F-\chi), \qquad (10)$$

and thus a vanishing of the salinity gradient at the bottom implies that $\partial F/\partial z$ and $\partial \chi/\partial z$ are equal at the bottom.

3. A deep estuary

Weisberg and Sturges (1976) estimated from velocity profiles taken in Narragansett Bay that the boundary layer thickness is on the order of 2–4 m, values which are consistent with eddy viscosities ranging from 4 to 12 $cm^2\ s^{-1}$. For an estuary of depth $>5\delta$ we may approximate the solutions to (4) and (5) by assuming the depth of the estuary is infinite. In this limit the solutions are

$$\chi = (r/p^{1/2}) \exp[-p^{1/2}(1+i)z+i\theta], \qquad (11)$$

$$F = r\exp[-(1+i)z+i\theta]. \qquad (12)$$

We have substituted for the shear in these equations the value $\tau = -r(1+i)e^{i\theta}$. This choice for the shear is a perturbation about $-(1+i)$, the shear which results with laminar flow conditions. The magnitude of the velocity shear is proportional to r and independent of θ.

Two examples serve to indicate the patterns of velocity profiles achieved by varying r and θ. In the first example (Fig. 1) θ is fixed at zero and r is permitted to vary. Shown in the figure is the phase of the velocity relative to the surface velocity as a function of depth. Decreasing r diminishes the shear and also diminishes the phase variations, whereas increasing r has the opposite effect. If θ is increased to 20°, for depths above 0.3δ, the pattern is the same as in Fig. 1a but with reduced phase variations. At depths below 0.3δ the phase reverses from that of a Stokes layer, with lag not lead at depth and at these depths the flow cannot be construed as a realistic representation of what might happen in a turbulent boundary layer. Accordingly, the region below 0.3δ is excluded from the flux calculations. Negative values of θ increase the phase variations for fixed r.

We illustrate in Fig. 2 profiles of velocity, salinity and salt flux for a flow of Schmidt number 1.5 and for a depth of 2.5 times the boundary layer thickness. The exact solution of (2) and (3) is used (i.e., the

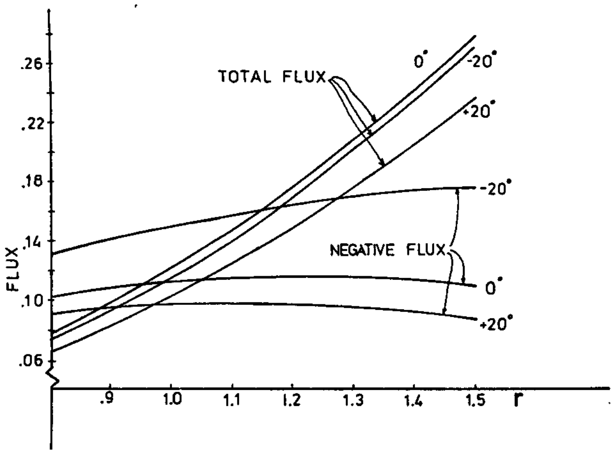

FIG. 3. Relative contribution of the negative flux to the total flux as a function of total shear r. Three curves are shown for each quantity based upon phase shifts θ of $-20°$, $0°$, $20°$ relative to a laminar flow.

boundary conditions are applied at the surface) and we use the Stokes solution which has a vanishing bottom velocity. Shown in the graph are two curves for velocity and salinity. The curves labeled 1 are the components in phase with the surface flow and those subscripted with 2 lag by 90°. The velocities are normalized to a unit surface current. The salt flux is negative in the lower half of the flow and positive in the upper half. The integral of the salt flux is positive. In Fig. 3 we show the trends in both the total salt flux and the negative contribution to the salt flux as a function of the magnitude r of the bottom shear. As the total shear is increased, r increases and the contribution of the negative flux becomes less important. For fixed shear the increase in phase variation which occurs from diminishing the value of θ has a great effect on the negative transport. Since the total transport is relatively unaffected by changes in θ, changes in the negative contribution are balanced by increased positive flux near the surface. In general, increasing the phase variations increases both the negative and positive contributions to the total flux for a fixed amount of shear.

In Fig. 4 we illustrate the time behavior of the salinity field during a tide cycle. In order to effect the contour plot the range of salinities was divided into 10 intervals and contours drawn for the mid range of each of these intervals. Near high slack an apparent instability is indicated, a result of the fact that the mean salinity profile is not included in the anomaly plot. The basic requirement for stability is that the estuary have sufficient stratification to compensate for the greater advection of dense water near the surface. The total transport of salt per unit width in an estuary of great

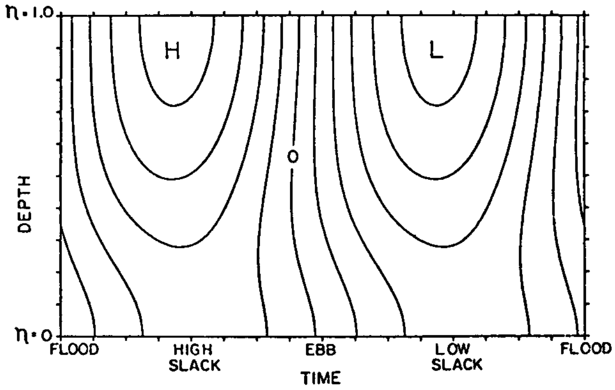

FIG. 4. Contour plot of salinity anomaly as a function of time for a Schmidt number of 1.5 and a boundary layer depth ratio of 2.5.

depth is

$$T=\frac{r^2 p^{\frac{1}{2}}}{8(p^{\frac{1}{2}}+1)(p+1)}. \qquad (13)$$

This result was derived by Mork (1970) who showed that the transport is maximized if the Schmidt number is approximately $\frac{2}{3}$.

4. Conclusion

The results presented here are based on constant coefficients for the eddy viscosity and diffusivity. The artifact of varying the bottom shear serves to create a range of velocity profiles. In no case is the conclusion that the flux is negative near to the bottom altered, if the flow leads the surface flow.

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