A Method for the Routine Analysis of Pitch-and-Roll Buoy Wave Data

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(Manuscript received 11 June 1987, in final form 6 January 1988)

ABSTRACT

A simple, computationally efficient method is proposed as a standard procedure for the routine analysis of pitch-and-roll buoy wave data. The method yields four directional model-free parameters per frequency: the mean direction, the directional width, the skewness, and the kurtosis of the directional energy distribution. For most applications these parameters provide sufficient directional information. The estimation procedure and error characteristics of the parameter estimates are discussed and illustrated with computer simulated data. An optional interpretation of the combination of skewness and kurtosis as an indicator of uni-modality of the directional energy distribution is suggested and illustrated with field observations.

1. Introduction

Information on the directional characteristics of wind waves is of great importance for a wide range of engineering and scientific purposes; e.g., calculation of wave loads on offshore structures, calculation of sediment transport, real-time swell forecasting or validation of wave prediction models. The desired degree of detail may range from the complete two-dimensional spectrum via a limited number of directional parameters per frequency to a very condensed description in terms of a few characteristic parameters, such as a dominant wave direction.

For many applications involving wave directionality the mean wave direction and the directional width per frequency provide quite sufficient information. To obtain this information, a large variety of measurement techniques have been developed. Each of these techniques has its own advantages with respect to analysis and operations. The technique based on the pitch-and-roll buoy has undoubtedly been used most frequently and has now reached an operational stage. This holds in particular for the directional waverider, the so-called WAVEC-buoy, which has been in operation since 1984 (Van der Vlugt 1984). For such a pitch-and-roll buoy, a large variety of analysis methods can be used which are all based on a cross-spectral analysis of the three basic heave and slope signals of the buoy (e.g., Longuet-Higgins et al. 1963; Mitsuyasu et al. 1975; Long and Hasselmann 1979; Long 1980; Hasselmann et al. 1980; Van Heteren 1983; Lawson and Long 1983). Two classes of methods can be distinguished. In the methods of the first class, attempts are made to reconstruct the directional energy distribution (at each frequency), whereas in the methods of the second class some characteristic directional parameters are estimated (at each frequency). It should be noted that these methods, including the one proposed in the present paper, apply equally well to other multivariate time series of wave-related quantities such as surface elevation and subsurface orbital velocity in two orthogonal directions (e.g., Forristall et al. 1978) to estimate wave directional characteristics.

The reconstruction methods are based on several principles including truncated Fourier series and parametric models. Longuet-Higgins et al. (1963) showed that only the first four Fourier coefficients of the directional energy distribution per frequency can be derived from the auto-, co-, and quadspectra of the buoy signals (appendix A). These four Fourier coefficients are used to approximate the directional distribution $D(\theta)$ with a truncated Fourier series (Longuet-Higgins et al. 1963):

$$D(\theta) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{2} \{a_n \cos(n\theta) + b_n \sin(n\theta)\}\right]. \quad (1)$$

However, this method may result in an estimated directional distribution with negative values, whereas $D(\theta)$ is positive by definition. To avoid these negative values Longuet-Higgins et al. (1963) suggested to convolve the directional distribution from Eq. (1) with a

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weighting function. This arbitrarily broadens the estimated directional distribution considerably. This estimation of $D(\theta)$ should therefore be avoided.

The methods of parametric models are based on an a priori assumed shape of the directional energy distribution. Many different shapes can be assumed but the number of independent shape parameters should not be more than four since only four Fourier coefficients are available. A frequently used two-parameter model is the one suggested by Longuet-Higgins et al. (1963):

$$D(\theta) = A \cos^{2s}\{(\theta - \alpha_0)/2\} \tag{2}$$

in which α_0 is the mean wave direction, s is a parameter which controls the directional width of $D(\theta)$, and A is a normalization constant. A variation on this model is the four-parameter double $\cos^{2s}$-model used by Hasselmann et al. (1980) and Van Heteren (1983). Another four parameter model has been suggested by Borgman and Yfantis (1978).

Other methods which attempt to reconstruct the directional distribution are the variational method of Long and Hasselmann (1979), the data adaptive method of Oltman-Shay and Guza (1984) and the maximum entropy method of Lygre and Krogstad (1986). These methods provide a relatively detailed reconstruction of the directional distribution but the interpretation of the results of these methods does not seem to be a routine matter. All these methods of the first class suffer from various shortcomings for routine applications: they may suggest a misleading directional resolution, the shape assumptions may not be justified or the results require skilled interpretation. Reconstruction of $D(\theta)$ from pitch-and-roll buoy data should therefore be undertaken only in a nonroutine manner, if the distribution of wave energy over directions is a strictly required quantity in any further processing.

The methods of the second class are not as well established as those of the first class. These methods are used to estimate characteristic parameters of $D(\theta)$ such as a mean direction and a directional width. Strictly speaking one can determine such parameters from an estimate of $D(\theta)$ which is obtained with one of the above indicated reconstruction methods. However, this would only compound the shortcomings of that type of analysis. We suggest in the present study a method which estimates the parameters directly from the Fourier coefficients without model assumptions thus either avoiding or reducing the above shortcomings. With this method we wish to arrive at directional wave parameters that:

(i) are descriptive,
(ii) are model-free,
(iii) can be expressed analytically in terms of the above mentioned four Fourier coefficients and,
(iv) are readily computed from given pitch-and-roll buoy data.

To further enhance the usefulness of these parameters we will explore their error characteristics. The parameters chosen are the mean wave direction θ_0, the directional width σ, the skewness γ and the kurtosis δ of the directional energy distribution per frequency. In section 2 of this paper we inspect various analytical expressions for these parameters (e.g. Longuet-Higgins et al. 1963; Borgman 1969; Mardia 1972) and we argue for a particular selection. The error characteristics of the parameters are investigated in section 3. Some aspects of interpretation are addressed in section 4. Our conclusions are formulated in section 5.

2. Model-free parameters

a. The directional distribution

Wind waves are adequately described for most purposes with the two-dimensional energy spectrum $E(f, \theta)$ which distributes the wave energy over frequency (f) and direction (θ). Per frequency the normalized distribution of energy over direction is given by $D_f(\theta)$, defined by

$$D_f(\theta) = \frac{E(f, \theta)}{E(f)} \tag{3}$$

in which $E(f)$ is the frequency spectrum

$$E(f) = \int_0^{2\pi} E(f, \theta) d\theta. \tag{4}$$

In the remainder of this paper the analysis takes place per frequency and the subscript f in $D_f(\theta)$ will be dropped. Longuet-Higgins et al. (1963) approximate $D(\theta)$ as a Fourier series with four terms, the coefficients of which can be determined from the heave-, pitch- and roll signals of a pitch-and-roll buoy (appendix A). However, instead of actually using these coefficients in an attempt to reconstruct $D(\theta)$ we prefer to characterize $D(\theta)$ in terms of descriptive parameters expressed in these Fourier coefficients. These parameters are essentially the lowest four moments of $D(\theta)$ as derived next.

The directional energy distribution $D(\theta)$ is a density function with the following properties:

$$\left.\begin{aligned} &D(\theta) \geq 0 \\ &D(\theta + 2\pi) = D(\theta) \end{aligned}\right\} \quad \text{for} \quad -\infty < \theta < +\infty \tag{5}$$

and for any interval with length 2π:

$$\int_{\alpha-\pi}^{\alpha+\pi} D(\theta) d\theta = 1 \quad \text{for} \quad -\infty < \alpha < +\infty. \tag{6}$$

If $D(\theta)$ is considered only in the interval $(\alpha - \pi, \alpha + \pi)$, these characteristics are very similar to those of a probability density function. Such a function (defined on a line and not on a circle) is conveniently and commonly characterized by its moments from which such conventional parameters as the mean, standard deviation, skewness, and kurtosis can be defined. However, such parameters are not conventionally defined for distributions on a circle such as $D(\theta)$. They should be based on circular moments. These in principle can be defined in a number of ways, analogous to the definition of moments on a line, using trigonometric weighting functions. Within the constraints (i) through (iv) mentioned in the Introduction we have chosen definitions that are closely related to, or even identical to, those in the literature on this subject. Longuet-Higgins et al. (1963) and Borgman (1969) suggest some definitions and Mardia (1972) addresses the problem of describing directional data in great detail. In the following we will exploit their suggestions.

b. Line moments

To use the analogy between line distributions and circular distributions we initially treat $D(\theta)$ as if it were a distribution defined on a line interval with length 2π centered around a mean direction θ_0'. This mean direction is defined such that

$$\int_{\theta_0'-\pi}^{\theta_0'+\pi} (\theta - \theta_0')D(\theta)d\theta = 0. \quad (7)$$

Suitable measures μ_i for the line moments centered at this mean wave direction are then written as

$$\mu_i = E\{(\theta - \theta_0')^i\} = \int_{\theta_0'-\pi}^{\theta_0'+\pi} (\theta - \theta_0')^i D(\theta)d\theta \quad (8)$$

in which $E\{\quad\}$ denotes the expected value operator. The corresponding line parameters, width σ_l, skewness γ_l and kurtosis δ_l can subsequently be defined as

$$\sigma_l = \mu_2{}^{1/2} \quad (9)$$

$$\gamma_l = \mu_3/\sigma_l^3 \quad (10)$$

$$\delta_l = \mu_4/\sigma_l^4. \quad (11)$$

These definitions are commonly accepted and intuitively appealing. For symmetrical and fairly narrow distributions Longuet-Higgins et al. (1963) show that σ_l is the rms angular deviation of energy from the mean direction [their Eq. (93)] and suggest for distributions that are not too broad a "peakedness" parameter equal to $\delta_l - 1$ [their Eq. (97)]. However, these parameters σ_l, γ_l and δ_l do not meet the condition that they can be expressed in terms of the first four Fourier coefficients of $D(\theta)$. To define similar parameters which do meet this condition requires the definition of circular moments such as suggested by Borgman (1969) and Mardia (1972).

c. Circular moments

Borgman (1969) pointed out that the centrality measure used by Gumbel et al. (1953), defined as the vectorial mean of the directional distribution $D(\theta)$, can be expressed in terms of the lowest Fourier coefficients of $D(\theta)$:

$$\theta_0 = \arctan(b_1/a_1) \quad (12)$$

in which a_1 and b_1 are the lowest Fourier coefficients in Eq. (1). Following Borgman (1969) and Mardia (1972) we interpret this measure as the mean wave direction. It follows from this definition that, in analogy with Eq. (7),

$$\int_{\theta_0-\pi}^{\theta_0+\pi} \sin(\theta - \theta_0)D(\theta)d\theta = \int_0^{2\pi} \sin(\theta - \theta_0)D(\theta)\,d\theta = 0. \quad (13)$$

Borgman (1969) therefore suggested to define circular moments in terms of $\sin(\theta - \theta_0)$ or better still, in terms of $2\sin\{(\theta - \theta_0)/2\}$ since this is a better approximation of $(\theta - \theta_0)$ than $\sin(\theta - \theta_0)$. We adopt both suggestions in a somewhat generalized form by defining (without implying that we approximate the line moments) circular moments $\eta_{i,j}$ as

$$\eta_{i,j} = 2^j \int_0^{2\pi} \sin^i(\theta - \theta_0)\sin^j\{(\theta - \theta_0)/2\}D(\theta)d\theta. \quad (14)$$

These moments in turn can be used to define measures of width, skewness and kurtosis. For reasons indicated below we consider only the following circular moments:

second order

$$\eta_{0,2} = 2(1 - m_1) \quad (15)$$

$$\eta_{2,0} = (1 - m_2)/2 \quad (16)$$

third order

$$\eta_{1,2} = 2n_1 - n_2 = -n_2 \quad (17)$$

fourth order

$$\eta_{0,4} = 6 - 8m_1 + 2m_2 \quad (18)$$

in which m_1, m_2, n_1 and n_2 are the centered Fourier coefficients given by

$$m_1 = \int_0^{2\pi} \cos(\theta - \theta_0)D(\theta)d\theta = a_1\cos(\theta_0) + b_1\sin(\theta_0) = (a_1{}^2 + b_1{}^2)^{1/2} \quad (19)$$

$$m_2 = \int_0^{2\pi} \cos\{2(\theta - \theta_0)\}D(\theta)d\theta = a_2\cos(2\theta_0) + b_2\sin(2\theta_0) \quad (20)$$

$$n_1 = \int_0^{2\pi} \sin(\theta - \theta_0)D(\theta)d\theta = b_1\cos(\theta_0) - a_1\sin(\theta_0) = 0 \quad (21)$$

$$n_2 = \int_0^{2\pi} \sin\{2(\theta - \theta_0)\} D(\theta) d\theta$$
$$= b_2 \cos(2\theta_0) - a_2 \sin(2\theta_0). \quad (22)$$

The coefficients a_1, a_2, b_1 and b_2 are the lowest four Fourier coefficients of $D(\theta)$ (appendix A). The centered Fourier coefficients have the following property [see Mardia 1972, his Eq. (3.3.4)]:

$$1 - m_2{}^2 - n_2{}^2 - 2m_1{}^2(1 - m_1{}^2) \geq 0 \quad (23)$$

which can be shown to follow from the fact that $D(\theta)$ is always positive. The indicated circular moments $\eta_{0,2}$, $\eta_{2,0}$, $\eta_{1,2}$ and $\eta_{0,4}$ are approximately equal to their respective line equivalents for narrow directional distributions as $\sin(\theta)$ and $2\sin(\theta/2)$ are approximately equal to θ for small θ.

Other values of i and j to define moments of $D(\theta)$ are not used in the present study since the corresponding circular moments [based on Eq. (14)] cannot be expressed in terms of the available four Fourier coefficients.

The above expressions for $\eta_{0,2}$, $\eta_{2,0}$ and $\eta_{0,4}$ are also given by Borgman (1969), except for a different normalization. Mardia (1972), using another method based on series expansions, obtains the same expressions for $\eta_{0,2}$, $\eta_{1,2}$, $\eta_{0,4}$ as Eqs. (15), (17) and (18), except for a different normalization. On the basis of these circular moments Mardia (1972) defines measures for directional width, skewness and kurtosis, which are partly based on results obtained with a wrapped normal distribution. They are therefore not entirely model-free. Instead of following Mardia (1972) further we therefore now continue with model-free formulations.

To obtain measures of skewness and kurtosis, we must normalize the third-order and fourth-order moment ($\eta_{1,2}$ and $\eta_{0,4}$). Since we find two second-order moments ($\eta_{0,2}$ and $\eta_{2,0}$) which can be used for this purpose we obtain two sets of definitions for directional width, skewness and kurtosis:

(i) based on $\eta_{0,2}$

$$\sigma_c = \eta_{0,2}^{1/2} = \{2(1 - m_1)\}^{1/2} \quad (24)$$

$$\gamma_c = \eta_{1,2}/\sigma_c{}^3 = \frac{-n_2}{\{2(1 - m_1)\}^{3/2}} \quad (25)$$

$$\delta_c = \eta_{0,4}/\sigma_c{}^4 = \frac{6 - 8m_1 + 2m_2}{\{2(1 - m_1)\}^2} \quad (26)$$

(ii) based on $\eta_{2,0}$

$$\sigma_c^* = \eta_{2,0}^{1/2} = \{(1 - m_2)/2\}^{1/2} \quad (27)$$

$$\gamma_c^* = \eta_{1,2}/\sigma_c^{*3} = \frac{-n_2}{\{(1 - m_2)/2\}^{3/2}} \quad (28)$$

$$\delta_c^* = \eta_{0,4}/\sigma_c^{*4} = \frac{6 - 8m_1 + 2m_2}{\{(1 - m_2)/2\}^2}. \quad (29)$$

These two sets of definitions [Eqs. (24)–(26)] and [Eqs. (27)–(29)] are equivalent to each other in the sense that in both sets the parameters are model-free and expressed in terms of the first four Fourier coefficients. The parameters from both sets are approximately equal to their respective line equivalents for narrow directional distributions. However, with a broadening of the directional distribution the differences between these circular parameters and their line equivalents obviously increase. We select from the above alternatives (24) through (29) those that remain closest to their line equivalent. To this end we investigate the differences in the following. For reasons of brevity in the text we will refer to such a difference between a circular parameter and its line equivalent as its "deviation".

d. Comparison between circular parameters and line parameters

The deviations of the circular parameters can be investigated analytically and numerically. In the analytical approach (which was suggested by one of the reviewers of this paper) these deviations are quantified by expressing the circular parameters in terms of the line moments with a Taylor series expansion, whereas in the numerical approach the deviations are quantified numerically for a large number of chosen directional distributions.

To express the circular width parameters σ_c and σ_c^* in terms of the second-order line moment μ_2, we write the term $\sin^i(\theta - \theta_0)\sin^j\{(\theta - \theta_0)/2\}$ in Eq. (14) as a Taylor series up to (and inclusive) fourth order in $(\theta - \theta_0)$ and we take $\theta_0 = \theta_0'$ in the definition of the line moment to have a common center of the distributions. We then find, correct to second order in σ_l, that the second-order circular moments are somewhat smaller than the second-order line moments: $\eta_{0,2} \approx \mu_2 - \mu_4/12$ and $\eta_{2,0} \approx \mu_2 - \mu_4/3$. The corresponding deviation in the circular widths σ_c and σ_c^* follows directly by taking the square root of these second-order circular moments. The result can be simplified with a two-term Taylor series expansion and the final result is

$$\left.\begin{aligned} \sigma_c &\approx \sigma_l - \sigma_l{}^3\delta_l/24 \\ \sigma_c^* &\approx \sigma_l - \sigma_l{}^3\delta_l/6 \end{aligned}\right\}. \quad (30)$$

Obviously, the deviation thus determined is always smaller for σ_c than for σ_c^*.

We have carried out a similar second-order analysis for the third- and fourth-order circular moments, i.e. an analysis in which the error is of second order in μ_3 and μ_4, respectively. It involves a Taylor series expansion of up to (and inclusive) sixth order in $(\theta - \theta_0)$. The results are consequently expressed in terms of all line moments up to (and inclusive) sixth order (squares and cross-products). However, the fifth- and sixth-order moments cannot be determined with pitch-and-roll buoy data and we will therefore not consider these corrections any further.

On the basis of the above analysis of σ_c and σ_c^*, one might be tempted to prefer the first set of definitions [Eqs. (24), (25) and (26)] over the second [Eqs. (27), (28) and (29)] since, after all, σ_c and σ_c^* are used to normalize the higher order moments. This analysis however is based on a Taylor series expansion which can be expected to produce reasonable results only for narrow directional distributions. Moreover, such a choice would ignore the differences in deviation of the higher-order moments. The numerical approach, addressed next, does not suffer from these drawbacks.

In the numerical approach we determine the circular moments and the line moments for a large number of fairly realistic, but otherwise arbitrarily chosen directional distributions. The circular moments and the value of θ_0 are determined from the first two Fourier components of the distributions following their above definitions, Eqs. (15), (16), (17) and (18). The line moments are determined following their definition, Eq. (8) taking θ_0' equal to θ_0. We carry out these computations for 222 directional distributions of the following three types: the $\cos^{2s}$-model defined by Eq. (2), a double $\cos^{2s}$-model defined by

$$D(\theta) = A_1 \cos^{2s_1}(\theta/2) + A_2 \cos^{2s_2}\{(\theta - \alpha_0)/2\} \quad (31)$$

in which A_1 and A_2 are normalization coefficients, and a skewed $\cos^{2s}$-model (analogous to the model used by Regier and Davis 1977) defined by

$$\left.\begin{aligned} D(\theta) &= A \cos^{2s}(\theta/2)\cdot\exp(-\mu\theta^2) \quad \text{for } 0 \leq \theta < \pi \\ D(\theta) &= A \cos^{2s}(\theta/2)\cdot\exp(-\nu\theta^2) \quad \text{for } -\pi \leq \theta < 0 \end{aligned}\right\} \quad (32)$$

in which A is a normalization coefficient. Per model all possible combinations of the parameter values given in the upper part of Table 1 (indicated with "comparison") are used. Some of the resulting distributions are the mirror-image of other distributions thus generated. Of each such mirror-pair one distribution has been ignored in the analysis. The circular moments and the circular parameters of these distributions are plotted against their line moments and line parameters in Fig. 1, except the second-order moments $\eta_{0,2}$ and $\eta_{2,0}$ to avoid cluttering the illustration with trivial information (the transformation from second-order circular moment to circular width is trivial). Our comments are as follows.

As for the circular width, Fig. 1, panels b and c show that the circular width σ_c is generally closer to the line width σ_l than the circular width σ_c^*. In fact, an inspection of the numerical values (not given here) shows that on the average the deviation in σ_c is about four times smaller than the deviation in σ_c^*, which is in agreement with the results of the analytical results, Eq. (30).

As for the skewness, panel d of Fig. 1 shows that the third-order circular moment $\eta_{1,2}$ is considerably lower (in absolute value) than the third-order line moment μ_3, at least for $|\mu_3| > 0.1$, say. To obtain the circular skewness this third-order circular moment is normalized with the circular width. Since the values of both these quantities are generally lower than the values of their line equivalent these deviations may compensate each other through this normalization. Such compensation does in fact occur (Fig. 1, panels e and f). The relatively large deviation of $\eta_{1,2}$ appears to be better compensated by the deviation of σ_c^*, which is also relatively large (Fig. 1, panel c) than by the relatively small deviation of σ_c (Fig. 1, panel b).

As for the kurtosis, similar effects as above occur, except that now the deviation of σ_c^* overcompensates the deviation of $\eta_{0,4}$ and that the normalization with σ_c provides a more favourable compensation than with σ_c^* (Fig. 1, panels g, h and i).

Considering the results of the above analyses, we choose the following circular parameters from the

TABLE 1. Parameter values and model number of the models used in the intercomparison of circular and line quantities and in the Monte-Carlo simulation.

	$\cos^{2s}$-model			Double $\cos^{2s}$-model					Skewed $\cos^{2s}$-model		
	α_0	s	s	Model	A_2/A_1	s_1	s_2	α_0	s	μ	ν
		all combinations				all combinations				all combinations	
Comparison	0	0.2	5.0		0.1	4	4	0	2	0	0
		0.5	10.0		0.2	20	20	30	4	0.1	0.1
		1.0	15.0		0.5	50	50	60	10	0.5	0.5
		1.5	20.0		1.0			90		1.0	1.0
		2.0	40.0					120		5.0	5.0
		4.0	50.0								
Simulation				I	1.0	12	12	0			
				II	1.0	12	12	90			
				III	0.1	12	12	90			
				IV	0.2	50	50	120			
				V	0.1	50	50	120			

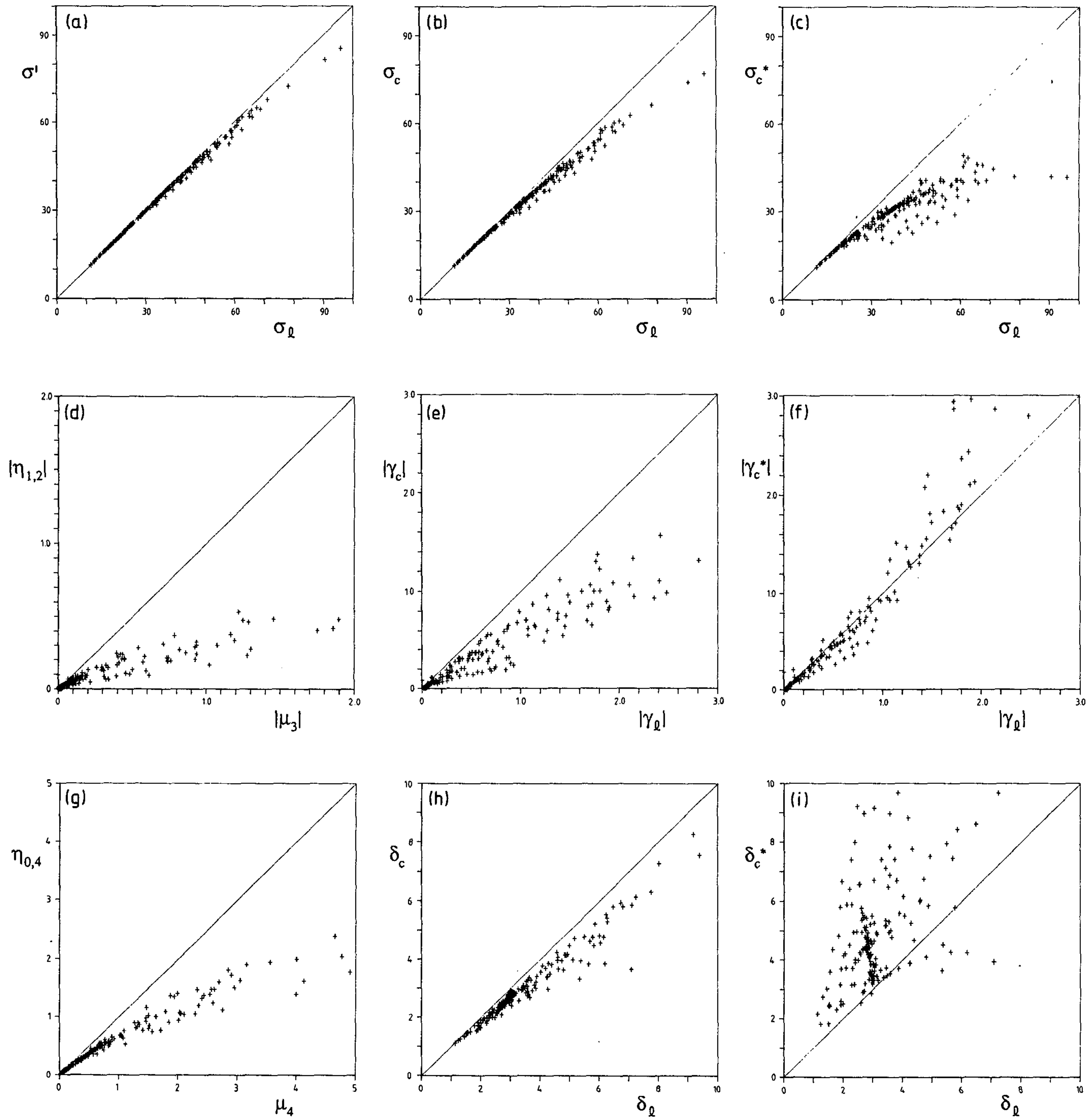

FIG. 1. The circular moments and optional circular parameters for a large number of model directional distributions as a function of the line moments and the line parameters.

above alternatives (dropping the suffix and the asterisk from the notation):

$$\theta_0 = \arctan(b_1/a_1) \tag{33}$$

$$\sigma = \{2(1 - m_1)\}^{1/2} \tag{34}$$

$$\gamma = \frac{-n_2}{\{(1 - m_2)/2\}^{3/2}} \tag{35}$$

$$\delta = \frac{6 - 8m_1 + 2m_2}{\{2(1 - m_1)\}^2}. \tag{36}$$

The dependencies of σ, γ and δ on the centered Fourier coefficients m_1, m_2 and n_2 are illustrated in Fig. 2 where it is seen that the values of the skewness γ and the kurtosis δ are sometimes very sensitive to variations in the values of m_1, m_2 or $|n_2|$. This occurs for low values of $|n_2|$ (less than 0.2, say) combined with high values of m_2 (higher than 0.4, say) and for high values of m_1 (higher than 0.7, say) combined with positive values of m_2. It is readily shown that the inequality given by Mardia (1972), Eq. (23), implies that $\delta \geqslant 1$ and $m_2{}^2 + n_2{}^2 \leqslant 1$. The values of γ and δ for physically realizable distributions are therefore always located within the regions indicated in Fig. 2 (full lines). The physically nonrealizable values are also indicated (dashed lines) but only because they occur in numerical simulations of the circular moments (see section 3*a*).

One might consider to introduce alternative circular parameters that are closer to the line parameters than σ, γ and δ by adding the second-order deviations to the original definitions. For instance, with the relationships between the circular moments $\eta_{0,2}$ and $\eta_{0,4}$ and the line moments μ_2 and μ_4, which are found in the above second-order analysis, and taking $\mu_4 \approx \eta_{0,4}$, it is readily shown that $\mu_2 \approx \eta_{0,2} + \eta_{0,4}/12$, so that the line width σ_l may be approximated with

$$\sigma' = (\eta_{0,2} + \eta_{0,4}/12)^{1/2}. \qquad (37)$$

To illustrate the correspondence between σ' and σ_l, the value of σ' is given in panel a of Fig. 1 as a function of σ_l for the above 222 distributions. Since $\sigma_c = \eta_{0,2}^{1/2}$, σ' may be seen as a corrected version of σ_c. A similar analysis may be carried out for γ and δ but the resulting expressions would include circular moments of fifth and sixth order which are not available from pitch-and-roll buoy data. Of the indicated potential alternatives one can therefore only consider σ' to be practical. However, a comparison between panel a and panel b of Fig. 1 shows that for the more common distributions (directional width typically less than 45°, say) the effect of the correction on the directional width is marginal and small compared to the sample error of σ_c and σ' (section 3a and Table 2). Moreover, we do not require that the circular parameters approximate

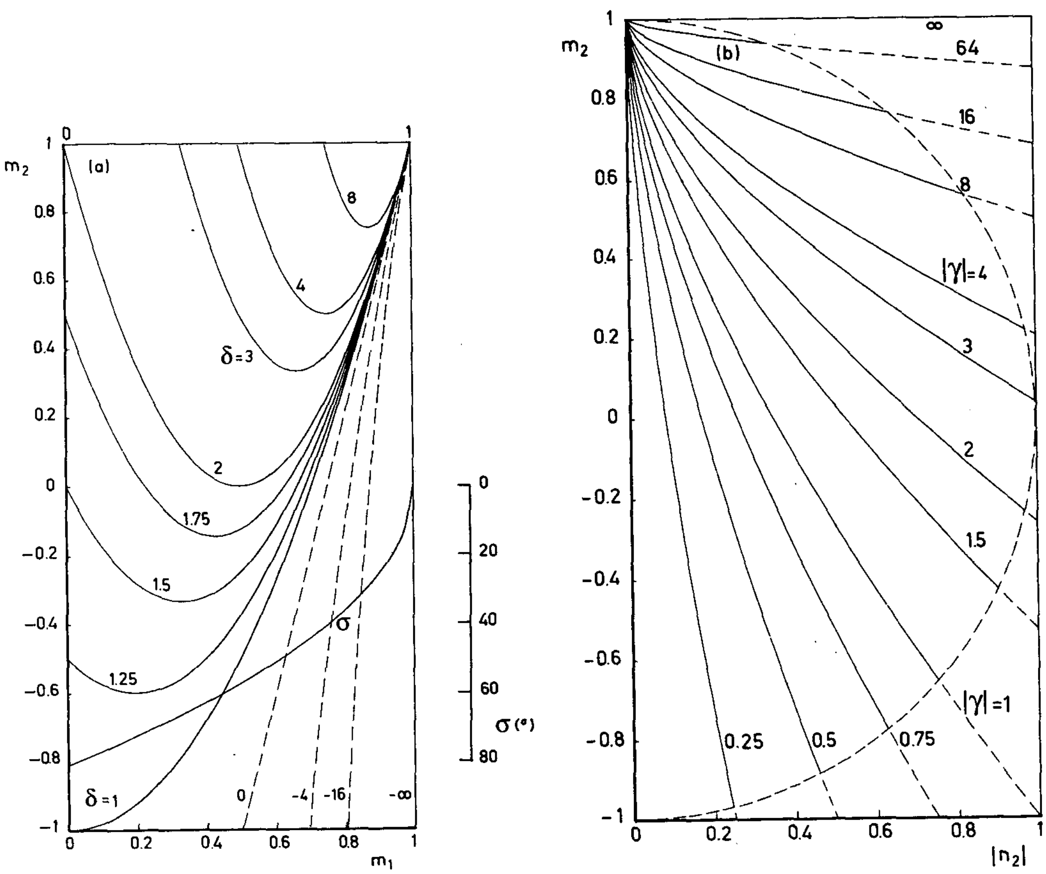

FIG. 2. The directional width σ (panel a), absolute value of skewness γ (panel b) and kurtosis δ (panel a) as a function of the centered Fourier coefficients m_1, m_2 and absolute value of n_2; physically nonrealizable values are indicated with dashed lines.

TABLE 2. Comparison of results of Monte-Carlo simulations with theoretical estimates of rms errors of directional parameters for five model distributions. Mean direction θ_0 and widths σ and σ' in degrees. For model numbers, see Table 1.

			Rms error			
			k_t		k_d	
Model	Parameter	True value	Eqs. (40) and (43) (Borgman et al.)	sim	Eqs. (40) and (41) (Long)	sim
I	θ_0	0	3.64	3.49	3.64	3.49
	σ	22.48	2.37	2.12	3.07	2.82
	σ'	22.88	—	2.35	—	2.86
	γ	0	—	0.38	—	0.44
	δ	2.78	—	2.59	—	1.35
II	θ_0	45.0	9.80	9.11	9.80	9.11
	σ	47.71	2.94	3.03	4.19	4.34
	σ'	49.87	—	3.45	—	4.61
	γ	0	—	0.43	—	0.45
	δ	1.60	—	0.53	—	0.42
III	θ_0	5.72	4.92	5.00	4.92	5.00
	σ	32.09	4.57	4.00	3.85	3.74
	σ'	33.82	—	4.44	—	3.97
	γ	1.23	—	0.69	—	0.59
	δ	4.24	—	1.38	—	1.58
IV	θ_0	100.86	5.45	5.77	5.45	5.77
	σ	40.64	6.22	5.36	4.91	5.08
	σ'	44.51	—	5.65	—	5.44
	γ	4.16	—	1.88	—	1.31
	δ	4.74	—	1.47	—	1.77
V	θ_0	95.19	3.60	3.70	3.60	3.70
	σ	31.33	6.68	5.63	4.15	4.30
	σ'	34.41	—	6.05	—	4.68
	γ	5.60	—	2.85	—	1.44
	δ	8.27	—	2.97	—	3.04

the line parameters (they do so only for narrow distributions; we only exploit the analogy) and we prefer to use a consistent set of definitions [i.e., based on Eq. (14)]. We therefore do not consider σ' as an alternative in the following.

e. Expressions for the $\cos^{2s}$-model

Since the $\cos^{2s}$-model of Eq. (2) is a widely used model for $D(\theta)$ it seems appropriate to give σ, γ and δ in terms of the parameters of this model. The $\cos^{2s}$-model is a symmetrical model so that the skewness γ is zero. The centered Fourier coefficients m_1 and m_2 can be determined in terms of the directional width parameter s by substituting Eq. (2) in Eqs. (A1) through (A4) of appendix A and subsequently using Eqs. (12), (19) and (20). The result for the directional width is

$$\sigma = \left(\frac{2}{s+1}\right)^{1/2}. \tag{38}$$

The theoretical maximum value of σ in this model is 81.03°. It occurs for a uniform directional distribution ($s = 0$). For the kurtosis we find:

$$\delta = \frac{3(s+1)}{(s+2)}. \tag{39}$$

This means that for a $\cos^{2s}$-model (for which $0 < s < \infty$), the kurtosis is limited to the interval (1.5, 3). Obviously, kurtosis and width are not mutually independent parameters for the $\cos^{2s}$-model.

3. Error characteristics of the proposed parameters

a. Sample properties of proposed parameters

An important aspect of the presentation and interpretation of the proposed parameters is their statistical sample variability. Formulas for the standard deviation and for the rms error of the estimated mean wave direction θ_0 and directional width σ have been derived by Long (1980) and Borgman et al. (1982). Borgman et al. (1982) also carried out a theoretical investigation of the statistical properties of the centered Fourier coefficients m_1, m_2, n_1 and n_2. Their work indicates that the bias of θ_0 and σ is usually an order of magnitude smaller than the rms error of θ_0 and σ, respectively. We therefore ignore in the following the theoretical estimates of the bias of θ_0 and σ.

The formulas of Long (1980) for the standard deviation (s.d.) of the mean wave direction θ_0 and the directional width σ can be simplified by a simple rotation of the coordinate system such that the mean wave direction is equal to zero. If this is done, the expressions of Long (1980) can be written as

$$\text{s.d.}\,(\theta_0) = \rho^{-1/2}\{(1 - m_2)/(2\,m_1^2)\}^{1/2} \quad (40)$$

$$\text{s.d.}\,(\sigma) = \rho^{-1/2}\left[\frac{m_1^2}{2(1-m_1)}\{m_1^2 + (K_2^2 - 1)/4 + (1 + m_2)(m_1^{-2} - 2)/2\}\right]^{1/2} \quad (41)$$

in which $K_2 = (m_2^2 + n_2^2)^{1/2}$ and ρ is the equivalent number of degrees of freedom of the spectral estimates at the frequency considered. In deriving his equivalent of expression (41), Long (1980) used the wavenumber estimated from the buoy data (k_d, see appendix A) rather than the wavenumber from the linear dispersion relation (k_t, appendix A) for the calculation of the Fourier coefficients. Incidently, the original equation of Long (1980) for the standard deviation of σ [his Eq. (19)], which is the basis of Eq. (41), was found to contain a misprint: the term $(1 - K_1)^{-1}$ should read $(1 - K_1)^{-4}$; this was confirmed by R. B. Long (personal communication, 1984).

We note that Eq. (40) for the standard deviation of the mean wave direction can be expressed in terms of σ and δ, using Eqs. (33) through (36):

$$\text{s.d.}\,(\theta_0) = \rho^{-1/2}\frac{\sigma}{(1 - \sigma^2/2)}\{1 - \delta\sigma^2/4\}^{1/2}. \quad (42)$$

Equation (41) for the standard deviation of the directional width cannot be written as a similarly simple expression.

Borgman et al. (1982) derived formulas for the rms error of the mean wave direction θ_0 and the centered Fourier coefficients m_1, m_2 and n_2. These formulas are very complex and they will not be repeated here. However, as with the formulas of Long (1980), they can be simplified by a rotation of the coordinate system. It should be noted that Borgman et al. (1982), in contrast to Long (1980) used the theoretical wavenumber k_t in their derivation. The expression for the rms error of θ_0 obtained from Borgman et al. (1982) after the rotation is identical to Eq. (40). However, Eq. (40) represents the standard deviation of θ_0 rather than the rms error. In spite of this we will refer in the following to the results of Long (1980) as rms error estimates consistent with neglecting the bias. To obtain from Borgman et al. (1982) an expression for the rms error of the directional width σ one can use their expression for the rms error of m_1 [their Eq. (91)] since it can be shown that the rms error of σ is approximately equal to $1/\sigma$ times the rms error of m_1 [Eq. (34)]. The result can be simplified by the above indicated rotation, resulting in

$$\text{rms}\,(\sigma) \approx \rho^{-1/2}\left\{\frac{1 + m_2 - 2m_1^2}{4(1 - m_1)}\right\}^{1/2} \quad (43)$$

or, in terms of σ and δ:

$$\text{rms}\,(\sigma) \approx \rho^{-1/2}\sigma(\delta - 1)^{1/2}/2. \quad (44)$$

From Eqs. (40), (41) and (42) we see that Borgman et al. (1982) use only the centered Fourier coefficients m_1 and m_2 whereas Long (1980) uses in addition the centered Fourier coefficient n_2.

To test the above expressions for the rms errors of θ_0 and σ and to obtain some estimates of the rms errors of the skewness γ, the kurtosis δ and the corrected circular width σ' (see section 2*d*), we carry out a Monte-Carlo simulation. This simulation is applied to five directional distributions of the double $\cos^{2s}$-type. The parameter values of these five distributions, denoted with I through V, are given in the lower part of Table 1 (indicated with "simulation").

Of these five distributions the ones with $\alpha_0 = 0°$ (i.e. a unimodal $\cos^{2s}$-model) and the ones with $\alpha_0 = 90°$ were also used by Borgman et al. (1982) in their Monte-Carlo study to test the theoretical estimates of the rms error of the mean wave direction θ_0 and the centered Fourier coefficients m_1, m_2 and n_2. The two other directional distributions are added to include larger values for the skewness and kurtosis than those used by Borgman et al. (1982).

For each of the above model distributions the Monte-Carlo simulation generated from a multidimensional Gaussian distribution a set of 400 unbiased auto- and cross-spectral values each with 40 degrees of freedom representing the pitch-and-roll buoy wave data. This simulation technique is described in Borgman et al. (1982) and Borgman (1982), but we added the inequality due to Mardia (1972), Eq. (23). Each set of auto- and cross-spectral values thus generated was analyzed to compute the corresponding values of θ_0, σ, σ', γ and δ based on the four Fourier coefficients [Eqs. (33) through (37)]. For the computation of these Fourier coefficients both k_d and k_t were used since the theoretically estimated rms errors are based on either k_d or k_t. The rms error of the directional parameter values thus obtained are shown in Table 2, the heading "sim" referring to the results of these simulations. The theoretical estimates of the rms error are given in the columns labeled "Eqs. (40) and (43)" and "Eqs. (40) and (41)" to indicate the formulas that are used.

Table 2 shows that there is a good agreement between the formulas for the rms errors of θ_0 and σ of Long (1980) and Borgman et al. (1982), on the one hand, and the results of the Monte-Carlo simulations on the other. The rms error of the mean direction θ_0 is about 5°–10°, independent of the type of wavenumber estimation (k_d or k_t), as may be expected since the estimate of θ_0 itself is independent of wavenumber. The other rms errors vary from about 10%–15% for the directional widths σ and σ' to about 30%–50% for the skewness γ (with a lower limit of about 0.4) to 25%–100% for the kurtosis δ. There seems to be no systematic difference between these results when using either k_d or k_t.

The inclusion of the above indicated criterion of Mardia (1972), is rather essential for the results of the computations based on k_t. When repeating the simulation with this criterion removed, we found that the rms errors of γ and δ based on k_t increased considerably. A closer inspection of these results showed that these large rms errors are due to the occurrence of some outliers in the value of γ and δ which is in turn due to the very high sensitivity of these parameters for small changes in the values of m_1, m_2 and $|n_2|$ for certain combinations of these moments (see Fig. 2 and section 2*d*). The application of the criterion of Mardia (1972) removes these outliers but also quite a few other values, in fact, with this criterion included we had to simulate about 600 sets of values to retain the above indicated 400 required values. This indicates that some of the original assumptions of Borgman et al. (1982) underlying the simulations may require modifications (e.g., use χ^2-distributions rather than Gaussian distributions). For our purposes we feel that the inclusion of Mardia's (1982) criterion in the simulations is sufficient. Incidentally, we verified that all 222 distributions used in section 2d comply with this criterion. The simulations based on k_d are only marginally affected by the inclusion of this criterion. Considering these results we recommend for the routine analysis of pitch-and-roll buoy data (if realistic values of γ and δ are required) to include the above criterion of Mardia (1972), and perhaps risk losing some data, and to use k_d rather than k_t in the computation of the four Fourier coefficients (appendix A). It is anyway more convenient to use k_d rather than k_t since the determination of k_t requires, in general, information on water depth and currents which is not always reliably available.

In the above we discussed the sampling error of the proposed parameters. However, in practice other effects also cause errors in the proposed parameters, such as instrument noise or errors in the computation of the wavenumber. These aspects are discussed below.

b. *Errors in estimates of θ_0 and σ due to noise*

The data of pitch-and-roll buoys will always be contaminated to some extent with noise (e.g., instrument noise, horizontal motion of the buoy which is not accounted for, spectral leakage). The effect of noise on the estimation of the mean wave direction and the directional width is illustrated here with a simple approach. We assume that only the autospectra are contaminated as follows:

$$\left.\begin{aligned}\tilde{C}_{zz} &= b_z \cdot C_{zz}\\ \tilde{C}_{xx} &= b_x \cdot C_{xx}\\ \tilde{C}_{yy} &= b_y \cdot C_{yy}\end{aligned}\right\} \qquad (45)$$

where b_x, b_y and b_z are greater than 1 and in which C refers to the true values and $\tilde{C}$ to the contaminated values of the autospectra. When the wavenumber k_d is used and $b_x = b_y$ is assumed, i.e. the noise in x-slope is equal to that in y-slope, we find that the estimated mean wave direction $\tilde{\theta}_0$ is not affected:

$$\tilde{\theta}_0 = \theta_0 \qquad (46)$$

whereas the estimated directional width $\tilde{\sigma}$ is affected:

$$\tilde{\sigma}^2 = \sigma^2 + (1 - \epsilon)(2 - \sigma^2) \qquad (47)$$

in which $\epsilon = (b_x b_z)^{-1/2}$ and σ is in radians. Equation (47) is obtained by substituting Eqs. (45) and (A5) in Eqs. (A1) and (A2), see appendix A, for the Fourier coefficients a_1 and b_1, and from the expression for the directional width, Eq. (34). Relation (47) is illustrated in Fig. 3. It shows a rapid increase of the estimated directional width when noise is present on one or more of the three buoy signals, in particular for low values of the directional width σ. For instance, for $\sigma = 20°$ and 1% to 5% noise on all signals ($b_x = b_y = b_z = 1.01$ to 1.05) we find a 7.5% to 30% increase in the estimated directional width ($\tilde{\sigma} = 21.5°$ to 26.4°).

c. *Errors in estimates of θ_0 and σ due to errors in the estimated wavenumber*

Errors in the estimated mean wave direction and the estimated directional width may also be introduced by errors in estimating the wavenumber. For instance, when the wavenumber is determined with the dispersion relationship from linear wave theory while information on depth or current is not correct.

To estimate the effect of such errors on the estimated

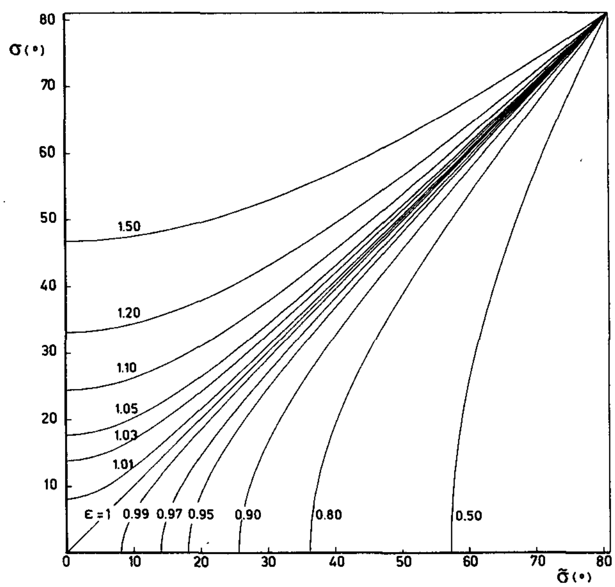

FIG. 3. Sensitivity of the directional width to errors (instrument or analysis); σ: true directional width, $\tilde{\sigma}$: directional width contaminated with noise; ϵ is a measure for the error (noise, wavenumber estimation or buoy response).

directional width we write the estimated wavenumber $\tilde{k}$ as

$$\tilde{k} = \mu k \quad (48)$$

where k is the wavenumber defined in appendix A. The estimated mean wave direction θ_0 is not dependent on the wavenumber and therefore it is not affected by errors in the estimation of k. For the effect on the directional width the same expression is found as Eq. (47) with ϵ replaced by μ^{-1} (Fig. 3). This figure clearly illustrates the strong sensitivity of the estimated directional width to small errors in the estimation of wavenumber, in particular for narrow distributions. Furthermore, for some values of σ it is not possible to compute $\tilde{\sigma}$, e.g. for $\sigma = 5°$ and $\mu = 0.99$ (i.e. $\epsilon = 1.01$).

d. Errors in the estimates of θ_0 and σ due to buoy response

Although pitch-and-roll buoys are designed to follow the sea surface as a free water particle, departures from this behavior will occur due to the finite dimension and mass of the buoy and the anchoring system. To study these effects we assume a linear system response for both heave and slope. Furthermore we assume the slope response to be isotropic. In the time domain we therefore adopt the following response model:

$$\tilde{z}(t) = h_z(t) * z(t) \quad (49)$$

$$\tilde{x}(t) = h_s(t) * x(t) \quad (50)$$

$$\tilde{y}(t) = h_s(t) * y(t) \quad (51)$$

where $z(t)$, $x(t)$ and $y(t)$ are the heave, x-slope and y-slope of the sea surface and $*$ is the convolution operator. The buoy signals for heave, x-slope and y-slope, indicated with $\tilde{z}(t)$, $\tilde{x}(t)$ and $\tilde{y}(t)$ respectively are obtained through the linear pulse response functions of the buoy for heave, $h_z(t)$ and slope, $h_s(t)$. After some algebra the nine auto-, co- and quadspectra of the buoy signals can be expressed in terms of the auto-, co- and quad-spectra of the sea surface heave and slopes and the spectral response functions corresponding to $h_z(t)$ and $h_s(t)$. Substitution of these spectra into Eqs. (A1) through (A4) of appendix A and using the wavenumber estimate from the three autospectra, k_d, leads to the following estimates of the Fourier coefficients:

$$\tilde{a}_1 = a_1 \cos(\phi_s - \phi_z) \quad (52)$$

$$\tilde{b}_1 = b_1 \cos(\phi_s - \phi_z) \quad (53)$$

$$\tilde{a}_2 = a_2 \quad (54)$$

$$\tilde{b}_2 = b_2 \quad (55)$$

where the tilde indicates the estimates of the Fourier coefficients as influenced by the buoy response. Equations (52) through (55) show that the Fourier coefficients of the first harmonic of $D(\theta)$ are only affected by the difference in phase shift of heave and slope introduced by the buoy response (ϕ_z and ϕ_s, respectively) whereas the Fourier coefficients of the second harmonic are not influenced at all (in the present approximation of a linear and isotropic buoy response). Substituting $\tilde{a}_1$ and $\tilde{b}_1$ into Eqs. (19) through (22) and (33) and (34) gives the same expressions for the corresponding estimates of θ_0 and σ as in the case of noise: Eqs. (46) and (47) with ϵ now given by $\epsilon = |\cos(\phi_s - \phi_z)|$. This implies that the estimated mean wave direction is not affected by an isotropic linear buoy response but that the directional width is (see Fig. 3). Since $|\phi_s - \phi_z|$ is roughly estimated to be about 5° (i.e. $\epsilon = 0.996$) for a WAVEC buoy, the corresponding error in σ may be realistically estimated to vary from less than 0.5° (for $\sigma > 25°$, young sea states) to about 1.0° (for $\sigma = 10°$, swell).

In the above analysis the use of k_d removes in a very straightforward algebraic manner some undesirable response characteristics which would not be removed if k_t were used. This reinforces our recommendation in section 3a to use k_d in the analysis of pitch-and-roll buoy data rather than k_t.

4. Interpretation of the proposed parameters

In section 2 we have chosen θ_0 as the mean wave direction following the suggestion of Mardia (1972) who shows that θ_0 can be interpreted as the vectorial mean of a set of unit vectors distributed as $D(\theta)$. We have also shown that the other chosen directional parameters σ, γ and δ can for many directional distributions be interpreted as line quantities for width, skewness and kurtosis. The first two parameters, θ_0 and σ, usually provide sufficient directional information on the waves. However, one may be tempted to consider $D(\theta)$ to be symmetric and unimodal if only these measures for mean direction and directional width are used. To warn against such an assumption we suggest a simple criterion based on the values of skewness γ and kurtosis δ. Such a warning is also relevant to the interpretation of the values of θ_0 and σ as these parameters are fairly sensitive to the occurrence of small, secondary maxima in $D(\theta)$ far from the mean wave direction (bimodality or multimodality). Bimodal distributions occur in practice often for a local sea with a swell coming from a distant source. The sensitivity of θ_0 and σ to bimodality or multimodality is illustrated with the double $\cos^{2s}$-model (Eq. 31). We take $s_1 = s_2 = 12$, with α_0 varying from 0° to 180° and the ratio A_2/A_1 varying from 0.05 to 0.4. The directional parameters θ_0 and σ for these distributions are shown in Fig. 4. In particular σ appears to be sensitive for the occurrence of bimodality. For instance, for a moderately small secondary lobe in the directional distribution ($A_1/A_2 \sim 0.2$, say) located far from the main lobe ($\alpha_0 \sim 120°$, say), the mean wave direction shifts by about 10 degrees whereas the directional width nearly doubles (from about 22° to about 45°).

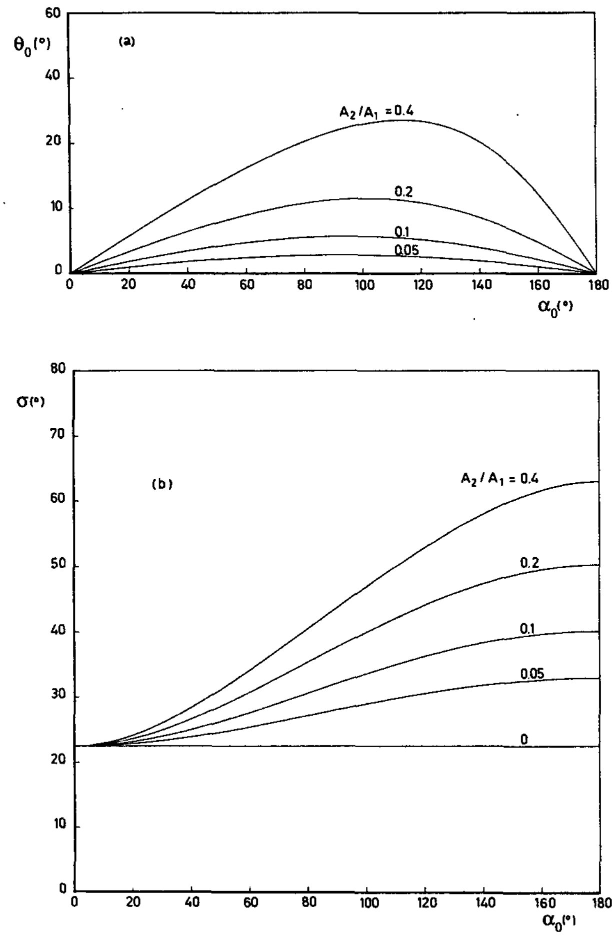

FIG. 4. Variation of (a) the mean wave direction θ_0 and (b) directional width σ for the double $\cos^{2s}$-model for various magnitudes of the secondary lobe (A_2/A_1) as a function of difference in peak directions (α_0).

The criterion to warn against the assumption of a unimodal, symmetric distribution is found from further analyzing the 222 model distributions of section 2d (see Table 1) which we supplemented with 48 model distributions of the $\cos^{2s}$-type to include very skewed distributions (by adding $\alpha_0 = 150°$ and $165°$ to Table 1). From these 270 distributions, which are each strictly speaking either unimodal or bimodal, we selected those that are (nearly) unimodal and symmetric with the procedure of appendix B. The absolute values of the skewness, $|\gamma|$ and the δ values of these distributions are indicated in Fig. 5, panel (a) whereas the $|\gamma|$ and δ values of the other distributions are given in panel (b). It is obvious that these two sets of $|\gamma|$ and δ values are fairly well separated by the following expression (also indicated in Fig. 5):

$$\left.\begin{aligned} \delta &= 2 + |\gamma| \quad \text{for} \quad |\gamma| \leq 4 \\ \delta &= 6 \quad\quad\quad \text{for} \quad |\gamma| > 4 \end{aligned}\right\}. \tag{56}$$

We therefore consider an arbitrarily shaped directional distribution to be approximately unimodal and symmetric only if its δ-value is larger than the δ-value from Eq. (56). To illustrate the use of this criterion we consider a situation with a bimodal frequency spectrum that occurred after a turn in wind direction (Fig. 6, location is off the Dutch coast near the research platform Noordwijk, 52°12′N, 4°15′E, and time is 1400–1430 UTC 8 July 1980). The measurements were carried out with a WAVEC pitch-and-roll buoy. These measurements indicate the existence of two distinct wavefields, one from northwest and one from northeast. The numerical values of the estimated directional parameters are given as a function of frequency in Table 3. A warning against the assumption of a unimodal, symmetric distribution, based on the proposed criterion is issued only for the transition frequencies between the two wavefields (see Fig. 6). Since the directional

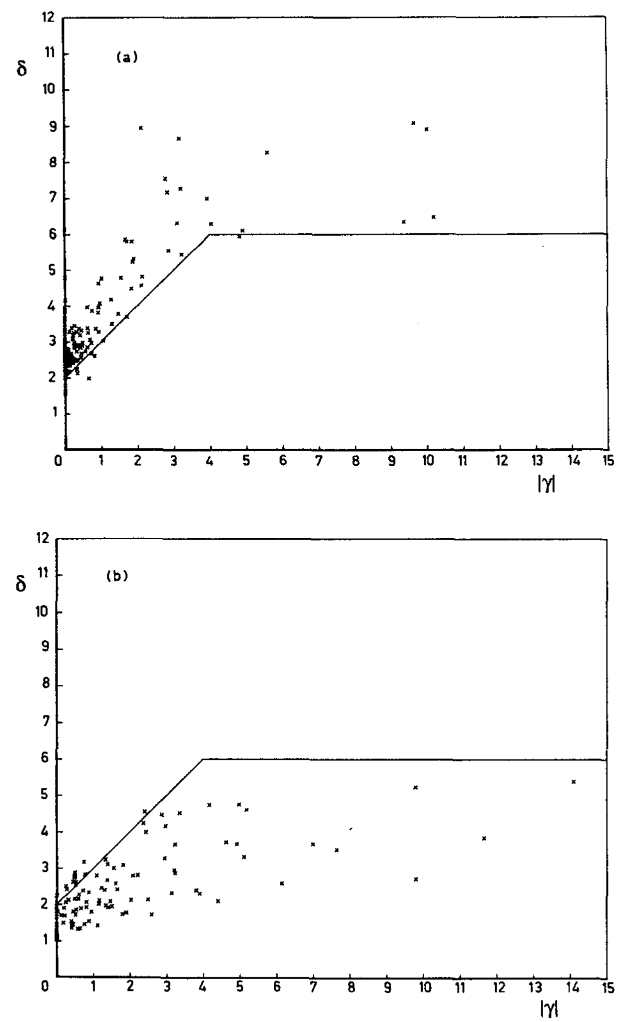

FIG. 5. Scatter diagram of ($|\gamma|$, δ)-values of a large number of model directional distributions. Panel a shows those directional distributions which are deemed to be (nearly) unimodal and symmetric. Panel b shows those directional distributions for which this is not the case. The solid line represents the chosen criterion of Eq. (56).

distribution in this transition region may be expected to be bimodal because of the probable shape of the two-dimensional spectrum, these warnings seem to be proper. This example illustrates that the combination of values of γ and δ provides more information than the individual values separately. It also illustrates that the method used in this paper provides more information than an analysis of the same buoy data based on an assumed symmetric and unimodal distribution such as the $\cos^{2s}$-model.

5. Conclusions

A method for the routine analysis of pitch-and-roll buoy data is described which yields four directional

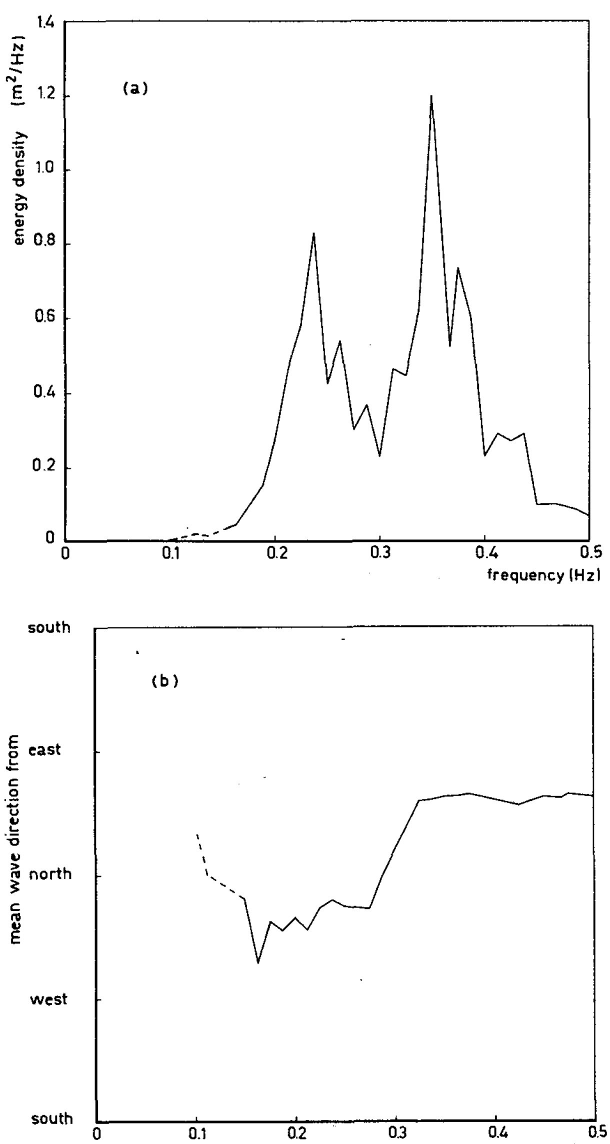

FIG. 6. Observed (a) energy density spectrum and observed (b) mean wave direction as a function of frequency. Observed with a WAVEC-buoy at 1400–1430 UTC 8 July 1980 near research platform Noordwijk. Dashed lines for some minor low-frequency swell.

TABLE 3. Values of directional parameters and warning against unimodal, symmetric model assumption as a function of frequency. Observed with a WAVEC-buoy at 1400–1430 UTC 8 July 1980 near research platform Noordwijk.

Frequency (Hz)	θ_0 (deg)	σ (deg)	γ	δ	Warning issued
0.200	329	33	0.91	3.61	no
0.225	336	30	1.41	4.40	no
0.250	337	38	0.65	3.91	no
0.275	335	50	0.28	1.79	no
0.300	17	55	1.12	1.97	yes
0.325	53	41	2.41	4.18	yes
0.350	56	21	2.22	11.59	no
0.375	58	27	2.40	9.45	no
0.400	54	28	0.65	5.92	no
0.425	50	25	1.54	7.08	no
0.450	56	34	1.30	4.12	no
0.475	58	27	1.12	8.35	no

parameters per frequency, viz. the mean wave direction θ_0, the directional width σ, the skewness γ and the kurtosis δ. These parameters are obtained analytically from the first four Fourier coefficients of the directional distribution $D(\theta)$ without any a priori assumption as to the shape of $D(\theta)$.

The combination of skewness and kurtosis values gives an indication as to whether or not the directional distribution is (nearly) unimodal and symmetric.

Based on the work of Long (1980) and Borgman et al. (1982), simplified expressions are given for the rms error of observations of the mean wave direction and the directional width due to statistical sample variability. These expressions agree well with the results of Monte-Carlo simulations. Order-of-magnitude estimates of the rms errors of skewness and kurtosis observations are also provided by Monte-Carlo simulations. Other errors in the observed mean direction and width introduced by various effects such as buoy response, measurement noise, and wavenumber determination are discussed and to a large extent quantified.

The method proposed here is model-free, readily implemented, and computationally efficient. The error characteristics of the main parameters are well understood and readily determined. The proposed method is therefore well suited for the routine analysis of pitch-and-roll buoy wave data.

Acknowledgments. This work was carried out in the framework of a joint project of the Ministry of Transport and Public Works of the Netherlands, Datawell B.V., the Institute of Applied Physics and the Delft University of Technology. Furthermore, part of these investigations was supported by the Technology Foundation (STW) in the Netherlands. We thank J. A. Battjes and N. Booij of the Delft University of Technology for their comments and suggestions. We thank one of the reviewers of this paper for suggesting the second-order analysis described in section 2d and for sharing some initial results of such analysis with us.

APPENDIX A

Basic Fourier Analysis

A pitch-and-roll buoy measures the vertical elevation and the two slope components of the buoy as a function of time. By applying standard cross-spectral analysis to these signals, resulting in 9 auto-, co- and quad-spectral density functions, the first four Fourier coefficients of the directional distribution per frequency can be estimated directly (Longuet-Higgins et al. 1963):

$$a_1(f) = \int_0^{2\pi} \cos(\theta) D_f(\theta) d\theta = \frac{Q_{zx}(f)}{k(f)C_{zz}(f)} \quad \text{(A1)}$$

$$b_1(f) = \int_0^{2\pi} \sin(\theta) D_f(\theta) d\theta = \frac{Q_{zy}(f)}{k(f)C_{zz}(f)} \quad \text{(A2)}$$

$$a_2(f) = \int_0^{2\pi} \cos(2\theta) D_f(\theta) d\theta = \frac{C_{xx}(f) - C_{yy}(f)}{k(f)^2 C_{zz}(f)} \quad \text{(A3)}$$

$$b_2(f) = \int_0^{2\pi} \sin(2\theta) D_f(\theta) d\theta = \frac{2C_{xy}(f)}{k(f)^2 C_{zz}(f)} \quad \text{(A4)}$$

where C represents the auto- and co-spectra and Q the quadspectra, respectively, with x and y indicating the two orthogonal components of the buoy slope and z indicating the heave. The wavenumber k, defined as 2π divided by the wave length of a harmonic wave traveling in (x, y)-space can be determined in two ways. It can be obtained with the dispersion relation of the linear wave theory (k_t); it can also be estimated from the autospectra (k_d):

$$k_d(f) = \left\{\frac{C_{xx}(f) + C_{yy}(f)}{C_{zz}(f)}\right\}^{1/2}. \quad \text{(A5)}$$

APPENDIX B

Shape Analysis

Consider the directional distribution of Fig. B1 as a characteristic example of the distributions inspected in section 4 of the main text (see also section 2d and Table 1). The distribution is deemed to be (nearly) symmetric if the ratio of the surface area of the left side of the distribution (indicated as A in Fig. B1) and the surface area of the right side (indicated as B in Fig. B1) is within certain limits, or in more formal terms,

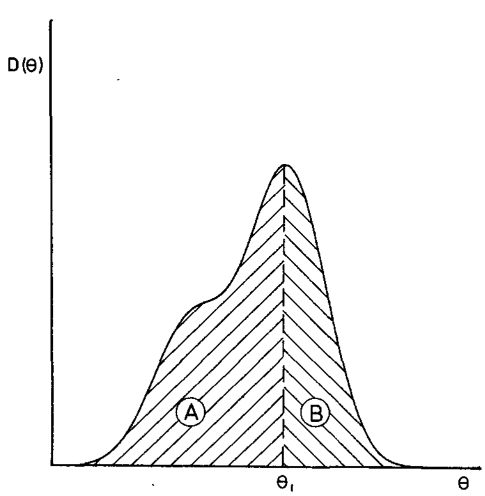

FIG. B1. Example directional distribution to illustrate symmetry.

$$\frac{1}{p} < \int_{\theta_1-\pi}^{\theta_1} D(\theta) d\theta \bigg/ \int_{\theta_1}^{\theta_1+\pi} D(\theta) d\theta < p \quad \text{(B1)}$$

in which θ_1 is the direction of the (highest) peak of $D(\theta)$.

If strictly speaking the distribution is bimodal (see Fig. B2 for a characteristic example), then it is deemed to be (nearly) unimodal if the modulation of $D(\theta)$ is relatively shallow, in other words, if

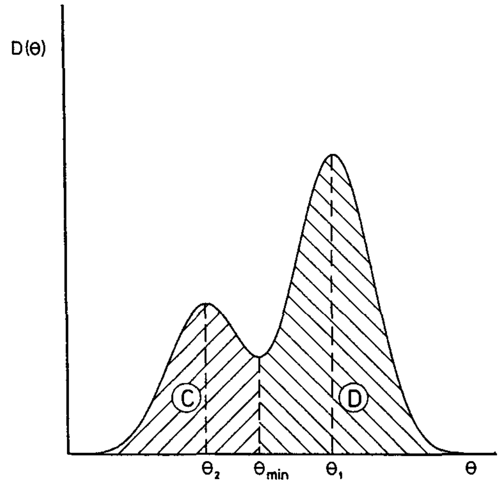

FIG. B2. Example directional distribution to illustrate unimodality.

$$D(\theta_{\min}) > qD(\theta_2) \quad \text{(B2)}$$

in which θ_2 is the direction of the lowest peak of $D(\theta)$, $\theta_{\min}$ is the direction of the secondary minimum of $D(\theta)$ (between θ_1 and θ_2) and the coefficient q is somewhat smaller than 1. It is also deemed to be (nearly) unimodal if the surface area of the secondary lobe (indicated as C in Fig. B2) is small compared with the surface area of the main lobe (indicated as D in Fig. B2), or in more formal terms,

$$\int_{\theta_{\min}-\pi}^{\theta_{\min}} D(\theta) d\theta \bigg/ \int_{\theta_{\min}}^{\theta_{\min}+\pi} D(\theta) d\theta$$

greater than r or less than $1/r$. (B3)

After some trial and error with values of p, q and r which we consider realistic, we found that the (nearly) unimodal, symmetric distributions were well separated from the others in the $(|\gamma|, \delta)$-plane if $p = 2.0$, $q = 0.8$ and $r = 0.2$ (Fig. 5).

REFERENCES

Borgman, L. E., 1969: Directional spectra models for design use. *Offshore Technology Conf.*, Houston, Amer. Inst. Mining, Metallurg. Petrol. Eng. and others, 721–746.

——, 1982: Techniques for computer simulation of ocean waves. *Topics in Ocean Physics,* A. R. Malanotte Rizzoli, Ed., Noord-Holland, 387–417.

——, and E. Yfantis, 1978: Measurement of directional ocean wave spectra from fixed platforms. *Proc. MTS-IEEE Conf., The Ocean Challenge (OCEANS 78),* Washington D.C., 634–638.

——, R. L. Hagan and A. J. Kuik, 1982: Statistical precision of directional spectrum estimation with data from a tilt-and-roll buoy. *Topics in Ocean Physics,* A. R. Malanotte-Rizzoli, Ed., Noord-Holland, 418–438.

Forristall, G. Z., E. G. Ward, V. J. Cardone and L. E. Borgman, 1978: The directional spectra and kinematics of surface gravity waves in tropical storm Delia. *J. Phys. Oceanogr.,* **8,** 888–909.

Gumbel, E. J., J. A. Greenwood and D. Durand, 1953: The circular normal distribution: Theory and tables. *Amer. Statistical Assoc. J.,* March, 131–152 (quoted in Borgman, 1969).

Hasselmann, D. E., M. Dunckel and J. A. Ewing, 1980: Directional wave spectra observed during JONSWAP 1973. *J. Phys. Oceanogr.,* **10,** 1264–1280.

Lawson, L. M., and R. B. Long, 1983: Multimodal properties of the surface-wave field observed with pitch-roll buoys during GATE. *J. Phys. Oceanogr.,* **13,** 474–486.

Long, R. B., 1980: The statistical evaluation of directional spectrum estimates derived from pitch/roll buoy data. *J. Phys. Oceanogr.,* **10,** 944–952.

——, and K. Hasselmann, 1979: A variational technique for extracting directional spectra from multicomponent wave data. *J. Phys. Oceanogr.,* **9,** 373–381.

Longuet-Higgins, M. S., D. E. Cartwright and N. D. Smith, 1963: Observations of the directional spectrum of sea waves using the motions of a floating buoy. *Ocean Wave Spectra,* Prentice-Hall, 111–136.

Lygre, A., and H. E. Krogstad, 1986: Maximum entropy estimation of the directional distribution in ocean wave spectra. *J. Phys. Oceanogr.,* **16,** 2052–2060.

Mardia, K. V., 1972: *Statistics of directional data,* Academic Press.

Mitsuyasu, H., F. Tasai, T. Suhara, S. Mizuno, M. Ohkusu, T. Honda and K. Rikiishi, 1975: Observations of the directional spectrum of ocean waves using a cloverleaf buoy. *J. Phys. Oceanogr.,* **5,** 750–760.

Oltman-Shay, J., and R. T. Guza, 1984: A data-adaptive ocean wave directional-spectrum estimator for pitch and roll type measurements. *J. Phys. Oceanogr.,* **14,** 1800–1810.

Regier, L. A., and R. E. Davis, 1977: Observation of the power and directional spectrum of ocean surface waves. *J. Mar. Res.,* **35,** 433–451.

Van Heteren, J., 1983: Estimation of multi-model directional wave spectra from tri-orthogonal measurements. *Coastal Eng.,* **7,** 205–231.

Van der Vlugt, A. J. M., 1984: Experiences with the WAVEC-buoy. *Proc. Symp. on Description and Modelling of Directional Seas,* Denmark, Dan. Hydraul. Inst. and Dan. Maritime Inst, Paper A3, 6 pp.