FIELD MEASUREMENT OF THE SPEED OF PROPAGATION OF WIND WAVES AS A FUNCTION OF WAVELENGTH[1]

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ABSTRACT

By means of an example it is shown that cross-spectral analysis may be used to obtain a direct empirical measurement of short wave velocity as a function of wavelength. The results indicate that the wave speeds obtained from the classical Stokes formula are too low, and that the theoretical results of Milne-Thompson are in better accord with observation. It is also indicated that, in this empirical study and many others, the wave steepness is nearly independent of wavelength for the high frequency side of the wave spectrum.

Many textbooks contain analytical expressions for the speed of propagation of short surface waves as a function of wavelength, but the literature of water waves contains few direct measurements. It is the purpose of this note to outline a technique for obtaining this relation empirically, and to present the results of one such measurement and some considerations arising from this measurement.

THE EXPERIMENT

Two fast response water level recorders were placed 38 cm apart on a line normal to the wave crests in the western part of Lake Mendota, Wisconsin on 6 December 1960. The observations were made at about 1430, with a wind velocity of 8 m/sec (2 m above the water), water temperature 3.4°C and air temperature –5°C at Truax Field about 7 miles away. Each water level recorder consisted of fine silver wires attached to lucite rods, the resistance of the water between the two wires providing one arm of an AC bridge. The output of the bridge was rectified and recorded on a Minneapolis-Honeywell Visicorder. This provided a visual record of the variations of water level as the wave train passed each recorder. Observation continued for 3 min (Fig. 1).

During the time of observation the lapse rate from 50 cm to 200 cm was –0.17°C, the roughness height Z_0 extrapolated from the measured wind profile (under adiabatic assumptions) was 0.0002 cm, and the extrapolated wind speed at a height of 1 cm was 5 m/sec.[2]

ANALYSIS

If there had been a single simple wave train of uniform frequency, the phase velocity of the waves could have been computed directly from the phase lag of the downwind recorder behind the upwind, the observed frequency and the spacing of recorders. Since the wave train was composed of many frequencies, however, a more sophisticated treatment was necessary. In particular cross-spectrum analysis was used.

For our purposes it will suffice to say that the variance spectrum of a particular time series shows us the distribution of the variance of the series as a function of frequency. Variance spectral analysis of wave records such as those described above have been discussed by Hicks (1960) and Kinsman (1960). For the comparison of two time series the cross-spectrum may be used. (See Panofsky, 1958, for a simple treatment.) The two parts of the cross-spectrum are the co-spectrum $\Phi(\omega)$ and the quadrature spectrum $\Psi(\omega)$, which give respectively the in-phase and out-of-phase covariance of the two series as a function of frequency. Since the phase angle $\Theta(\omega)$ between the two series is given by

$$\Theta(\omega) = \arctan\,[\Psi(\omega)/\Phi(\omega)] \qquad (1)$$

[1] The substance of this paper was presented at the First National Coastal and Shallow Water Research Conference, Los Angeles, October, 1961.

[2] This wind speed corresponds very closely to the observed speed of the waves at the spectral peak (Fig. 3).

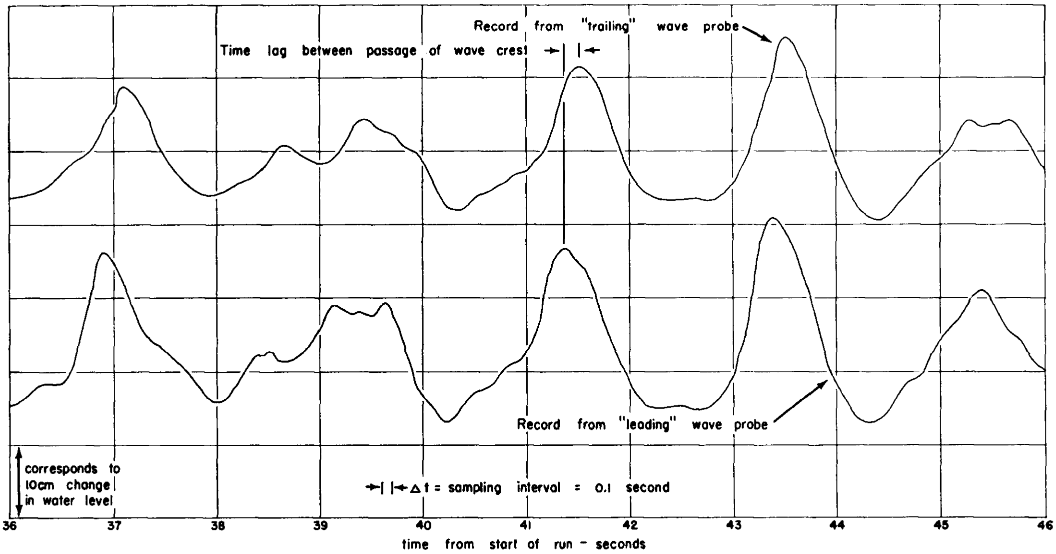

FIG. 1. Simultaneous records of water level at two probes 38 cm apart on a line normal to the wave crest, Lake Mendota, 6 December 1960. The water level was recorded continuously, with an overall system time constant of about 0.06 sec. For digital computation of the spectra values were read from the record at 0.1-sec intervals.

and the time lag, $\tau(\omega)$, between the two series is defined by

$$\tau(\omega) = \Theta(\omega)/\omega \qquad (2)$$

we can use the cross-spectrum of the "upwave" and "downwave" records to obtain the phase velocity or wave speed as a function of wavelength. This technique has been used by Flexer (1960) to obtain the speed of atmospheric temperature waves but apparently has not been used to study water waves.

From expression (2) above and the basic wave definitions we obtain

$$c(\omega) = D/\tau(\omega) \qquad (3)$$

and

$$\lambda(\omega) = c(\omega)/f \qquad (4)$$

where $c(\omega)$ is the phase velocity, $\lambda(\omega)$ the wave length, D the probe spacing, and f the wave frequency. All necessary parameters thus come from the known experimental dimensions and the cross-spectral estimates.

For digital computation of the cross-spectrum the water level values were read off at intervals of 0.1 sec. This resolution was justified by the overall measured time con-

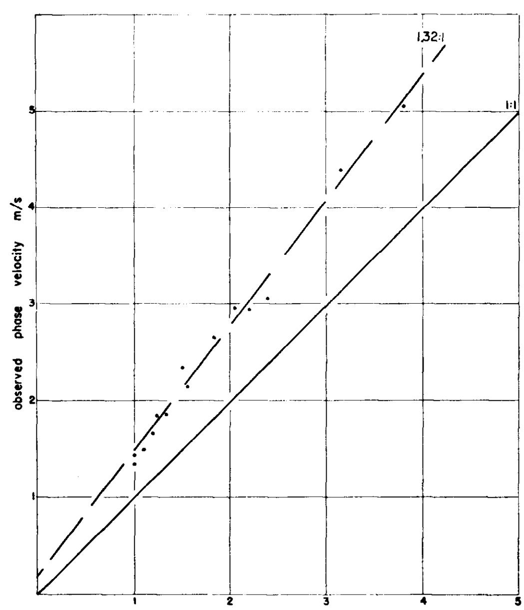

FIG. 2. Plot of the observed *vs.* computed wave speeds. The line of best fit (dashed line) intersects the ordinate above the origin at the velocity of the water on which the waves were superimposed. The 1.32:1 slope of the line shows that the classical phase velocity formula must be multiplied by an approximately constant factor to yield the observed phase velocities.

stant of 0.06 ± 0.04 sec experimentally determined for the system.

RESULTS

As a result of the preceding analysis a plot of wave speed as a function of wavelength was obtained which showed the square relationship contained in the classical expression for short waves

$$c^2 = (g\lambda/2\pi) \tag{5}$$

As shown in Figure 2, the correlation be-

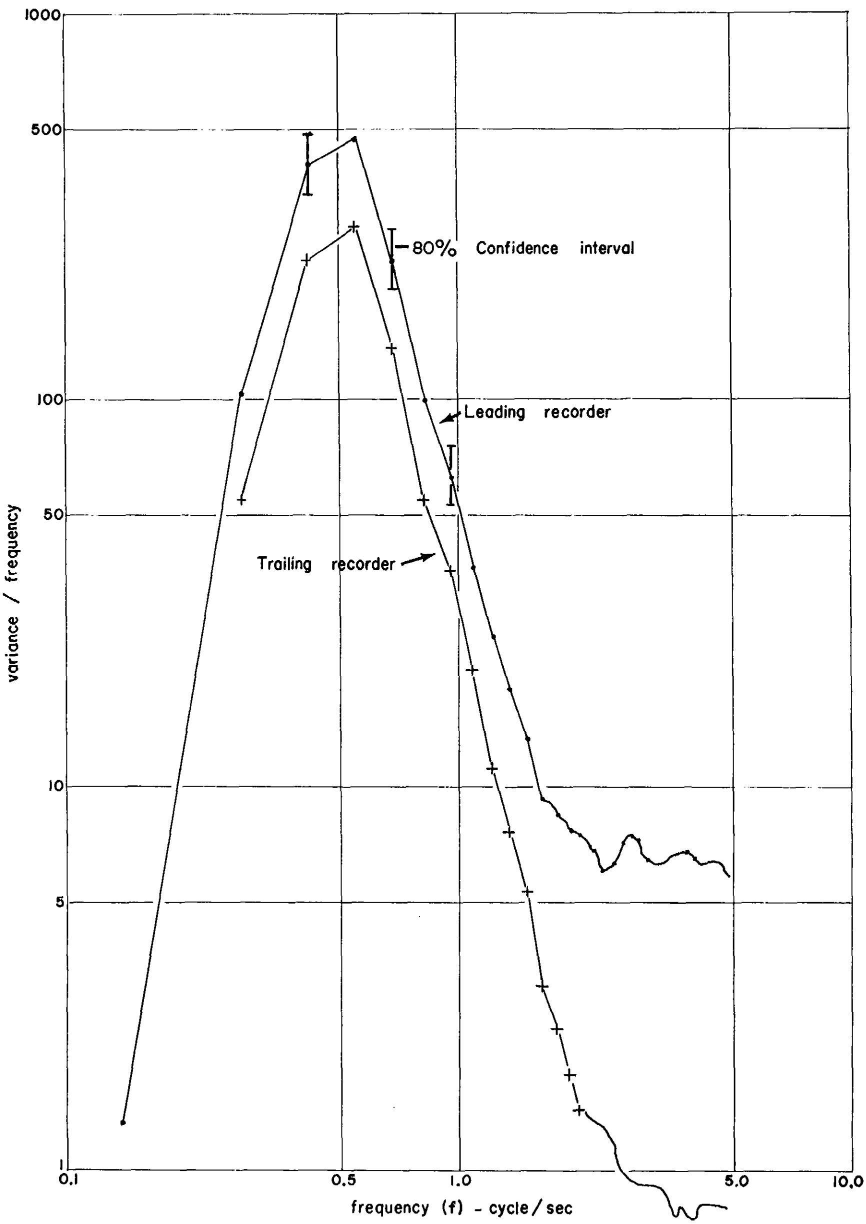

FIG. 3. Variance spectra of the waves recorded. It is unlikely that the trailing record contains less variance due to deterioration of the waves in 38 cm travel. It is probable that the trailing recorder was somewhat less sensitive than the leading recorder, despite efforts to make them identical. There is, however, no essential difference in the form of the spectra.

tween the observed wave speed and that given by expression (5) is nearly perfect, but with a nearly constant multiplicative factor of 1.32.[3] The wave train was moving from water over 14 m deep onto a 4-m shoal but correction for the depth of water was negligible and for the purposes of this discussion it seems reasonable to treat the waves as though they were in deep water.

Since the classical expression was developed for infinitesimal, sinusoidal waves, and the observed wave train was clearly finite and roughly trochoidal, it would appear that an expression for finite waves should be used. One such, given by Milne-Thompson (1960) for finite trochoidal waves is

$$c^2 = (g\lambda/2\pi)\ \exp\ (3\pi H/\lambda) \qquad (6)$$

where H is the wave height, and thus H/λ is the wave steepness. Using the empirical factor of 1.32 to compute H/λ from expression (6) gives a wave steepness of 0.06. Reference to Figure 1 shows that this is clearly the right order of magnitude.

Since the factor 1.32 is independent of wave speed and thus wavelength or frequency, there is an implied independence of wave steepness and wavelength, or

$$H/\lambda = \text{constant} \qquad (7)$$

An independent test of this result may be obtained from the variance spectra of waves. Many of these have been discussed in the literature. On the high frequency side of the spectral peak, in the wavelength range used here, most of these spectra have slopes on log-log plots of about –5. This in turn implies a relation of the form

$$H^2/(1/T) \sim T^5 \qquad (8)$$

where T is the wave period, but since

$$T = \lambda/c \text{ and } c^2 \sim \lambda$$

expression (8) reduces to the statement (7), $H/\lambda = \text{const.}$

The spectra of the wave train used in this study also have slopes near –5, implying a wave steepness independent of wavelength, just as did the results of the wave speed analysis.[4]

SUMMARY

The use of cross-spectral analysis of the output of a pair of fast response wave recorders allows the direct empirical determination of wave speed as a function of wavelength. The results of one such analysis show that use of the classical expression for the phase velocity of short waves yields values that are too low by about a quarter. Application of Milne-Thompson's formula for the speed of finite trochoidal waves gave results more in accord with observation but also suggested that the wave steepness was independent of wavelength in the particular case observed. The average variance spectrum of short waves suggests the same independence of wavelength and steepness.

From these results one might conclude that a fruitful line of investigation might be to examine the slope of the high frequency side of the wave spectra to see whether those with a relative excess of high frequency variance (slope less than –5) might not be associated with a growing wave train, while those with steeper slopes might represent decaying wave systems.

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[3] This analysis is restricted to those waves longer than about 0.5 m and shorter than 9 m.

[4] Phillips (1962) has also reasoned on the basis of dimensional considerations, but with a different argument, that the exponent must be minus five.