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A Simple Derivation of Hasselmann's Nonlinear Ocean–Synthetic Aperture Radar Transform

HARALD E. KROGSTAD

SINTEF Industrial Mathematics, Trondheim, Norway

We present a simple derivation of Hasselmann's nonlinear spectral transform for the synthetic aperture radar imaging of ocean wave fields and discuss some of its implications.

1. Introduction

There are a number of mechanisms affecting the imaging of an ocean surface with a synthetic aperture radar (SAR) [*Hasselmann et al.*, 1985]. The modulation in backscatter is first of all due to long-wave-induced varying surface tilt (tilt modulation) and straining (hydrodynamic modulation). However, the most important effect, and up to recently also the most difficult contribution to treat analytically, has been the velocity bunching caused by the apparent shift in the location of moving scatterers on the surface.

Recently [*Hasselmann and Hasselmann*, 1991], K. and S. Hasselmann gave a new closed form nonlinear spectral transform resulting from the velocity bunching. Judging from the experience with recent field data [*Hasselmann and Hasselmann*, 1991; *Krogstad and Schyberg*, 1991], it seems clear that the transform represents a major step forward in our understanding of the SAR-imaging process of ocean waves.

Below we shall present a simple derivation of the nonlinear transform and discuss some of its implications. It turns out that the degree of nonlinearity is strongly dependent on the azimuth wavenumber scaled by the root-mean-square azimuth shift. In particular, it seems that at least most of the azimuth cutoff observed in all measured SAR-spectra is an intrinsic part of the fully nonlinear theory.

2. Derivation

Following in essence *Hasselmann et al.* [1985], we describe the nonlinear mapping from an ocean wave spectrum Ψ to a SAR image spectrum S as a sequence of linear and nonlinear transformations:

$$S = V\{T[S(\Psi)]\}. \tag{1}$$

Here S denotes the scanning distortion due to the finite velocity scanning of the moving surface, T is the real aperture radar (RAR) modulation consisting of the tilt and hydrodynamic modulations, and V is the velocity bunching.

Let the ocean surface $\eta(\mathbf{G}, t)$ have the spectral representation

$$\eta(\mathbf{x}, t) = \int_{\mathbf{k},\omega} e^{i(\mathbf{kx} - \omega t)}\, dZ(\mathbf{k}, \omega) \tag{2}$$

and the wavenumber-frequency spectrum

$$d\chi(\mathbf{k}, \omega) = E[dZ(\mathbf{k}, \omega)dZ(\mathbf{k}, \omega)^*], \tag{3}$$

see *Phillips* [1977]. We shall assume that the surface satisfies Gaussian linear wave theory (GLWT) in which case ω and $\mathbf{k}$ are connected by the dispersion relation $\omega^2 = g|\mathbf{k}|$ (deep water is considered for simplicity). In this case the ocean surface wavenumber spectrum is defined by

$$\Psi(\mathbf{k})d\mathbf{k} = 2\int_{\omega > 0} d\chi(\mathbf{k}, \omega). \tag{4}$$

Within GLWT, any collection $\mathbf{W}(\mathbf{x}, t)$ of space- and time-varying fields like surface slope, surface velocity, or related quantities is obtained from the surface elevation by a linear transformation

$$\mathbf{W}(\mathbf{x}, t) = \int_{\mathbf{k},\omega} \mathbf{T}_W(\mathbf{k}, \omega)e^{i(\mathbf{kx} - \omega t)}dZ(\mathbf{k}, \omega), \tag{5}$$

where $\mathbf{T}_W$ is the corresponding vector of transfer functions [*Borgman*, 1979].

Consider now an instrument moving along the x axis with velocity V while scanning $\mathbf{W}$ parallel to the y axis. The observed field is thus $\mathbf{Y}(\mathbf{x}) = \mathbf{W}(\mathbf{x}, t = x/V)$ with a covariance function

$$\rho_{\mathbf{Y}}(\mathbf{x}) = \int_{\mathbf{k},\omega} e^{i(\mathbf{kx} - \omega x/V)}\mathbf{T}_W(\mathbf{k}, \omega)\mathbf{T}_W(\mathbf{k}, \omega)^H d\chi(\mathbf{k}, \omega), \tag{6}$$

where H denotes the complex conjugated, transposed vector. Using the symmetry of χ, we may rewrite the integration over positive values of ω and apply the definition of the ocean wavenumber spectrum,

$$\rho_{\mathbf{Y}}(\mathbf{x}) = \int_{\mathbf{k}} e^{i\mathbf{kx}}[\mathbf{T}_W(\mathbf{k}_1, \omega_1)\mathbf{T}_W(\mathbf{k}_1, \omega_1)^H\Psi(\mathbf{k}_1))|\partial\mathbf{k}_1/\partial\mathbf{k}|$$
$$+ \mathbf{T}_W(\mathbf{k}_2, -\omega_2)\mathbf{T}_W(\mathbf{k}_2, -\omega_2)^H\Psi(-\mathbf{k}_2)|\partial\mathbf{k}_2/\partial\mathbf{k}|]d^2k/2. \tag{7}$$

Here $\omega_i = +(g|\mathbf{k}_i|)^{1/2}$ and $\mathbf{k}_1$ and $\mathbf{k}_2$ are functions of $\mathbf{k}$ obtained as solutions of

$$\mathbf{k} = \mathbf{k}_1 - \omega_1/V\mathbf{i},$$
$$\mathbf{k} = \mathbf{k}_2 + \omega_2/V\mathbf{i}, \tag{8}$$

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where i denotes the x axis unit vector. The spectrum of $\mathbf{Y}$ is therefore

$$\Phi_{\mathbf{Y}}(\mathbf{k}) = \mathbf{H}_1(\mathbf{k}_1, \omega_1)|\partial\mathbf{k}_1/\partial\mathbf{k}| + \mathbf{H}_2(\mathbf{k}_2, \omega_2)|\partial\mathbf{k}_2/\partial\mathbf{k}|, \tag{9}$$

where

$$\begin{aligned}\mathbf{H}_1(\mathbf{k}, \omega) &= \mathbf{T}_W(\mathbf{k}, \omega)\mathbf{T}_W(\mathbf{k}, \omega)^H \cdot \Psi(\mathbf{k})/2,\\ \mathbf{H}_2(\mathbf{k}, \omega) &= \mathbf{T}_W(\mathbf{k}, -\omega)\mathbf{T}_W(\mathbf{k}, -\omega)^H \cdot \Psi(-\mathbf{k})/2,\end{aligned} \tag{10}$$

Note that whereas the ocean wavenumber spectrum is nonsymmetric and indicates the propagation direction of the waves, the spectrum of the two-dimensional field $\mathbf{Y}$ is symmetric with respect to $\mathbf{k}$.

The scanning distortion transformation is well known (see, for example, the discussion by *Raney and Lowry* [1978]). The spectra of all linearly surface-related quantities needed below are obtained from (8)–(10) by inserting the appropriate transfer functions.

One basic assumption for the nonlinear transform is that the RAR modulation including both tilt and the hydrodynamic modulations is a linear transformation in the above sense. Let $I^R(\mathbf{x})$ denote the RAR image. In accordance with *Hasselmann et al.* [1985], we write $I^R(\mathbf{x}) = I_0[1 + \delta I^R(\mathbf{x})]$, where I_0 is the mean radar cross section, and below spectra and covariance functions for I^R refer to the nondimensional modulation $\delta I^R(\mathbf{x})$.

Let $\xi(\mathbf{x})$ denote the field of apparent azimuth shifts due to the surface motion. Then $\xi(\mathbf{x}) = (R/V)u_r(\mathbf{x})$, where R is the distance between the scatterer and the radar, V is the radar velocity, and u_r is the velocity component of the surface toward the radar. If the surface wave particle velocity from GLWT is used, u_r and hence ξ are also linear transformations.

Denoting the image after the velocity-bunching transformation by I^V, we have, assuming that the number and the intensity of the point scatterers are conserved during the velocity bunching transformation [*Hasselmann and Hasselmann*, 1991],

$$I^V(\mathbf{x}) = \sum_{\mathbf{x}'} I^R(\mathbf{x}')\left|\frac{d\mathbf{x}'}{d\mathbf{x}}\right|, \tag{11}$$

where the summation extends over all solutions of $\mathbf{x} = \mathbf{x}' - \mathbf{i}\xi(\mathbf{x}')$. Since I^R and ξ according to the assumptions in fact are jointly strictly stationary fields, I^V will be stationary as well. Considering furthermore only second-order ergodic fields, let A be a square region with sides increasing to infinity. Then,

$$\begin{aligned}E[I^V(\mathbf{o})] &= \lim_{|A|\to\infty} |A|^{-1}\int_A I^V(\mathbf{x})\,d\mathbf{x}\\ &= \lim_{|A|\to\infty} |A|^{-1}\int_A \sum_{\mathbf{x}'} I^R(\mathbf{x}F)\left|\frac{d\mathbf{x}'}{d\mathbf{x}}\right| d\mathbf{x}\\ &= \lim_{|A|\to\infty} |A|^{-1}\int_{A'} I^R(\mathbf{x}')\,d\mathbf{x}' = E(I^R) = I_0.\end{aligned} \tag{12}$$

Here we have used that the transformed region A' deviates from A only in negligible narrow strips near the azimuth borders. Furthermore, the spectrum of I^V is obtained from

$$S_{\mathrm{SAR}}(\mathbf{k}) = \lim_{|A|\to\infty} |A|E[|\widehat{I_A^V}(\mathbf{k})|^2] - I_0^2\delta(\mathbf{k})$$

where $\widehat{I_A^V}(\mathbf{k})$ is the "finite Fourier coefficient"

$$\begin{aligned}\widehat{I_A^V}(\mathbf{k}) &= |A|^{-1}\int_A e^{i\mathbf{kx}}I^V(\mathbf{x})\,d\mathbf{x}\\ &= |A|^{-1}\int_A e^{i\mathbf{kx}}\sum_{\mathbf{x}'} I^R(\mathbf{x}')\left|\frac{d\mathbf{x}'}{d\mathbf{x}}\right| d\mathbf{x}\end{aligned} \tag{13}$$

$$\widehat{I_A^V}(\mathbf{k}) = |A|^{-1}\int_A e^{i\mathbf{k}(\mathbf{x}-\mathbf{i}\xi(\mathbf{x}))}I^R(\mathbf{x})\,d\mathbf{x} \tag{14}$$

(In the last expression, $\mathbf{x}'$ has been replaced by $\mathbf{x}$ and the transformed region has been replaced by A). The mean square expectation of the Fourier coefficient is

$$E(\widehat{I_A^V}(\mathbf{k})|^2 = |A|^{-2}\int_{\mathbf{x}\in A}\int_{\mathbf{x}'\in A'} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}G(\mathbf{x}, \mathbf{x}', k_x)\,d\mathbf{x}\,d\mathbf{x}', \tag{15}$$

$$G(\mathbf{x}, \mathbf{x}', k_x) = E\{I^R(\mathbf{x})I^R(\mathbf{x}')e^{-ik_x(\xi(\mathbf{x})-\xi(\mathbf{x}'))}\}. \tag{16}$$

By the stationarity of the fields, G is only a function of $\mathbf{x} - \mathbf{x}'$ which we write, with a slight abuse of notation. $G(\mathbf{x}, k_x)$. Thus,

$$\begin{aligned}S_{\mathrm{SAR}}(\mathbf{k}) &= \lim_{|A|\to\infty} |A|^{-1}\int_{\mathbf{x}\in A}\int_{\mathbf{x}'\in A'} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}G(\mathbf{x} - \mathbf{x}', k_x)\,d\mathbf{x}\,d\mathbf{x}' - I_0^2\delta(\mathbf{k})\\ &= \int_{\mathbf{x}} e^{i\mathbf{kx}}G(\mathbf{x}, k_x)\,d\mathbf{x} - I_0^2\delta(\mathbf{k}).\end{aligned} \tag{17}$$

In order to evaluate the expectation in (16), we consider the four-dimensional Gaussian vector

$$\mathbf{X} = [I^R(\mathbf{x}), I^R(0), \xi(\mathbf{x}), \xi(0)]^t \tag{18}$$

with mean $\mu = (I_0, I_0, 0, 0)^t$ and covariance matrix, Σ,

$$\Sigma(\mathbf{x}) = \begin{pmatrix} I_0^2\rho_{II}(\mathbf{o}) & I_0^2\rho_{II}(\mathbf{x}) & I_0\rho_{I\xi}(\mathbf{o}) & I_0\rho_{I\xi}(\mathbf{x})\\ I_0^2\rho_{II}(\mathbf{x}) & I_0^2\rho_{II}(\mathbf{o}) & I_0\rho_{I\xi}(-\mathbf{x}) & I_0\rho_{I\xi}(\mathbf{o})\\ I_0\rho_{I\xi}(\mathbf{o}) & I_0\rho_{I\xi}(-\mathbf{x}) & \rho_{\xi\xi}(\mathbf{o}) & \rho_{\xi\xi}(\mathbf{x})\\ I_0\rho_{I\xi}(\mathbf{x}) & I_0\rho_{I\xi}(\mathbf{o}) & \rho_{\xi\xi}(\mathbf{x}) & \rho_{\xi\xi}(\mathbf{o}) \end{pmatrix}. \tag{19}$$

The characteristic function of $\mathbf{X}$ is defined $K(\mathbf{t}) = E(e^{i\mathbf{t}^t\mathbf{X}})$ and we observe that

$$G(\mathbf{x}, k_x) = -(\partial^2K/\partial t_1\partial t_2) \qquad \mathbf{t} = (0, 0, -k_x, k_x)^t. \tag{20}$$

For multivariate Gaussian distributions $K(\mathbf{t}) = \exp(i\mathbf{t}^t\mu - (1/2)\mathbf{t}^t\Sigma\mathbf{t})$ [*Anderson*, 1984, theorem 2.6.1] and by carrying out the straightforward differentiation, we arrive at

$$\begin{aligned}G(\mathbf{x}, k_x) = I_0^2e^{-k_x^2(\rho_{\xi\xi}(\mathbf{o})-\rho_{\xi\xi}(\mathbf{x}))}\{1 + \rho_{II}(\mathbf{x}) + ik_x[\rho_{I\xi}(\mathbf{x})\\ - \rho_{I\xi}(-\mathbf{x})] + k_x^2[(\rho_{I\xi}(\mathbf{o}) - \rho_{I\xi}(\mathbf{x}))(\rho_{I\xi}(\mathbf{o}) - \rho_{I\xi}(-\mathbf{x}))]\}\end{aligned} \tag{21}$$

which is identical to equation (42) of *Hasselmann and Hasselmann* [1991].

3. Properties of the Nonlinear Transform

We first note that the G function is general with respect to RAR and shift transformations as long as they are linear in the sense stated in (5).

The function is furthermore homogeneous in the mean radar cross section, and the terms may be scaled by the nondimensional RAR modulation standard deviation $\mu = \rho_{II}(\mathbf{o})^{1/2}$ and the nondimensional azimuth wavenumber $\kappa = k_x\rho_{\xi\xi}(\mathbf{o})^{1/2}$;

$$G(\mathbf{x}, k_x) = I_0^2 e^{-O(\kappa^2)}[1 + O(\mu^2) + O(\kappa\mu) + O((\kappa\mu)^2)]. \tag{22}$$

For a characteristic ocean wavenumber k_0, $k_0\rho_{\xi\xi}(\mathbf{o})^{1/2}$ was denoted the nonlinearity parameter by *Brüning et al.* [1990].

Because of the dependence of k_x, equation (17) is not a straightforward Fourier transform, and the result turns out to be strongly dependent on the magnitude of κ.

For common ocean wave spectra and a RAR modulation given by the tilt modulation [*Monaldo and Lyzenga*, 1986], a rough estimate gives

$$\mu \approx s(\alpha)H_s k_0/2 \tag{23}$$

$$\kappa \approx \frac{k_x}{k_0}\frac{H\omega_0 H_s k_0}{4V} \tag{24}$$

where α is the SAR look angle, H is the SAR height, V is the SAR velocity, $\omega_0^2 = gk_0$, k_0 is the dominant wavenumber, H_s is the significant wave height, and $s(\alpha)$ is the look angle dependence in the tilt modulation transfer function [*Monaldo and Lyzenga*, 1986]. The combination $H_s k_0$ signifies the overall steepness of the sea, typically in the range 0.2–0.4. For an H/V ratio equal to 50 s, $\omega_0 = 1$ rad s^{-1}, and incidence angles around 25°, $\mu = O(0.3)$ and $\kappa = (k_x/k_0) \times O(3)$. Thus, we cannot in general assume that κ is a small quantity.

When $|\kappa| \ll 1$, G is approximately equal to $I_0^2[1 + \rho_{II}(\mathbf{x})]$, and $S_{\mathrm{SAR}}(\mathbf{k}) = S_{II}(\mathbf{k})$, the RAR image spectrum.

For κ up to the size of μ we may expand the G function to second order, giving

$$G(\mathbf{k}, k_x)/I_0^2 = 1 + \rho_{II}(\mathbf{x}) + ik_x[\rho_{I\xi}(\mathbf{x}) - \rho_{I\xi}(-\mathbf{x})] + k_x^2\rho_{\xi\xi}(\mathbf{x}) + O(\kappa^3). \tag{25}$$

The corresponding linear SAR spectrum is obtained after some manipulations from (9):

$$S_{\mathrm{lin}}(\mathbf{k})/I_0^2 = |T(\mathbf{k}_1, k_x, \omega_1)|^2 \left|\frac{\partial \mathbf{k}_1}{\partial \mathbf{k}}\right| \Psi(\mathbf{k}_1)/2 + |T(\mathbf{k}_2, k_x, -\omega_2)|^2 \left|\frac{\partial \mathbf{k}_2}{\partial \mathbf{k}}\right| \Psi(-\mathbf{k}_2)/2, \tag{26}$$

where T now is a combined transfer function

$$T(\mathbf{k}_j, k_x, \omega) = T_I(\mathbf{k}_j, \omega) + ik_x T_\xi(\mathbf{k}_j, \omega),\ j = 1, 2. \tag{27}$$

Here T_I and T_ξ are the RAR and shift transfer functions, respectively, and $\mathbf{k}_1$ and $\mathbf{k}_2$ are solutions of (8).

This form is well known [*Hasselmann et al.*, 1985], and was also noted to follow from the full nonlinear transform of *Hasselmann and Hasselmann* [1991].

When $|\kappa|$ increases further, the nonlinear character of the transform becomes ever more dominant. By applying the formula to a sine wave, characteristic higher-order harmonics occur as most easily seen from a Taylor expansion of the exponential factor in G. When $|\kappa|$ becomes larger than 1, the same exponential factor will tend to suppress all other contributions in G unless when $\rho_{\xi\xi}(\mathbf{x})$ is close to $\rho_{\xi\xi}(\mathbf{o})$, which for realistic sea states will occur only near $\mathbf{x} = \mathbf{o}$. Since $|\rho_{\xi\xi}(\mathbf{x})| \le \rho_{\xi\xi}(\mathbf{o})$, a Taylor expansion around $\mathbf{x} = \mathbf{o}$ gives to the leading order $\rho_{\xi\xi}(\mathbf{o}) - \rho_{\xi\xi}(\mathbf{x}) \approx \mathbf{x}^t\mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a positive definite matrix reflecting the curvature of $\rho_{\xi\xi}(\mathbf{o}) - \rho_{\xi\xi}(\mathbf{x})$ at $\mathbf{o}$. (For the derivatives of $\rho_{\xi\xi}$ to exist in the mathematical sense, it is necessary to assume that the wave spectrum decays fast enough for large frequencies, or that we only integrate up to a certain cutoff frequency.) Thus approximating G by

$$G(\mathbf{x}, k_x) \approx I_0^2[1 + \rho_{II}(\mathbf{o})]e^{-k_x^2\mathbf{x}^t\mathbf{A}\mathbf{x}}, \tag{28}$$

a straightforward Fourier transform yields that $S_{\mathrm{SAR}}(\mathbf{k}) \sim k_x^{-2}$ for large-azimuth wavenumbers, that is, a power decay. However, in practice the decay must eventually be faster because of the SAR antenna pattern and other effects like a limited scene coherence time.

If we write $r(\mathbf{x}) = \rho_{\xi\xi}(\mathbf{x})/\rho_{\xi\xi}(\mathbf{o})$ and expand the $\mathbf{x}$-dependent part of the exponential factor, we have

$$G(\mathbf{x}, k_x) = I_0^2 e^{-\kappa^2} \sum_{n=0}^{\infty} \frac{(r(\mathbf{x})\kappa^2)^n}{n!} K(\mathbf{x}, \kappa), \tag{29}$$

where K is the part in curly brackets in (21). Taking the Fourier transform term by term, we obtain

$$S_{\mathrm{SAR}}(\mathbf{k}) = e^{-\kappa^2}(S_{II}(\mathbf{k}) + O(|\kappa|)) + \sum_{n=1}^{\infty} e^{-\kappa^2}\frac{\kappa^{2n}}{n!} S_{r^nK}(\mathbf{k}). \tag{30}$$

For $|\kappa| < 1$, the first term and hence the RAR spectrum dominates. Since the functions $\exp(-\kappa^2)\kappa^{2n}/n!$ are concentrated around $\kappa = n^{1/2}$ with maxima approximately equal to $(2\pi n)^{-1/2}$, the corresponding contributions to S_{SAR} shift outward along the azimuth wavenumber axis as n increases; the main contribution for a certain κ comes from terms around $n = \kappa^2$. However, when n increases, $r(\mathbf{x})^n$ also becomes very localized around $\mathbf{o}$, and with the same reservations as were mentioned above, we may approximate $r(\mathbf{x})^n$ by $\exp\{-n\mathbf{x}^t[\mathbf{A}/\rho_{\xi\xi}(\mathbf{o})]\mathbf{x}\}$. The corresponding Fourier transform, $S_{r^nK}(\mathbf{k})$, will then be approximately proportional to

$$\frac{1}{n}\exp\left(-\frac{\rho_{\xi\xi}(\mathbf{o})\mathbf{k}^t\mathbf{A}^{-1}\mathbf{k}}{n}\right)[1 + \rho_{II}(\mathbf{o})]. \tag{31}$$

Since this function extends out to $O(n^{1/2})$, we infer $S_{r^nK}(\mathbf{k}) = O(1/n)$ at $\kappa = n^{1/2}$. A rough indication of the behavior of the sum in (30) is therefore given by the function

$$H(\kappa) = e^{-\kappa^2}\sum_{n=1}^{\infty}\frac{\kappa^2}{n!n} = e^{-\kappa^2}\int_0^{\kappa^2}\frac{e^t - 1}{t}\,dt. \tag{32}$$

Elementary calculus yields $H(\kappa) \propto 1/\kappa^2$ for large values of $|\kappa|$, in accordance with the asymptotic behavior found before. The function is drawn in Figure 1, where we first note the pronounced maxima for $|\kappa| \approx 1.2$. Equally noticeable,

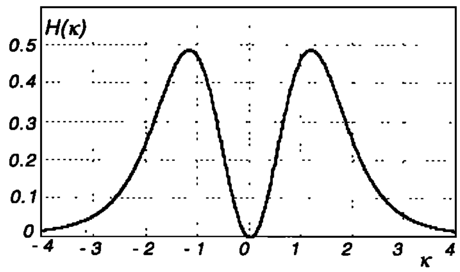

Fig. 1. The H function defined in (23) as a function of the dimensionless azimuth wavenumber κ.

however, is the significant drop when $|\kappa|$ reaches 2–3, suggesting that the nonlinear transform leads to an implicit azimuth cutoff in this range. We note that $\kappa = \pi$ corresponds to a wavelength $2\rho_{\xi\xi}(\mathbf{o})^{1/2}$ which is about the cutoff seen in measured SAR spectra.

We note further that RAR modulation only contributes for large values of κ through the constant $\rho_{II}(\mathbf{o})$, implying that the detailed form of the RAR modulation is of no importance in that region.

To summarize, the preceding analysis leads to the following qualitative picture of the transform. For $|\kappa| \ll 1$, $S^V(\mathbf{k}) = S_{II}(\mathbf{k})$. In the range $|\kappa| \leq O(\mu) < 1$, $S^V(\mathbf{k}) = S_{\text{lin}}(\mathbf{k})$, the linearized SAR spectrum. Around $|\kappa| = 1$ there is a transition range where the nonlinearities become evident. For $|\kappa| > 1$, the transformation is strongly nonlinear, and the detailed RAR modulation is only of minor importance.

For a numerical computation of (17) it is convenient [*Hasselmann and Hasselmann*, 1991] to rewrite the series expansion as a power series in k_x:

$$G(\mathbf{x}, k_x) = I_0^2 e^{-k_x^2\rho_{\xi\xi}(\mathbf{o})} e^{k_x^2\rho_{\xi\xi}(\mathbf{x})}\{A(\mathbf{x}) + k_x B(\mathbf{x}) + k_x^2 C(\mathbf{x})\}$$

$$= I_0^2 e^{-k_x^2\rho_{\xi\xi}(\mathbf{o})} \sum_{n=0}^{\infty} k_n^x D_n(\mathbf{x}), \qquad (33)$$

where

$$D_n(\mathbf{x}) = \frac{1}{i!} A(\mathbf{x})\rho_{\xi\xi}(\mathbf{x})^i + \frac{1}{(i-1)!} C(\mathbf{x})\rho_{\xi\xi}(\mathbf{x})^{i-1} \qquad \text{for} \quad n = 2i,$$

(the last term vanishes for $i = 0$ and $0! = 1$) and

$$D_n(\mathbf{x}) = \frac{1}{i!} B(\mathbf{x})\rho_{\xi\xi}(\mathbf{x})^i \quad \text{for} \quad n = 2i + 1,$$

and A, B, and C are the expressions involving the correlation functions in (21). Since A, B, and C need to be computed only once, one Fourier transform is necessary for each power of k_x.

The necessary number of terms in the expansion to cover a certain azimuth wavenumber range may be estimated by studying the Taylor series of the H function. Figure 2 shows the necessary order of the Taylor expansion of H for a relative error less than 10% in the interval $[-\kappa_0, \kappa_0]$. The maximum error occurs at the end of the interval, and by order is meant the maximum power of κ in (32). Covering the range up to $\kappa = \pi$ thus requires around 20 terms.

Some care must be exercised when selecting the spatial resolution (Δx) and the size (N) of the two-dimensional fast Fourier transform in a numerical evaluation of (17). First of all, the spectral computation should cover an azimuth wavenumber range up to $\kappa = \pi$, which requires an azimuth spatial resolution of at least $\rho_{\xi\xi}(\mathbf{o})^{1/2}$. Moreover, $N \times \Delta x$ should be significantly larger than the extension of the correlation functions in (21) in order to avoid aliasing in the spectrum. However, the most serious restriction on the size of Δx follows by considering the behavior of the G function on a finite numerical grid. The value of G at the origin is $I_0^2(1 + \rho_{II}(\mathbf{o}))$ regardless of the value of $|k_x|$. At the rest of the points, where $|\rho_{\xi\xi}(\mathbf{x})| < \rho_{\xi\xi}(\mathbf{o})$, $G(\mathbf{x}, k_x)$ converges to 0 when $|k_x|$ increases. Because of the rapid variation of G when $|\kappa|$ is large, it is necessary to choose Δx so small that $\exp[-k_x^2(\rho_{\xi\xi}(\mathbf{o}) - \rho_{\xi\xi}(\mathbf{x}))]$ is sufficiently resolved on the grid for the whole range of azimuth wavenumbers. This typically necessitates a finer spatial resolution than merely $\rho_{\xi\xi}(\mathbf{o})^{1/2}$ (The numerical resolution necessary for computation of (17) should not be confused with the actual resolution of the SAR we are considering.)

Applications of the nonlinear transform to real SAR spectra are so far limited, but very encouraging [*Hasselmann and Hasselmann*, 1991; *Krogstad and Schyberg*, 1991]. In particular, the observed azimuth cutoff seems to be close to the cutoff predicted by the transform. It is conceivable, however, that additional azimuth cutoff, for instance, due to a finite scene coherence time, sometimes must be applied to fit actual data.

In conclusion, it seems that with a final "fine tuning" of, say, the RAR modulation transfer function and the effect of multiple looks, the current analytical theory essentially explains the ocean to SAR spectral transform.

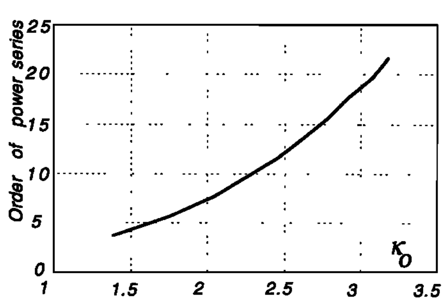

Fig. 2. Approximate necessary order of the series expansion for the G function for covering an azimuth wavenumber range $-\kappa_0 < \kappa < \kappa_0$.

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H. E. Krogstad, SINTEF Industrial Mathematics, Central Building II, NTH, N-7034 Trondheim, Norway.

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