Q

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Abstract. Measurements of the specific attenuation factor $1/Q$ in homogeneous materials in the laboratory and in the field show overwhelmingly that $1/Q$ is substantially independent of frequency, whereas $1/Q$ varies as the first power of frequency in liquids. This conclusion, the result of observations over a wide range of frequencies in metals and nonmetals, in rocks in the laboratory and in the field, suggests that the mechanism for attenuation in solids is substantially different from that of liquids; a proposed nonlinear mechanism for attenuation is reviewed. The available data on attenuation of body waves, surface waves, and free oscillations are reviewed. An inversion method is described whereby the intrinsic Q in shear of the earth's mantle is computed from the surface wave and free oscillation data. The restrictions and assumptions in the calculation are (1) Q must be positive, (2) Q is independent of frequency, and (3) the mechanism of energy dissipation is through a complex modulus. The results show that, in shear, the upper mantle has a much higher attenuation than the lower mantle. Q for the upper mantle is estimated to be 110 from the surface to a depth of 650 km; for the lower mantle, below 650 km, it is much higher than this, but the exact value cannot be estimated with precision. There are hints of a fine structure for Q in the upper mantle, but present accuracy of the data and the assumptions used do not permit the literal use of this result. Partial melting in a low velocity layer at shallow depth is considered, and a small amount of partial melting is not inconsistent with the above result and the data.

1. INTRODUCTION

Were it not for the intrinsic attenuation of sound in the earth's interior, the energy of earthquakes of the past would still reverberate through the interior of the earth today. The chaos resulting from this awesome prospect is a speculation which lies outside the scope of this paper. Rather, it is the task here to investigate where in the earth seismic energy is converted into heat; is this performed with equal efficiency everywhere in the interior, or are some parts of the interior more capable of performing this operation than others? If the answer to this question can be obtained, a new factor will have been found which can help to indicate the physical and chemical state of the earth's interior and may act as support for models of the state of the interior derived from other evidence. But our task is complicated because of the problems introduced by the familiar heterogeneity of the earth's elastic, nondissipative properties.

Except for nonlinear behavior near earthquake foci, seismic strains are small and seismic oscillations take place in the linear domain of elasticity. Attenuation of harmonic signals is therefore exponential, and the magnitude of the attenua-

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tion is describable by the exponential rates of decay. Experiments cited below show that nonlinearity is introduced for strains in excess of 10^{-5} or 10^{-6}. Seismic strains are generally well below this value. The linear region of excitation corresponds to attenuation factors which are independent of the amplitude of the excitation.

The plan of this investigation is as follows: In section 2 we summarize the experimental evidence obtained in laboratory measurements of attenuation in solids, both metals and nonmetals. We indicate that at low frequencies, with certain exceptions, it is reasonable to anticipate that Q for a homogeneous sample is substantially independent of frequency. In the third section some of the models will be explored that have been suggested to explain the frequency dependence of the attenuation factor. In section 4 a summary of the field measurements of attenuation within the earth will be given. In section 5 some of the theoretical problems associated with the interpretation of the experimental results will be presented. In section 6 we present some preliminary results that have been obtained in an attempt to interpret the experimental data in terms of a distribution of the attenuation factor as a function of depth.

Before embarking upon this program, it is appropriate to give a working definition of Q, the specific attenuation factor. This is a reduction to a dimensionless form of the more usual measures of attenuation. Several measures of attenuation are available, all more or less in common use. All these definitions are related to the familiar expression in electrical circuit theory

$$2\pi/Q = \Delta E/E \tag{1}$$

In this definition ΔE is the amount of energy dissipated per cycle of a harmonic excitation in a certain volume, and E is the peak elastic energy in the system in the same volume. Some of the difficulties that will be encountered in later sections of this paper are concerned with the type of energy E to be used in the definition. In the cases of interest, it will be seen that the elastic energy stored locally is not equal to the kinetic energy of motion, and hence not equal to one half the total energy, again locally.

Another measure of attenuation is obtained from the logarithmic decrement of a harmonic wave. The logarithmic decrement is equal to the quantity π/Q. The spatial attenuation factor for a wave function observed throughout space at a fixed time will be of the form exp $(-\alpha x)$, where $\alpha = \omega/2cQ$, and c is the phase velocity. For comparison, the observation of this wave function as a function of time at a fixed point in space yields the damping factor $e^{-\gamma t}$, where $\gamma = \omega/2Q$. In homogeneous systems without dispersion, the various definitions are equivalent. It is in strongly dispersed systems such as the earth, in which the dispersion is produced by geometrical heterogeneity, that some difficulty is encountered in relating the several definitions.

In this study we shall assume that the interior of the earth is locally homogeneous and isotropic. Imagine that we extract a sample of the material of the interior of the earth, take it to the laboratory, and make some measurements on it. In this case, all the definitions are equivalent. A slab of the material placed

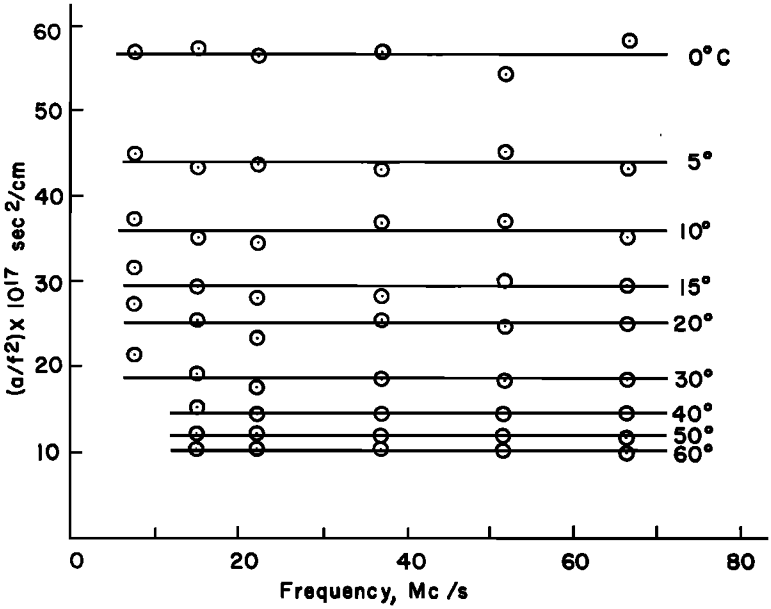

Fig. 1. Attenuation of sound in water [after *Pinkerton*, 1947]. The ratio of the attenuation factor to the square of frequency is plotted as a function of frequency and temperature.

between parallel plates can be studied in a standing wave experiment; we obtain a Q corresponding to a decay rate of standing waves given by γ. If we take this same sample and measure the distance attenuation factor α, we obtain an identical value for Q. If we measure the ratio of the energy stored to the energy dissipated in this sample by either a standing wave or a propagating wave technique, the energy removed per unit volume will be related to the peak strain energy stored through (1); since the maximum strain energy stored is also equal to the maximum kinetic energy, all four of the calculations yield identical results. A fifth definition relating to a measure of the sharpness of the resonance curve of a material undergoing forced vibrations is also equivalent in the case in which the material is homogeneous and isotropic; $Q = \omega/\Delta\omega$, where ω is the resonant frequency, and $\Delta\omega$ is the line width.

In the technique used here for the determination of Q in the solid part of the interior of the earth, we will have to assume that Q is substantially independent of frequency. It is not clear that this assumption can be made rigorously. Nevertheless, laboratory experiments on many solids have shown that, up to moderately high frequencies, the dimensionless quantity Q is indeed independent of frequency to a very good approximation. This result indicates that the mechanism whereby energy is removed from elastic waves in solids is not the same as the mechanism for attenuation in liquids. In most liquids the attenuation factor is found to vary as the square of the frequency (Figure 1); in solids the attenuation factor α varies as the first power of the frequency. Hence in liquids Q^{-1} is proportional to the first power of the frequency.

TABLE 1

Material	Q	Frequency Range	Type of Excitation	Reference
Copper	2,140	10 to 18 cps	Longitudinal	*Lindsay* [1914]
Steel	5,000	5 to 10 cps	Longitudinal	*Lindsay* [1914]
Copper (cold rolled)	640	11 to 25 cps	Bending	*Kimball and Lovell* [1927]
Molybdenum (swaged)	465	8 to 30 cps	Bending	*Kimball and Lovell* [1927]
Nickel (cold rolled)	960	12 to 33 cps	Bending	*Kimball and Lovell* [1927]
$3\frac{1}{2}$% nickel steel (swaged)	1,360	8 to 25 cps	Bending	*Kimball and Lovell* [1927]
Monel (cold rolled)	1,400	8 to 32 cps	Bending	*Kimball and Lovell* [1927]
Copper	2,180	2.5 to 30 kc/s	Longitudinal resonance	*Wegel and Walther* [1935]
	4,380	3 to 30 kc/s	Shear resonance	*Wegel and Walther* [1935]
Lead	36	1.6 to 15 kc/s	Longitudinal resonance	*Wegel and Walther* [1935]
	34	1 to 9 kc/s	Shear resonance	*Wegel and Walther* [1935]
Copper	980	1 to 6 cps	Bending	*Gemant and Jackson* [1937]
Steel	1,850	2 to 8 cps	Bending	*Gemant and Jackson* [1937]
Aluminum (polycrystalline)	5,900	3.1 to 7.5 Mc/s	Longitudinal pulses	*Mason and McSkimin* [1947]
(two samples)	7,630	5 to 15 Mc/s		
Aluminum (polycrystalline)	19,400	3.5 to 4.5 Mc/s	Shear pulses	*Mason and McSkimin* [1947]
(two samples)	17,200	3 to 6.8 Mc/s		
Magnesium	965	7 to 76 Mc/s	Longitudinal pulses	*Roth* [1948]
Copper (unannealed)	1,770	15 to 65 Mc/s	Longitudinal pulses	*Lücke* [1956]
Copper (annealed)	5,830	25 to 75 Mc/s	Longitudinal pulses	*Lücke* [1956]
Aluminum (single crystal, 110 direction)	1,090	15 to 60 Mc/s	Longitudinal pulses	*Lücke* [1956]
Aluminum	200,000	1 to 200 kc/s	Longitudinal resonance	*Zemanek and Rudnick* [1961]

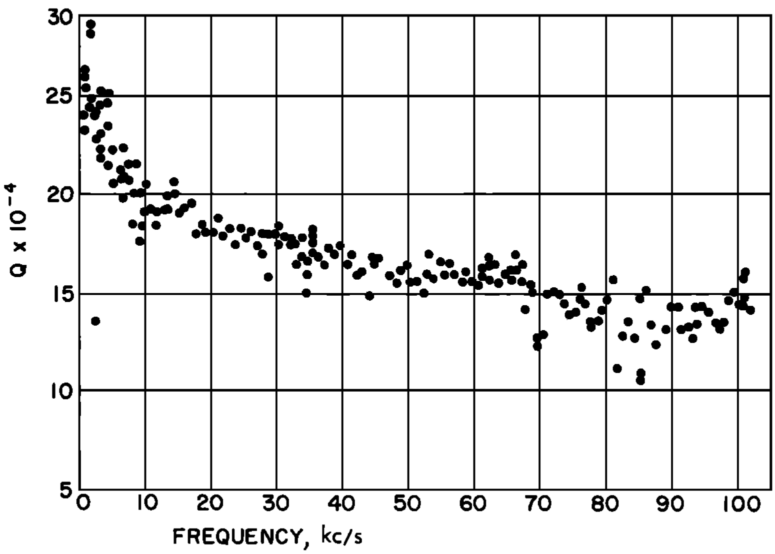

Fig. 2. Q for longitudinal waves in aluminum rod [after *Zemanek and Rudnick,* 1961].

2. ATTENUATION OF SOUND IN SOLIDS IN THE LABORATORY

One of the earliest suggestions in the literature that Q is substantially independent of frequency in solids is found in the work of *Lindsay* [1914]. Since that time there have been a significant number of investigations which have verified that the specific dissipation function $1/Q$ is independent of frequency. A list of values of Q's observed for metals is given in Table 1. Values of Q for other metals are tabulated by *Birch* [1942], but the range of frequencies over which Q is found to be independent of frequency is not given.

Of particular interest in Table 1 is the work of *Zemanek and Rudnick* [1961], who measured the attenuation in a long aluminum rod under longitudinal excitation (Figure 2). In these experiments the range of frequencies was much broader than in the other experiments reported in Table 1. Occasional anomalous low points were reported by Zemanek and Rudnick; these could be correlated closely to torsional modes at frequencies very close to the particular harmonic under investigation in the sequence of longitudinal modes. Of further interest is the notably high value of Q for aluminum reported by Zemanek and Rudnick. These two observations, that of coupling to modes unintentionally excited and the general increase in the value of Q reported, indicate the possibility that many of the observations made heretofore on attenuation in solids may have been misleading principally because losses in the supports or in adjacent modes may not have been carefully eliminated. If this interpretation is correct, however, it would appear that losses in the supports have the same frequency dependence as in the materials under study.

In all the results listed in Table 1, the values of Q are substantially independent of frequency. In ferromagnetic materials [*Bozorth,* 1951; *Wegel and*

TABLE 2

Material	Q	Frequency Range	Type of Excitation	Reference
Soda-lime glass	1,450	5.6 to 6.1 kc/s	Longitudinal resonance	*Wegel and Walther* [1935]
Soda-lime glass	1,340	3.6 to 64 kc/s	Shear resonance	*Wegel and Walther* [1935]
Ebonite	37	1 to 7 cps	Bending	*Gemant and Jackson* [1937]
Ebonite	107	0.3 to 3 cps	Shear	*Gemant and Jackson* [1937]
Trolitol	67	2 to 6 cps	Shear	*Gemant and Jackson* [1937]
Soft glass	330	1 to 3 cps	Bending	*Gemant and Jackson* [1937]
Soft glass	210	1 to 6 cps	Shear	*Gemant and Jackson* [1937]
Wood	120	1-1/2 to 8 cps	Bending	*Gemant and Jackson* [1937]
KCl (single crystal) (110 direction)				
Before heat treatment	4,770	20 to 180 Mc/s	Longitudinal pulse	*Lücke* [1956]
After heat treatment	15,700	20 to 140 Mc/s	Longitudinal pulse	*Lücke* [1956]
Celluloid	7	1/2 to 18 cps	Bending	*Kimball and Lovell* [1927]
Glass	490	12 to 27 cps	Bending	*Kimball and Lovell* [1927]
Corning C-1-172 glass	1,970	4 to 6 Mc/s	Shear pulses	*Mason and McSkimin* [1947]
Corning 012 glass	4,200	4 to 7-1/2 Mc/s	Shear pulses	*Mason and McSkimin* [1947]
Corning 790 Vycor	8,520	4 to 11 Mc/s	Shear pulses	*Mason and McSkimin* [1947]
Amersil fused SiO_2	44,500	5 to 19 Mc/s	Shear pulses	*Mason and McSkimin* [1947]

Walther, 1935; *Roderick and Truell,* 1952] a considerable part of the loss is associated with hysteresis effects, and $1/Q$ as a function of frequency has a substantial component which is not independent of frequency.

Further reference should be made to the work of *Mason and McSkimin* [1947], who carried the investigations on polycrystalline aluminum out of the low frequency range and into the range of frequencies where the attenuation factor is no longer proportional to the first power of frequency. Here a significant fourth power component is found which strongly suggests the possibility that Rayleigh scattering is present.

Similar observations can be made for nonmetals. In the low frequency range, for most polycrystalline materials and for glasses, Q is substantially independent of frequency. Pertinent summaries are offered in Table 2. In amorphous materials, such as fused quartz, Q is also substantially independent of frequency. Again it can be remarked that *Birch* [1942] gives values of Q for a number of other nonmetals, but the range of frequencies over which Q is independent of frequency is not given.

For earth materials observed in the laboratory, the conclusion is the same. Laboratory observations for rocks are summarized in Table 3. Again reference can be made to *Birch's* [1942] summary for values of Q at selected frequencies for a number of earth materials. *Krishnamurthi and Balakrishna* [1957] obtained a similar frequency dependence for the attenuation factors for four rocks, but they did not report the velocities, and so values of Q cannot be calculated.

The results of *Knopoff and Porter* [1963] show that in granite the attenuation of Rayleigh waves over a wide frequency range has a behavior similar to that in polycrystalline aluminum as reported by Mason and McSkimin. In the frequency range 50–400 kc/s, Westerly granite appears to have a Q substantially independent of frequency; at higher frequencies, a fourth power law of attenuation becomes predominant (Figure 3). As in the case of aluminum, this suggests that Rayleigh scattering is an important process. *Knopoff and Hudson* [1964] have shown that the scattering of elastic waves in random heterogeneous media indeed has a fourth power dependence of the type predicted from single Rayleigh scattering.

Peselnick and Outerbridge [1961] have measured Q in limestone over a wide range of frequencies and find that Q diminishes by perhaps a factor of 5 from 4 cps to 10 Mc/s. Thus over a wider range of frequencies than in any other specimens Q is substantially independent of frequency. It is not clear whether the diminution in Q at high frequencies is due to compositional differences among specimens or due to Rayleigh scattering.

Born [1941] has studied sandstone in the laboratory which has had varying amounts of interstitial water injected into the sample. The dry rock has a Q that is independent of frequency; as varying amounts of water are injected, the quantity $1/Q$ has increasing linear dependence on frequency (Figure 4). The linear behavior is, of course, what is expected for the attenuation in a fluid. At the laboratory frequencies used, the attenuation due to the interstitial fluid soon becomes dominant over the attenuation in the dry specimen even for small amounts of fluid.

TABLE 3

Material	Q	Frequency Range	Type of Excitation	Reference
Diabase glass	3,500	50 to 90 kc/s	? (0°C)	*Birch* [1942]
Silica	1,250	1 to 10 cps		*Gemant and Jackson* [1937]
Quincy granite	100	140 cps to 1.6 kc/s	Longitudinal resonance	*Birch and Bancroft* [1938]
Quincy granite	150–200	140 cps to 1.6 kc/s	Torsional, flexural resonance	*Birch and Bancroft* [1938]
Amherst sandstone	52	930 cps to 12.8 kc/s	Longitudinal resonance	*Born* [1941]
Hunton limestone	65	2.8 to 10.6 kc/s	Longitudinal resonance	*Born* [1941]
Sylvan shale	73	3.4 to 12.8 kc/s	Longitudinal resonance	*Born* [1941]
Core from Cockfield-Yequa formation	67	3.6 to 10.9 kc/s	Longitudinal resonance	*Born* [1941]
Orchard, Texas, cap rock core	45	1.1 to 6.6 kc/s	Longitudinal resonance	*Born* [1941]
Orchard, Texas, cap rock core	52	100 to 2000 cps	Transverse resonance	*Born* [1941]
Sandstone	21	50 to 120 cps	Flexural excitation	*Bruckshaw and Mahanta* [1954]
Oolitic limestone	45	50 to 120 cps	Flexural excitation	*Bruckshaw and Mahanta* [1954]
Shelly limestone	63	50 to 120 cps	Flexural excitation	*Bruckshaw and Mahanta* [1954]
Granite	57	50 to 120 cps	Flexural excitation	*Bruckshaw and Mahanta* [1954]
Dolorite	90	50 to 120 cps	Flexural excitation	*Bruckshaw and Mahanta* [1954]
Diorite	125	50 to 120 cps	Flexural excitation	*Bruckshaw and Mahanta* [1954]
Westerly granite	79	50 to 400 kc/s	Rayleigh wave pulses	*Knopoff and Porter* [1963]
Solenhofen limestone	110	3 to 15 Mc/s	Compressional pulses	*Peselnick and Zietz* [1959]
Solenhofen limestone	190	3 to 9 Mc/s	Shear pulses	*Peselnick and Zietz* [1959]
I-1 limestone	165	5 to 10 Mc/s	Compressional pulses	*Peselnick and Zietz* [1959]
I-1 limestone	≈400	3 to 15 Mc/s	Shear pulses	*Peselnick and Zietz* [1959]
H-1 limestone	190	5 to 10 Mc/s	Compressional pulses	*Peselnick and Zietz* [1959]
Solenhofen limestone	920–185	4 cps to 10 Mc/s	Shear	*Peselnick and Outerbridge* [1961]

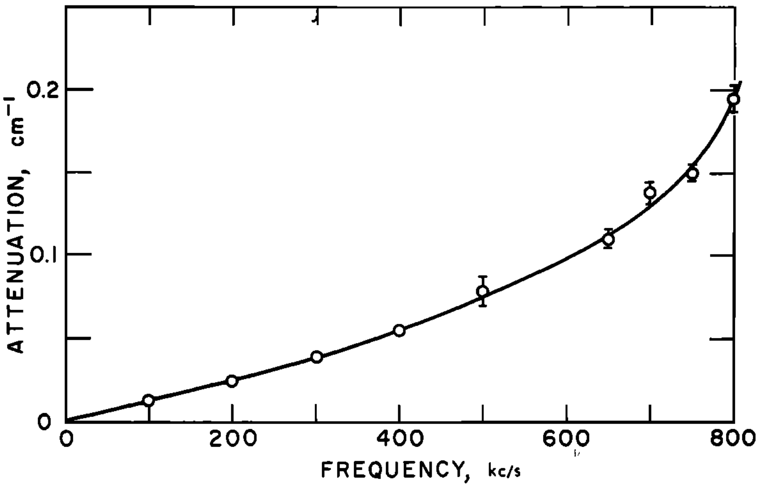

Fig. 3. Attenuation factor for Rayleigh waves in fine-grained (Westerly, R. I.) granite [after *Knopoff and Porter,* 1963].

Of importance is the fact that *Q*'s for rocks are an order of magnitude and more lower than in other materials. *Peselnick and Zietz* [1959] indicate that *Q* for calcite is about 1900, a factor of about 10 greater than in limestone (polycrystalline calcite). This suggests that grain boundary effects are likely to be important and that grain boundary effects show the same frequency dependence for *Q* as do the terms for single crystals and amorphous materials.

A number of anomalous observations of *Q* can be cited. The observations on ferromagnetic materials and the observations of Rayleigh scattering at high

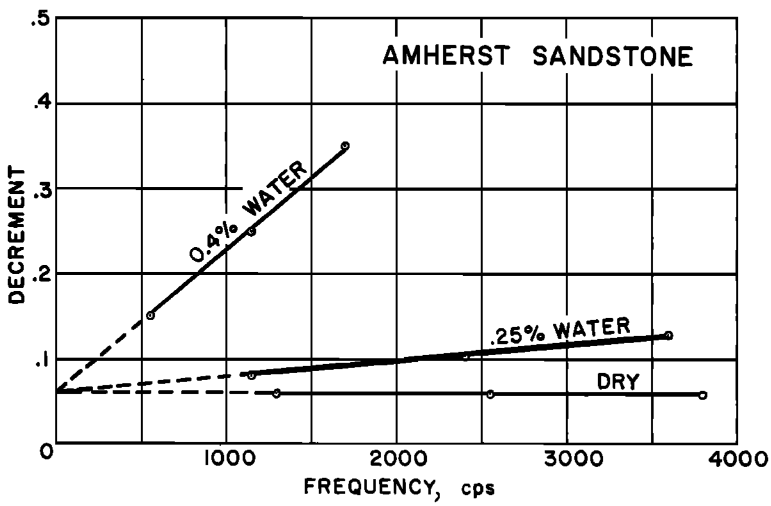

Fig. 4. Logarithmic decrement in sandstone with varying amounts of interstitial water [after *Born,* 1941]. Logarithmic decrement is proportional to Q^{-1}.

frequencies have already been noted. We cite the work of *Auberger and Rinehart* [1961], who have obtained some irregular variation of Q with frequency. *Fitzgerald* [1958*a*, *b*] has reported anomalous dispersion at very low frequencies corresponding to wavelengths that are very much larger than his samples; corresponding to such dispersion, significant absorption is often observed. In addition, in single crystals at ultra high frequencies, the attenuation due to dislocations and impurities has a behavior completely different from that described here for low frequencies. *Krishnamurthi and Balakrishna* [1957] have reported attenuation factors for limestones that are substantially independent of frequency; these results are in conflict with those Peselnick and Zietz obtained on similar materials. Finally, we note that all the observations described here are for very small strains; for finite strains, the attenuation is considerably greater and does not have this simple linear behavior. For strains less than 10^{-5}, *Mason* [1956] has found that the attenuation is substantially independent of the amplitude of the strain. *Peselnick and Outerbridge* [1961] place this limit at 10^{-6}. This suggests that linear theories, in which the dissipative terms are proportional to the amplitude, are appropriate to describe the mechanism which produces the attenuation.

3. MODELS OF LOSS FOR CONSTANT Q

A number of attempts have been made to find modifications of Hooke's law that would account for the deviations from perfect elasticity which have been observed. In the following discussion, we restrict the description to one-dimensional models; the extension to three-dimensional models is not difficult.

The attempts to account for the observed law of variation of Q with frequency can be divided into two groups, those invoking linear models and those invoking nonlinear models. Since the observed attenuation is substantially independent of the magnitude of the strain, for small strains, the processes must be linear in amplitude; any cases of nonlinearity involve nonlinearities in other properties.

The most significant generalization among the early attempts to explain the nature of acoustic loss is that of *Boltzmann* [1876]. In Boltzmann's theory the strain due to an applied stress is delayed by some sort of 'memory' behavior in the material. For a stress excitation, which is a complicated function of time, this can be expressed as a convolution with an elementary memory function which expresses the nature of the delay. Thus

$$P(t) = \int_{-\infty}^{t} E(\tau)M(t - \tau)\, d\tau \tag{2}$$

where $E(\tau)$ and $P(t)$ are the strain and stress, respectively; the equation is considered to be schematic, and the actual tensor nature of the functions is not written explicitly. The memory function is $M(t)$. If we take the Fourier transform of (2), we see that the transform of $M(t)$, $m(\omega)$ takes the nature of a transfer function which describes the loss mechanism:

$$p(\omega) = e(\omega)m(\omega) \tag{3}$$

Thus, if the nature of the loss is given in real space-time as an operator describing a real mechanism such as viscosity, internal friction, etc., presumably the Fourier transform of this function can be found and comparison can be made with the experimental results of harmonic excitation of the material under study. In a similar way, the reverse procedure can be followed. If the transfer function $m(\omega)$ is known, its inverse can be taken, and thus the mechanism for the nature of the attenuation can be described in real space-time.

Maxwell [1866] suggested a model in which viscosity was introduced to describe creep under large deformations. Under small amplitudes of excitation, viscosity becomes the causative mechanism for loss. The Maxwellian relation, in one dimension as before, is

$$(dE/dt) = (1/\mu)(dP/dt) + (P/\eta) \tag{4}$$

where μ is an elastic constant and η is a viscosity. In terms of lumped circuit parameters, we construct a series combination of mass, spring, and dashpot. The transfer function is

$$m(\omega) = \frac{\mu}{1 - i\mu/\eta\omega} \tag{5}$$

The complex wave number is $k = \omega(m/\rho)^{1/2}$, where ρ is the density. We write

$$Q = \text{Re}\, k/2 \,\text{Im}\, k. \tag{6}$$

For the Maxwell value of $m(\omega)$,

$$Q \approx \omega\eta/\mu \tag{7}$$

Meyer [1874*a*, *b*], *Kelvin* [1878], and later *Voigt* [1892] suggested a model, often called a viscoelastic model, in which the stress and strain are related by

$$P = \mu E + \eta(dE/dt) \tag{8}$$

again in one dimension. The lumped circuit for this model consists of spring and dashpot in parallel. This model has a transfer function

$$m(\omega) = \mu(1 + i\omega\eta/\mu) \tag{9}$$

Hence the acoustic loss for this solid is, by (6),

$$Q^{-1} = \omega\eta/\mu \tag{10}$$

In fact, the Kelvin-Voigt model shows a frequency dependence that corresponds exactly to the attenuation of sound in liquids. This suggests that viscous damping is very likely the predominant mechanism for attenuation of sound in liquids.

The attenuation must be an even function of frequency (Figure 5) so that energy will be dissipated for both positive and negative frequencies. Over a rather broad range of frequencies, the experimental evidence requires that the curves in the two quadrants be mainly linear.

From the two solutions above, however, the attenuation factor as a function of frequency varies as the square of the frequency for the Kelvin-Voigt model or is independent of frequency for the Maxwell model.

It is necessary to demand that a relatively complex combination of masses, springs, and dashpots in intricate series-parallel combinations be used to approximate linear relations. That this combination of physical parameters occurs in a large number of materials with exactly the same ratio of values for all components would be highly fortuitous indeed. *Knopoff and MacDonald* [1958] have shown that any linear combination of constant parameters consisting of purely elastic elements and purely viscous elements can only lead to attenuations which are even functions of frequency *in each quadrant;* odd powers of ω such as the linear relations described in Figure 5, in each quadrant, can be obtained only as an approximation. Before rejecting a complex system of linear parameters as representing the mechanism required to explain the observed first power frequency dependence of the attenuation factor, we investigate the possibility that this is in fact an approximation over the entire frequency band.

We start by noting that the dynamic ratio of stress to strain will be an elastic modulus, and, again ignoring the tensor relations involved, we find that the function $m(\omega)$ is indeed the complex modulus of elasticity.

$$m(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} M(t)\, dt \tag{11}$$

Following *Derjaguine* [1934], we take the real and imaginary parts of (11). These are evidently the cosine and sine transforms of $M(t)$.

$$\begin{aligned} C(\omega) &= \int_0^{\infty} M(t) \cos \omega t\, dt \\ S(\omega) &= \int_0^{\infty} M(t) \sin \omega t\, dt \end{aligned} \tag{12}$$

If we further assume that density remains unchanged and that all the attenuation is associated with the complex modulus, then a complex velocity can be written as

$$v = (m/\rho)^{1/2} = \{C + iS\}^{1/2} \rho^{-1/2} \tag{13}$$

Since the wave function is $e^{i\omega[x/v - t]}$, the phase velocity c and the attenuation factor α become

$$1/c = \mathrm{Re}\ 1/v \qquad \alpha = \omega\ \mathrm{Im}\ 1/v \tag{14}$$

Thus the specific attenuation factor Q^{-1} can be written as

$$Q^{-1} = \frac{2\alpha c}{\omega} = 2\left\{\frac{(C^2 + S^2)^{1/2} - C}{(C^2 + S^2)^{1/2} + C}\right\}^{1/2} \tag{15}$$

For very small loss, $Q \gg 1$, $C(\omega) \gg S(\omega)$, we see that (15) can be approximated by

$$Q^{-1} \approx S/C \tag{16}$$

A function with a transform pair $S(\omega)$ and $C(\omega)$ can now be found which fits the observations of the frequency dependence of Q. There is an infinite set of such functions.

Lomnitz [1957] has indicated that the function log t has a transform pair which fits the linear behavior of Q quite well over a rather large frequency range. He cites evidence for a logarithmic memory function from experiments on creep in rocks [*Lomnitz,* 1956]. However, the creep evidence on rocks is obtained at strains five or six orders of magnitude larger than those used in the acoustic measurements. The acoustic strains are probably of the order of 10^{-10} in seismic waves in the earth, and are very likely of the same order of magnitude in the laboratory. As noted above, laboratory experiments [*Mason,* 1956; *Peselnick and Outerbridge,* 1961] have shown that, for strains in excess of 10^{-6}, nonlinearity is introduced and the samples may no longer be in a linear domain of response.

A further difficulty encountered with the Lomnitz model is that the function log t is singular at time $t = 0$. To remove this singularity, a characteristic time t_0 must be introduced so that the function $M(t)$ will be $\log(t + t_0)$. An estimate of t_0 can be obtained from causality conditions.

Futterman [1962] has computed possible transform pairs $S(\omega)$ and $C(\omega)$ as consistent with the observations as possible, if the causality conditions, written as the Kramers-Kronig relations, are obeyed. Futterman's work shows that the condition $C(\omega) =$ constant, $S(\omega) =$ constant is inconsistent with causality. If this condition did hold, the curves of Figure 5 could be extended to the origin (Figure 6); there would obtain a discontinuous slope at the origin and a phase velocity independent of frequency [*Knopoff,* 1956, 1959].

In view of Futterman's result, some cutoff frequency must be used in the relations so that the condition $Q \neq Q(\omega)$ holds only for frequencies higher than the cutoff frequency (Figures 6 and 7). Futterman has investigated a number of models, all involving a characteristic lower cutoff frequency. If we allow the phase velocity to be frequency dependent, a reasonable model is a logarithmic one in which the dispersion introduced by casuality is of the order of Q^{-1}. The specific attenuation factor is then roughly constant over a rather broad frequency range, and it is possible to determine the nature of the transfer function $m(\omega)$. In this case the transform pair is

$$c = c_0\left[1 - \frac{1}{\pi Q_0}\log\left(\gamma\frac{\omega}{\omega_0}\right)\right]^{-1} \tag{17}$$

$$Q = Q_0\left[1 - \frac{1}{\pi Q_0}\log\left(\gamma\frac{\omega}{\omega_0}\right)\right]^{-1} \qquad \omega \gg \omega_0 \qquad \gamma = 0.57721\cdots$$

where c_0, Q_0, ω_0 are constants for the system.

Lomnitz's work can be criticized on the basis of Futterman's observation that the cutoff must be placed at low frequencies rather than at high. This indicates that the characteristic time is long, longer than any characteristic time in the excitation.

If we search for a mechanism which localizes the attenuation in some microstructure, such as imperfect elasticity in the bonds between atoms and groups of atoms, the value of an approach to this study through a linear transform pro-

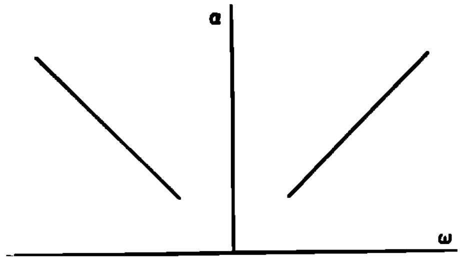

Fig. 5. Schematic diagram of attenuation factor as a function of frequency for most solids at low frequencies.

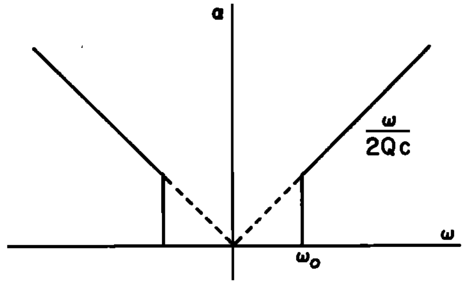

Fig. 6. Attenuation factor as a function of frequency with a low frequency cutoff.

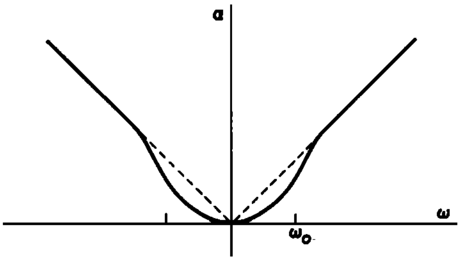

Fig. 7. Attenuation factor as a function of frequency with a low frequency cutoff.

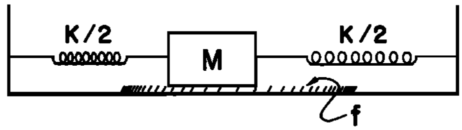

Fig. 8. Lumped parameter model for the nonlinear frictional solid

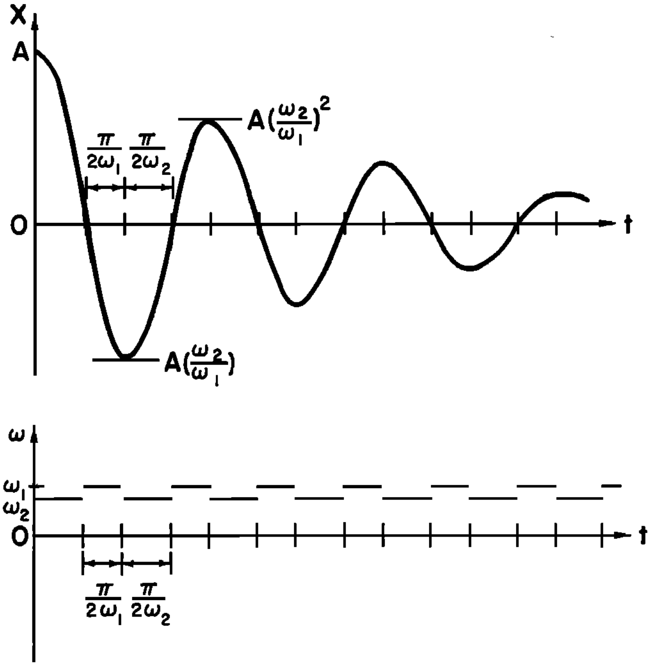

Fig. 9. *Top:* Oscillator position as a function of time for the model of (19). Each quarter cycle is a sinusoid with frequency $\omega(1 \pm b)^{1/2}$. *Bottom:* Instantaneous frequency.

cedure seems questionable. Obviously no characteristic lengths are available which far exceed dimensions of the specimens. On the other hand, characteristic times of creep *are* of the magnitudes required by Futterman's theory; however, no evidence of permanant deformation under small acoustic strains is present at laboratory frequencies. No causal statement in terms of a differential equation in space-time is yet available.

Another mechanism that has been suggested for obtaining a Q independent of frequency is one associated with hysteresis [*Krasilnikov,* 1963]. Consider a material with a non-Hookian stress-strain relation. Cycling of the material yields a series of curves enclosing a net area. The area under any loop represents the energy removed through cycling. The claim is usually made that the energy removed per cycle is a constant, and hence a Q independent of frequency will be obtained. What is ignored in this statement is that the area of the hysteresis curve is frequency dependent. The Boltzmann memory relationship described above shows that for cyclic strains the stress and strain are out of phase and hence can be described by an ellipse in the stress-strain plane. The phase factor is given by the ratio of imaginary to real parts of $m(\omega)$, and thus it and the width of the ellipse are frequency dependent in the models above. In fact the phase factor is precisely Q^{-1}. The problem has merely been restated if hysteresis is used as an explanation for a Q independent of frequency. What stress-strain model leads to a hysteresis cycle whose width is independent of frequency? From the above, linear stress-strain models are insufficient to meet this condition. The hysteretic construction suggests that cusping upon strain reversal is a likely explanation for the observation of Q independent of frequency; this, in turn, suggests nonlinearity in the model.

Knopoff and MacDonald [1958, 1960] have suggested that nonlinear mechanisms may be the resolution of the difficulty of the frequency dependence. The nonlinear mechanisms are linear in the amplitudes but are nonlinear in the directionality of the damping forces. This suggests that the nonlinear damping mechanism is of the nature of a frictional force. Coulomb friction has been suggested [*Förtsch,* 1956], but this can be shown to give an improper frequency dependence for the attenuation factor. In terms of lumped parameters, *Knopoff and MacDonald* [1960] have suggested that the linear model be represented by a mass-spring system which is attenuated by some sort of 'sandpaper' in which the attenuation depends not only on the sign of the displacement but also on the sign of the velocity (Figure 8). In their model, the differential equation of motion, for the one-dimensional case, can be written as

$$\rho \frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial^2 u}{\partial x^2} = -f_0 \left| \frac{\partial^2 u}{\partial t^2} \right| \operatorname{sgn} \frac{\partial u}{\partial t} \tag{18}$$

where u is the displacement of a particle, and f_0 is a constant. This type of frictional term is reasonable for the following reason: The energy per cycle is of the order of $\frac{1}{2} \rho \dot{u}^2 = \frac{1}{2} \rho\omega^2 A^2$, where A is the amplitude of the motion; the work done against friction will be of the order of $f_0 u \; |\partial^2 u/\partial^{t2}| = f_0 \omega^2 A^2$, and hence the ratio of energy lost per cycle to the total energy is independent of frequency.

The differential equation of motion for the harmonic oscillator of Figure 8 is

$$\ddot{x} + \omega^2(x + b\,|x|\,\mathrm{sgn}\,\dot{x}) = 0 \tag{19}$$

or

$$\begin{array}{lll} \ddot{x} + \omega^2(1 \pm b)x = 0 & x \gtrless 0 & \dot{x} \gtrless 0 \\ & x \gtrless 0 & \dot{x} \lessgtr 0 \end{array}$$

In each of the quarter cycles $x \gtrless 0, \dot{x} \gtrless 0$, the equation is harmonic; the complete solution is piecewise continuous. If we impose the conditions of continuity of displacement and velocity at the turning points among the four regions, the solution is a damped wave (Figure 9). The generalization of the relationship (18) to three dimensions, in which the tensor nature of the stresses and strains can be taken into occount, can be made quite easily.

The discontinuities in the wave form associated with the nonlinear nature of the internal friction mechanism, as suggested above, indicate that there should be some experimental evidence concerning this point. It is known that for wave propagation through a nonlinear filter [*Wiener*, 1958] there is continual degradation of the energy from the low frequencies into the high. Therefore, differential power spectra measured at several points should show relative regeneration of energy in the high frequencies. This is best done by bispectral analysis [*Hasselmann et al.*, 1963; *MacDonald*, 1964]. To date, bispectra of elastic pulses have not been calculated because of the high resolution required of the observations for the numerical calculations in bispectral analysis. The ultimate computation of bispectra should resolve the question between the use of linear and nonlinear models for the attenuation.

Mason [1958, p. 279] claims that a microscopic model for a frequency-independent Q can be obtained if the breakaway of pinning points of dislocations under acoustic excitation at normal temperatures is investigated. His result depends on the length of the Burgers vector, the density of dislocations, and a thermal activation term exp $(-E/kT)$. However, after a serious algebraic error is corrected, he obtains Q^{-1} proportional to ω^{-1}, rather than the ω^0 that is required.

4. ATTENUATION OF SEISMIC WAVES

We first consider the measurement of attenuation by the methods of explosion seismology and seismic prospecting. The measurements of interest for the purposes of this part of the discussion are those which have been made in seismic media that are relatively homogeneous. At least two critical seismic measurements have been performed. The data of *Collins and Lee* [1956], who have made measurements in the Pottsville, Maryland, sandstone formation, are of interest. The seismic pulses have been observed at a small number of stations ranging from 10 to 27.5 feet in a range sufficiently short that the material can be considered homogeneous. There appears to be no interaction with heterogeneous matter outside the boundaries of the formation in question. The seismic pulses at the various seismometers were Fourier analyzed, and the attenuation factor was found to be linear with frequency in the frequency range 100–1000 cps

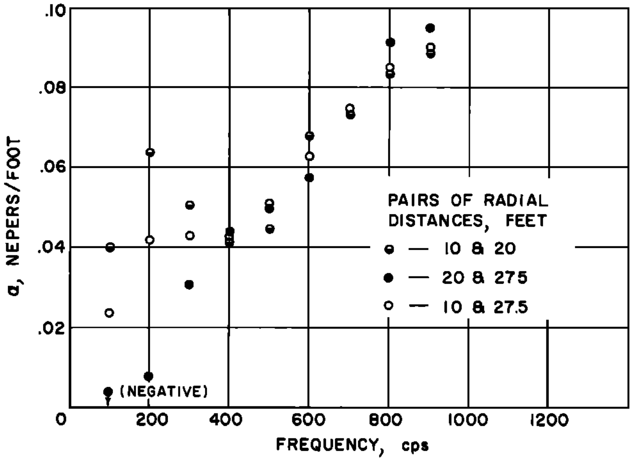

Fig. 10. Attenuation factor for compression waves in Pottsville sandstone [after *Collins and Lee,* 1956].

(Figure 10). This frequency range is comparable to that of the laboratory observations in the resonance experiments of Born (Figure 4).

An experiment performed over a somewhat larger range but in a more extensive body is the set of observations of *McDonal et al.* [1958] on the Pierre, Colorado, shale. By a technique similar to that of Collins and Lee, a Fourier analysis was made of the pulses recorded at seismometers placed at several positions in holes drilled into the formation. The attenuation factor as a function of frequency appears to be linear over the frequency range 50 to 550 cps (Figure 11). The Pierre shale formation had been studied earlier, in experiments analyzed by *Ricker* [1941], but the interpretation was based on a method in

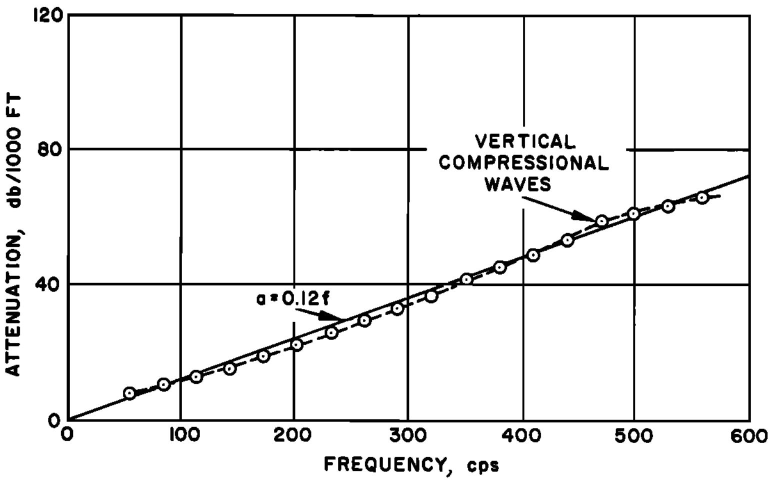

Fig. 11. Attenuation factor for compression waves in Pierre shale [after *McDonal et al.,* 1958].

TABLE 4

Formation	Q	Frequency Range	Type of Excitation
Pottsville sandstone*	7	100 to 900 cps	Compressional pulses
Pierre shale†	23	50 to 450 cps	Compression waves
Pierre shale†	10	20 to 125 cps	Shear waves
Loose Martite ore‡	12	300 to 500 cps	Compression waves
Jaspilite‡	13	400 to 1000 cps	Compression waves
Magnetite-hematite‡	53	600 to 1500 cps	Compression waves
Aegirite hornstone‡	22	450 to 900 cps	Compression waves

* *Collins and Lee* [1956].
† *McDonal et al.* [1958].
‡ *Karus* [1958].

which the broadening of seismic pulses with time and distance are correlated. Unfortunately, the correlation was strongly gaged by a prediction of the broadening of a pulse in a viscoelastic medium of the Kelvin-Voigt type, computed earlier [*Jeffreys,* 1931]. The amount of broadening observed is quite small, and hence the correlation is not discriminatory with regard to the model if the analysis is done in space-time as in the case of the analysis by Ricker. The analysis by McDonal et al. in space-frequency is considerably more reliable.

Karus [1958] has measured the attenuation factor in a number of soils at low frequencies using seismic pulses in more or less homogeneous formations. His results, too, yield values of Q independent of frequency. The results of these three sets of observations in the field are summarized in Table 4.

It is perhaps not surprising that these analyses of pulses in relatively homogeneous formations in situ should give results which are comparable in magnitude, frequency dependence, etc., to the results obtained in the laboratory on homogeneous rocks. However, the strains from explosion sources are, in many cases, somewhat larger than those obtained in the laboratory.

The results of all the laboratory measurements on nonmetals, metals, and rocks, and the few field measurements on rocks, are that the observation that Q is substantially independent of frequency is valid for most solids. Thus, if data were available from observations made at the surface that gave the attenuation of seismic waves in the earth as a function of frequency, an interpretation could be made using model that Q is assumed to be a function of depth in the earth but independent of frequency. The fact that the earth is inhomogeneous will make an interpretation of the distribution of Q as a function of depth relatively complicated; this interpretation will be reserved for the last section. The rest of this section will be devoted to a report of the observations made of attenuation in the real and therefore inhomogeneous earth.

Attenuation studies have been made on seismic body waves and surface waves propagated through and on an inhomogeneous earth. Several attempts have been made to measure the attenuation of seismic body waves [*Gutenberg,* 1958]. Gutenberg's work is unfortunately subject to the same criticism as Ricker's; he has studied the rate at which seismic pulses broaden from seismic observatory to seismic observatory. The data show much scatter, mainly because of the variation in the frequency response characteristics of instruments, the variation in the kind of foundation on which the seismographs are placed, and finally because of the uncertainities in the procedure introduced by making space-time correlations. This latter criticism can of course be resolved through the use of Fourier analysis; Fourier analyses of large numbers of data have become possible only recently through the application of large electronic computers to the problem. The influence of scattering due to heterogeneity in the upper layers of the earth has not been removed. Despite the uncertainties in the procedure, and although a frequency dependence was obtained inconsistent with the above discussion, Gutenberg obtained a value for the attenuation factor for P waves at 12 seconds of the order of $Q = 400$ and for S waves for the same period $Q = 700$; these values are not unreasonable. *Karnik* [1956] has studied this problem by a similar procedure, that is through measurements of space-time pulse broadening; his results are subject to the same criticism.

Anderson and Kovach [1964] have recently observed multiple reflections from a deep focus earthquake in Brazil recorded at a nearby station in Peru. The seismic shear waves traveled along several almost radial paths of multiple reflection between the focus, the core-mantle boundary, and the surface of the earth. From a combination of the amplitudes of these multiple reflections, an estimate of the reflection coefficient at the core-mantle boundary is possible. If a simplifying assumption is made about the directivity of the seismic source so that we can estimate the amount of shear wave energy radiated upward as well as downward from the earthquake focus, then a comparison of the amplitudes of these two types of seismic events at the observatory yields an estimate of Q for the two regions. The mean value of Q obtained in this way for the mantle in shear is of the order of 500, a value in agreement with that obtained earlier by *Press* [1956] by similar methods. Anderson and Kovach's results indicate that for the upper mantle Q is about 160 and for the lower mantle about 1450. These values are in good agreement with Q's for an inhomogeneous earth as computed from the interpretation of surface wave data (see below). Q is inferred to be roughly independent of frequency over the range 11 to 25 seconds for the entire mantle. This same event has also been observed by *Steinhart et al.* [1963], and results comparable to those reported by Anderson and Kovach have been obtained.

Both Rayleigh and Love surface waves have been used by a number of seismologists to obtain data on attenuation. The greater reliability of these data, when compared with the body wave technique, is considerable; however, the interpretation is considerably more involved, as will be seen in the next section. The use of only one seismograph for recording data suitable for the interpretation of the attenuation eliminates the possible influence of variation in instrumental characteristics from observatory to observatory. This difficulty has been avoided by Anderson and Kovach for body waves through the use of multiple reflections and can also be avoided in seismic surface wave observations. The general technique is to make observations at one observatory of the seismic surface waves from a very large earthquake as recorded on a long period seismograph. The seismic surface waves will execute several circuits of the earth if the magnitude of the energy released at the focus is sufficiently great. If Fourier analysis of the several passes of the same phase are made, the spectra of the passes can be compared. Before 1962, direct visual comparison of amplitudes at certain periods was made using the natural spectrograph that occurs in the earth due to the dispersion of surface waves at long range. In more recent years techniques of Fourier analysis have been applied to these observations as well.

Ewing and Press [1954*a*, *b*] analyzed the attenuation of mantle Rayleigh waves generated by the Kamchatka earthquake of 1952 by this method, and found a Q in the period range 250–350 seconds of about 350. In the period range 140–215 seconds, $Q = 170$.

Satô [1958] has measured the value of Q by a rough Fourier analysis for the mantle wave *LG* from both the Kamchatka (1952) and the New Guinea (1938) earthquakes. The *LG* phase is a wave of pure shear; the Rayleigh wave involves, of course, a mixture of compressional and shear motions. Satô's value of Q for Love waves ranged between 85 and 220 in the period range 360 to 450 seconds.

Båth and López-Arroyo [1962] have made attenuation measurements of Love waves from the Peruvian earthquake of January 13, 1960. The value of Q in the period range 75 to 300 seconds is approximately 90.

Additional evidence from measurements of the attenuation of Rayleigh waves from the Chilean earthquake of May 21, 1960, can be found in the work of *Press et al.* [1961]; values of Q comparable in magnitude to those summarized above were obtained. It is quite clear that there is significantly more attenuation in Love waves than in Rayleigh waves.

Perhaps the most extensive analysis of attenuation of both Love and Rayleigh waves has been made by *Ben-Menahem* [1964], who measured the attenuation of Love and Rayleigh waves from four great earthquakes from observations of multiple circuits around the earth past one station. The great circle paths from the Assam (1950), Mongolia (1957), Kamchatka (1952), and Alaska (1958) earthquakes to Pasadena, the reporting observatory, are all more or less the same. By Fourier analysis he has obtained a curve of specific attenuation factor covering the period range roughly 75 to 300 seconds for Love waves and 100 to 340 seconds for Rayleigh waves. Figure 12 is a composite of the values for these earthquakes as well as those summarized above. The experimental accuracy is indicated by the scatter in the graphs of the attenuation factors γ. The Love waves show a rather gentle increase in Q^{-1} toward the longer periods; the Rayleigh waves have a decrease in Q^{-1} toward longer periods. As noted above, the Love waves show a greater attenuation than do the Rayleigh waves. It should be noted that all the values of Q as reported here are dimensionless values of attenuation as observed at the surface of an inhomogeneous earth. The interpretation as to an intrinsic Q as a function of depth remains to be made.

At still longer periods in the earth, observations of attenuation can be made from the widths of the lines in the spectral analyses of the standing waves in the free modes of oscillation of the earth. To date, two major earthquakes, the Chilean earthquake of May 21, 1960, and the Alaskan earthquake of March 27, 1964, have been strong enough to excite the free modes of oscillation of the

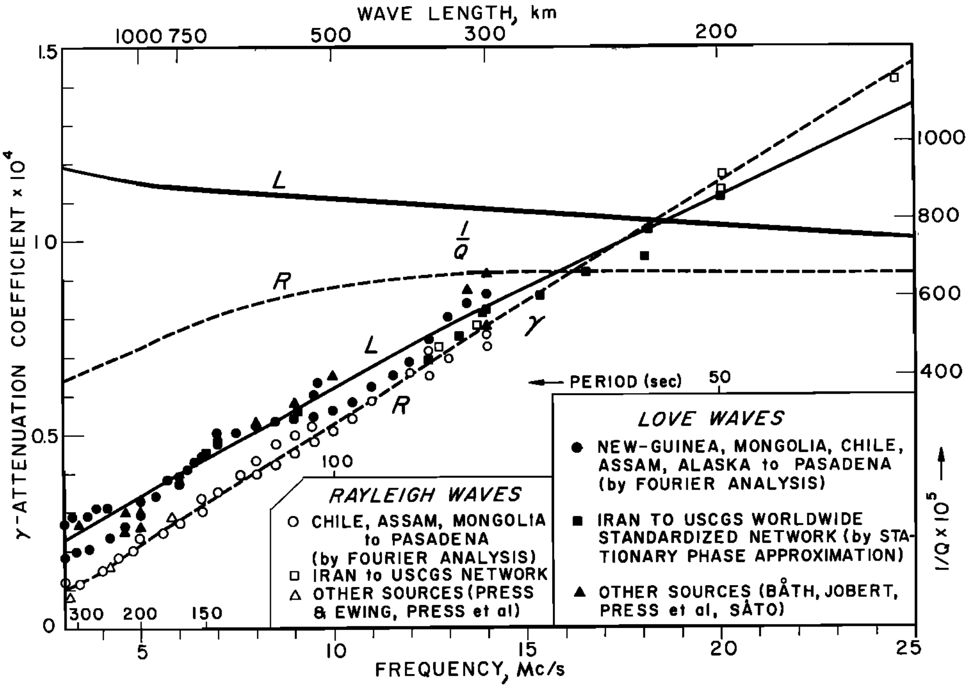

Fig. 12. Attenuation factor γ for Love and Rayleigh waves as a function of period for a number of earthquakes [after *Ben-Menahem,* 1964]. The smoothed values of γ are reduced to yield values of Q_T^{-1} in the upper curves.

earth since sensitive seismographs capable of recording seismic wave motions at very long periods have been in operation. The attenuation of the standing waves set up in the free modes of oscillation of the earth have been studied for the Chilean earthquake by several authors, notably *Alsop et al.* [1961], *Benioff et al.* [1961], *Ness et al.* [1961], *Smith* [1961], and *Connes et al.* [1962]. Values of attenuation factor reduced to dimensionless terms have been observed for a small number of lines of the spectrum. These results are summarized in Table 5. The values of Q in torsional or shear oscillations and those in spheroidal oscillations are of the order of 300. It should be noted that as the discrete spectra of the free oscillations approach the continuous spectra at shorter periods, the torsional oscillations are observed as Love or shear waves; the discrete spectra of the spheroidal oscillations, where matter is both compressed and sheared, approach the continuous Rayleigh wave spectra. Interpretation of these results can now be attempted.

5. ASSUMPTIONS USED IN INTERPRETATION

The interpretation of the observations of the attenuation of seismic waves at the surface of the earth in terms of a distribution of intrinsic absorption in the interior of the earth will now be undertaken. In the analysis which follows, we are concerned with the large scale features of the distribution of attenuation prop-

TABLE 5

Mode	Period, sec	Q_T	Reference	Comments
			Spheroidal Modes	
${}_0S_2$	3230	350	*Smith* [1961]	Possibility of energy being transferred into mode
${}_0S_2$	3230	370	*Alsop et al.* [1961]	Line splitting resolved
${}_0S_2$	3230	>300	*Benioff et al.* [1961]	Limited by length of record
${}_0S_3$	2133	380	*Benioff et al.* [1961]	
${}_0S_0$	1227	900	*Smith* [1961]	Radial mode, no splitting; high Q due to purely compressional deformation
${}_0S_0$	1227	7500	*Ness et al.* [1961]	
${}_0S_9$	634	366	*Ness et al.* [1961]	
${}_0S_{12}$	502	280	*Ness et al.* [1961]	
${}_0S_4$ to ${}_0S_{28}$	1551 to 274	>200	*Connes et al.* [1962]	Limited by length of record
			Torsional Modes	
${}_0T_2$	2599	400	*Smith* [1961]	Depends upon unlikely identification of split lines
${}_0T_2$	2576	160	*Smith* [1961]	Alternative identification of split lines
${}_0T_3$	1711	~400	*Smith* [1961]	Splitting factor considerably larger than theoretical value
${}_0T_5$	1074	300	*Alsop et al.* [1961]	(*a*) Possibility of coupling to ${}_1S_3$ mode (*b*) Marginal length of record
${}_0T_4$ to ${}_0T_{28}$	1296 to 273	>200	*Connes et al.* [1962]	

erties in the interior of the earth. Accordingly we confine our attention, in the case of surface waves, to the long period data only, such as those reported in the preceding section. To focus on the gross features, it is assumed that the earth has radial symmetry. Therefore, in this coarse analysis, any differences between continents and oceans are ignored; that is, the physical properties of the earth over all radii are assumed to be the same. Since continental and oceanic crustal thicknesses are known to differ, the approximation certainly fails when wavelengths of the order of the continental thickness are considered for the problem of the transmission of surface waves across continental margins. Therefore only wavelengths are considered that are longer than those for which this difficulty is demonstrable.

Kovach and Anderson have derived values of Q in the interior of the earth from the observation of body waves. The interpretation of body wave data is simpler than that for surface wave data, because the dispersion is not nearly as pronounced as for surface waves. However, the data are subject to the effects of scattering by inhomogeneity as a function of radius and of the nature of the focus in a much more pronounced way than are the data on surface waves. The rest of this section is confined to the interpretation of surface wave attenuation observations. The results of body wave and surface wave interpretations will be compared.

For the surface wave analysis, advantage is taken of the fact that the surface waves are strongly dispersed. The same inhomogeneity which may cause scattering of body waves, under the assumption of radial symmetry, allows surface waves to be transmitted without scattering, except for the effects of curvature. Since there is a variation of the penetration of surface waves of different periods into the interior of the earth, the attenuation at the surface of the earth, measured as a function of period, represents an integration of the contributions of the intrinsic attenuations throughout the entire earth in varying degree, depending on the period. For short periods the wavelength and the penetration are small, and hence the major contribution to the attenuation is from the uppermost layers of the earth; by the same token, for very long period waves, the attenuation as observed at the surface must correspond to the removal of energy from seismic waves by the material in the entire cross section of the earth, the uppermost layers as well as the deeper layers. The curve of attenuation as a function of frequency obtained from surface observations therefore reflects both the spatial variation and frequency dependence of the intrinsic attenuation in the interior of the earth.

From the discussion above it seems reasonable that, if the interior of the earth is everywhere solid, it is quite appropriate to assume that the intrinsic attenuation is everywhere independent of frequency. On the other hand, if there are parts of the earth that are liquid, it probably must be assumed that the attenuation factor varies as the square of the frequency (or Q^{-1} as the first power of the frequency), and this feature must be taken into account appropriately.

First the assumption that Q can be considered independent of frequency in the interior of the solid parts of the earth is considered. For most solid materials that have been measured in the laboratory, Q does appear to be independent of frequency. Except for the measurements of Zemanek and Rudnick, the data are usually taken over a relatively small range of frequencies. In the case of shear motions in the earth, values of the attenuation as a function of period are given from about 50 to 300 seconds for Love waves (Ben-Menahem), and a fairly reliable value is known for the torsional oscillations (Alsop et al.) at a period of 1074 seconds. In the case of Rayleigh waves and the spheroidal modes, the data are again given from about 50 to 300 seconds by Ben Menahem for Rayleigh waves; from the free modes of oscillation the longest period available is approximately 3000 seconds. Thus, for Love waves the available evidence encompasses a frequency range of about ten to one, and for Rayleigh waves of about sixty to one. The data of Zemanek and Rudnick show that for aluminum Q is not constant over such a wide frequency range, but in fact Q drops by about 40% over a frequency range of sixty to one. Since rocks have not been tested in the laboratory over such a broad frequency range, we shall assume that these numbers, taken for aluminum, are at least indicators of the reliability of the assumption that Q is a constant over such broad frequency ranges. If Q in the earth must vary by less than 20% over a frequency range of ten to one, it is not clear that this variation should be ascribed to structural features; it may in fact be associated with some intrinsic frequency variation of the attenuation.

In this interpretation it is further assumed that the earth is completely solid. Obviously this is an invalid assumption, since the core of the earth is liquid. In the case of Love waves or the torsional modes of vibration of the earth, only the mantle of the earth is set into oscillation, since there is no coupling in shear to the fluid core. Thus we need not be concerned with the core of the earth, except for the possibility of viscous damping of the gravest shear modes at the core-mantle interface. In the case of Rayleigh waves, however, the influence of attenuation in the liquid core is undoubtedly an important feature; account must be taken of the different dependence of attenuation on frequency in a liquid region when compared with the corresponding dependence for the mantle of the earth. In the interpretation below, we only compute the distribution of Q in shear in the mantle.

A further complication is associated with the low velocity layer. Several authors have suggested that the low velocity layer, a region falling roughly between 100 and 250 km and not found at uniform depth or with uniform thickness at all geographic locations, may be due to partial melting of the solid material composing the upper mantle. Since shear waves are transmitted through this region, it cannot be completely molten. The low velocity layer has a more pronounced minimum in the shear velocity than it does in the compression wave velocity. If the low velocity layer is the result of partial melting of the mineral constituents, due to some anomalous condition of temperature and pressure, we shall have to consider the influence of the attenuation of long period earthquake surface wave motions in this region due to a composite of solid material and of small amounts of interstitial liquid. Here perhaps the best guide is to be found in the work of *Born* [1941] mentioned earlier (Figure 4). When small amounts of water were injected into the interstitial region in sandstone, Born found an additional component of the attenuation that appears to be correlated with the viscous attenuation in fluids. Partial melting in the interior of the earth very likely will have the character of interstitial fluid in the presence of the host matrix, since shear waves can be transmitted through the solid parts.

We can estimate from dimensional arguments, in a rough way, the periods at which the attenuation due to melting will become important. From Born's data it can be seen that there is a characteristic frequency at which the attenuation is essentially doubled over the zero frequency value. We construct a dimensionless number relating this frequency ω_0, the shear modulus of the host material μ, and the shear viscosity of the fluid η, and demand that this be the same both in the sandstone-water experiment and in the earth:

$$(\omega_0 \eta/\mu)_{\text{sandstone}} = (\omega_0 \eta/\mu)_{\text{mantle}} \tag{20}$$

For water, the shear viscosity is 10^{-2} cgs; the characteristic frequency in the laboratory experiment is of the order of 10^3 cgs. For order of magnitude calculations, the modulus of rocks at great depths and of sandstone can be taken to be the same. The period at which the influence of interstitial fluid in the interior of the earth becomes important depends crucially on an estimate of the viscosity of molten rock; this value of viscosity depends strongly on the temperature and pressure of the molten material. *Birch* [1942] gives the viscosity of molten basalt as being perhaps of the order of 10^3 to 10^4 cgs. Thus, from (17) the period at which the effect of the interstitial fluid should become equal to the effect of the solid material is of the order of 100 to 1000 seconds, a range crucial to the surface wave measurements. Later we shall determine the influence on the attenuation of a region of the order of 100 to 150 km thick consisting of a small amount of molten material.

Born's results should hold for both compressional and shear waves. Even though there is no shear modulus in the interstitial fluid, shear waves in the solid are converted to compression waves in the fluid through the boundary conditions at the surfaces of the fluid regions, and energy is dissipated through the compressional or bulk viscosity in the fluid. Thus, although the laboratory experiment has been performed for compression waves in sandstone, the square law frequency dependence for attenuation is anticipated for shear waves in a solid with interstitial fluid, as long as the dimensions of the pockets of fluid are small compared with the wavelength. Because of the coupling of compression and shear in Rayleigh wave motions, the same result must hold for Rayleigh waves.

Can the attenuation observed in experiments using propagating waves, such as those using multiple transits past a given seismic station, be compared with the standing wave data in which line widths are measured at very long periods? The answer is that this comparison cannot be made directly without some corrections. This was demonstrated by *Brune* [1962], who gave a heuristic proof that the specific attenuation factors measured in the two kinds of experiments are related by the ratio of group and phase velocities in a heterogeneous medium. A rigorous proof of this statement was given by *Knopoff et al.* [1964]. It has been shown that, if Q_X is the dimensionless attenuation factor measured in propagating wave experiments and Q_T is the dimensionless attenuation factor measured in standing wave experiments in the same medium, these two quantities are related by the expression

$$cQ_X = UQ_T \tag{21}$$

where c is the phase velocity at the period in question and U is the group velocity at this period. The reader is reminded that

$$U = d\omega/dk \qquad k = \omega/c$$

where ω is the angular frequency, and k is the wave number. Thus the two kinds of data can be compared only when modified by the ratio c/U. Values of the group velocity are difficult to obtain in the period ranges corresponding to the free modes of oscillation of the earth, since the group velocity depends on a derivative property of the dispersion relation, and observations of the phase velocities are made only at discrete points corresponding to the periods of the free modes. Thus the values of Q_X obtained at shorter periods are reduced to values of Q_T of this period range by correcting by the ratio of the two velocities, since the two velocities are known in the continuum. Later we shall have need of the ratio U/c in the discrete spectrum, and, where required, it will be computed for specific models of the interior structure of the earth. The data plotted in Figure 12 are values of $1/Q_T$.

One last problem remains to be solved before the data can be interpreted. If the Fourier transform of the one-dimensional elastic wave equation is written as

$$\mu k^2 = \rho\omega^2 \tag{22}$$

then it is seen that the wave number k can be made complex by introducing either a complex shear modulus with real density or a complex density with real modulus or a mixture of both. Evidently a relationship exists connecting the two end-member processes such that the same values of complex wave number are obtained. Since the moduli of elasticity appear explicitly in the equations expressing the boundary condition involving continuity of stress across adjacent layers of a heterogeneous medium, it is clear that the two processes are not equivalent; energy transferred across boundaries governed by a complex modulus will be dissipated at the boundaries, whereas energy transferred across these boundaries with a real modulus will not be dissipated at the boundaries. In the latter case, the only dissipation can come from the body of the material. The influence of this difference of mechanism can be most easily seen if we write the phase velocity for Love waves as a function of frequency and the physical parameters in the interior of the system:

$$c = c(\omega, \mu(r), \rho(r)) \tag{23}$$

where we have assumed that the modulus and the density vary only with the radius. If we now assume that both the shear modulus and the density are complex, expand the above equation in a Taylor series about the lossless condition, and take the imaginary part of the result, we find that

$$-\frac{c}{k} \operatorname{Im} k = \operatorname{Im} c \doteq \int_{\text{core}}^{\text{surface}} \mu \frac{\partial c}{\partial \mu} \operatorname{Im} \frac{1}{\mu} \frac{\partial \mu}{\partial r}\, dr + \int_{\text{core}}^{\text{surface}} \rho \frac{\partial c}{\partial \rho} \operatorname{Im} \frac{1}{\rho} \frac{\partial \rho}{\partial r}\, dr + \cdots \tag{24}$$

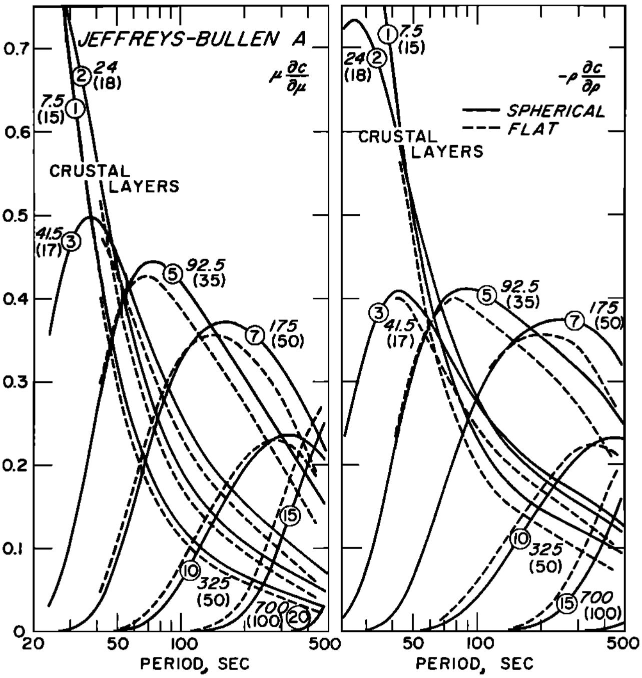

Fig. 13. Curves of μ $(\partial c/\partial \mu)$ and ρ $(\partial c/\partial \rho)$ for a number of layers in the interior of a spherical earth [after *Anderson,* 1964].

Thus the imaginary part of the wave number for $Q \gg 1$ has a different ratio connecting it with variations in shear modulus than it does with variations in density. From the wave equation 22 for a homogeneous medium it is easy to see that the quantities μ $(\partial c/\partial \mu)$ and ρ $(\partial c/\partial \rho)$ are equal to the negative of each other. *Anderson* [1964] has shown that in the interior of the earth the quantities μ $(\partial c/\partial \mu)$ and ρ $(\partial c/\partial \rho)$ are not equal to the negative of each other but are roughly so (Figure 13).

Further support for this conclusion comes from an investigation of the energies involved in the dissipation process. For a homogeneous material, the logarithmic decrement is the ratio of the energy lost per cycle to the energy stored per cycle. In a homogeneous material these energies can be potential energies, kinetic energies, or the sum of these, since the kinetic and potential energies are equal. However, in dispersive media, the kinetic and potential energies are not locally equal, since the phase velocity is not equal to the local intrinsic velocity. It can be shown that if μ is always real and all the attenuation is associated with complex ρ, then, for surface waves in an inhomogeneous flat-lying medium, all of whose properties vary with only one Cartesian coordinate z, the apparent attenuation of propagating waves as measured on the surface is related to the intrinsic attenuation in the interior of the medium through the expression

$$\frac{1}{Q_T} = \int_a^b T(z)/Q(z)\, dz \Big/ \int_a^b T(z)\, dz \tag{25}$$

where $T(z)$ is the kinetic energy density, and $Q(z)$ is the specific dissipation factor for that layer. If, however, the attenuation is completely associated with complex μ and not at all with complex ρ, the corresponding expression is

$$\frac{1}{Q_T} = \int_a^b V(z)/Q(z)\,dz \Big/ \int_a^b V(z)\,dz \tag{26}$$

where $V(z)$ is the potential energy density. It can be shown further that

$$\int_a^b V(z)\,dz = \int_a^b T(z)\,dz \tag{27}$$

namely that Rayleigh's principle applies for all the layers (but not for each one). Thus we see that the weighting factors operating on the intrinsic Q's in each layer are different according to the way that the kinetic and potential energies are partitioned among the layers. The above statements are easily proved by setting up the multiply reflected elementary wave functions in each layer and solving for the kinetic and potential energies in each layer. The solution for large numbers of layers follows quite easily if a matrix procedure such as that of *Haskell* [1953] or *Knopoff* [1964] is used. It is clear that models in which losses are due to mixtures of both complex μ and ρ are also possible, and the interpretation therefore allows a complete spectrum of possible results. In the work which appears below, we assume that only μ is complex and that ρ is everywhere real; this corresponds to a model with imperfect elastic bonds as the principal mechanism of dissipation.

6. INTERPRETATION

With this background we are in a position to evaluate the distribution of intrinsic Q in shear or torsional motions in the interior of the earth. We write the first term of (24) for the case in which only the shear modulus is complex:

$$\frac{\operatorname{Im} k}{k} = -\int_{\text{core}}^{\text{surface}} \frac{\mu}{c}\frac{\partial c}{\partial \mu} \operatorname{Im} \frac{1}{\mu}\frac{\partial \mu}{\partial r}\,dr \tag{28}$$

At a fixed real frequency it is easily seen that the observed value of Q_T is

$$\frac{1}{Q_T(\omega)} = \int_{\text{core}}^{\text{surface}} 2\mu(r)\,\frac{U(\omega)}{c^2(\omega)}\,\frac{\partial c}{\partial \mu}(\omega, r)\,\frac{1}{Q(r)}\,dr \tag{29}$$

where $Q(r)$ is the value of the intrinsic Q at any depth r.

The procedure of expanding the phase velocity as a perturbation in the various quantities was first outlined by *Knopoff* [1961] and developed later in a second paper [1962]; *Jeffreys* [1961] has indicated a development along similar lines. Values of μ $(\partial c/\partial \mu)$ for a number of models of the interior of the earth are given by *Anderson* [1964].

Anderson's values have been used to compute the dimensionless factor $1/Q_T$ to be expected for Love waves in a homogeneous spherical earth. The data to be fit include an isolated point at a period of 1074 seconds as determined by *Alsop et al.* [1961] and the continuous curve for attenuation as determined by

Ben-Menahem [1964]. Ben-Menahem's curve represents smoothing through the data points shown in Figure 12 which include those of his own calculations and those of Båth and López-Arroyo, Satô, Press et al., and *Jobert* [1962]. It will be seen below that the results of this interpretation depend strongly on the method of smoothing the scatter among the data.

The principal evidence for the value of attenuation in the free modes comes from the data of Alsop et al. referred to above. Alsop et al. have indicated that their result is open to several criticisms. The line splitting due to the earth's rotation [*Backus and Gilbert,* 1961; *Pekeris et al.,* 1961] expected in the fifth torsional mode may be sufficiently narrow so that an incorrect breadth will be measured in one of the components. In addition *MacDonald and Ness* [1961] have suggested that the measurement at the period of the fifth torsional mode may actually be the result of a superposition of the fundamental fifth torsional mode and the first overtone of the third spheroidal mode. This could provide an interference between the two modes which could again lead to an incorrect value of the line width. If the value of Alsop et al. is adopted, it is quite clear that Q_T at very long periods is considerably higher than Q_T at propagating wave periods. This can only be taken into account by a model of the earth which introduces a somewhat higher Q in the lower mantle than in the upper mantle. The point at 1074 seconds has a large probable error for the reasons noted above. The determinations of Q_T from the free modes by other authors are similarly subject to criticism in the matter of the length of the record analyzed, the line splitting, etc. Only the result of Alsop et al. will be used here.

The inversion is carried out as follows. The integral of (29) is a Fredholm integral equation of the first kind. It is solved by a quantization so that the vectors Q_T^{-1} (ω), $Q^{-1}(r)$ are related through the matrix $2\mu U c^{-2}\, \partial c/\partial\mu$ (ω, r):

$$(Q_T^{-1})_i = \left(\frac{2\mu U}{c^2}\frac{\partial c}{\partial \mu}\right)_{ij} (Q^{-1})_j \tag{30}$$

The problem is then simply one of inverting to solve for the vector $(1/Q)_j$, the other two matrices being given. If this is performed for the smoothed data of Ben-Menahem and the isolated point of Alsop et al., negative values for the quantities $(1/Q)$ in some of the layers are obtained. Thus the additional constraint $1/Q(r) > 0$ must be imposed on the experimental data $1/Q_T$ as a condition on the smoothing. Thus the smoothed data as given here are inconsistent with any plausible earth model and the condition $1/Q(r) > 0$, and hence the experimental results must be modified. Numerical experiments have shown that one way in which the smoothing of the data may be modified is to require a slight reduction of $1/Q_T$ at the long period end of the continuum and a slight increase in $1/Q_T$ at the discrete period. Ben-Menahem (private communication) indicates that the probable error in his data at 300 seconds may be as much as 10%.

Although an exact inversion is not possible, it is nevertheless possible to approximate the data with several models. Three such solutions are shown in Figure 14. In model I, an upper mantle with a Q of 110 extending to 650 km and a lossless ($Q = \infty$) lower mantle is seen to provide a reasonable fit to the

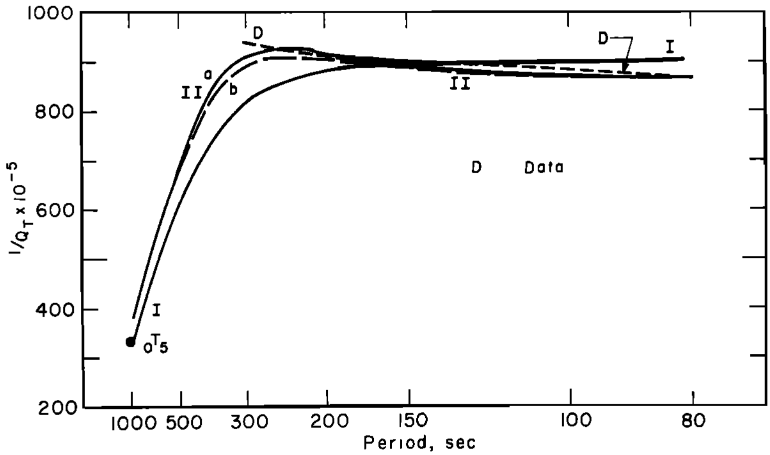

Fig. 14. Computed values of Q_T^{-1} for a number of models of the distribution of Q in the interior of the earth. Curve D gives the continuum data of Ben-Menahem.

data. However, the increase in $1/Q_T$ in the continuum toward the longer periods cannot be approximated by such a model (Table 6).

In model II, an intermediate layer of greater absorption is introduced at depth to account for the increase in $1/Q_T$ in the continuum toward the long periods. Models II*a* and II*b* provide two examples of such a fit. The solution fails at the longest periods of the continuum and at the period of the free mode for the reason given above.

The details of the calculation must not be taken as representing a final solution to the problem. As noted above, the failure of the data to be realizable physically requires that a readjustment be made. However, the following general features seem to be required by the data now at hand.

1. The lower mantle, below 650 km or so, must have a very much higher Q than the upper mantle. The numerical solutions cannot differentiate between an infinite Q for this region and a finite but high Q, say of the order required by the body wave results of Anderson and Kovach. The depth to the upper boundary

TABLE 6

	Model			
Depth below Surface	I	II*a*	II*b*	III
0 to 110 km	$Q = 110$	$Q = 115$	$Q = 114$	$Q = 120$
110 to 250 km	$Q = 110$	$Q = 120$	$Q = 121$	$Q = 120$ + viscous layer with $\eta = 25{,}000$ cgs at ¼% melting
250 to 325 km	$Q = 110$	$Q = 120$	$Q = 121$	$Q = 120$
325 to 650 km	$Q = 110$	$Q = 75$	$Q = 78$	$Q = 75$
650 to 2900 km	$Q = \infty$	$Q = \infty$	$Q = \infty$	$Q = \infty$

of this region cannot be given with precision, but it seems to be roughly at the depths below which earthquake foci cease to be observed.

2. The increase in $1/Q_T$ toward the long period end of the continuum requires that an intermediate layer be introduced in the upper mantle having more absorption than the regions immediately below or above. Solutions II*a* and *b* have placed this region at 325 to 650 km. As above, the depth, or thickness, of this layer cannot be given with precision; further refinement of the data may raise this layer closer to the surface and change its dimensions.

3. The values of Q in the topmost parts of the upper mantle depend strongly on attenuation in the short period part of the continuum, at periods shorter than 50 seconds. In models II, we have indicated that the region above 325 km has a higher Q than that below it; the region above 110 km may have a lower Q than the region immediately below. These values are highly speculative. The scattering of surface waves by lateral inhomogeneities becomes quite important at periods shorter than 50 seconds or so; the data at these periods include scattering effects and cannot be used in this interpretation.

4. The precise values of Q in shear are again unknown. Nevertheless, a mean Q in the upper mantle of 110 is not unreasonable, as is seen from model I. A mean Q for the entire mantle of $110 \times 2900/650 = 490$ is not inconsistent with the body wave measurements of Press and of Anderson and Kovach.

Through the use of Anderson's curves, we can also investigate the effect of a partly molten layer in the region 110 to 250 km. Here all that need be done is to add the contribution of such a region to the value of Q already computed. Evidently this must have the appropriate frequency dependence for the partly molten region. We assume a model of $Q(r)$ independent of frequency throughout the entire upper mantle and superimpose on it an additional viscous damping term for the region 110 to 250 km in which Q^{-1} is taken to be proportional to the first power of the frequency. This evidently has the effect of increasing the apparent Q in the upper mantle due to the solid parts, since some of the attenuation at high frequencies can now be attributed to the partly molten region. Consistent with the observations of Love wave attenuations as before, we find that model III (see Table 6) yields a curve of $1/Q_T(f)$ which is indistinguishable from that for model II*a* in Figure 14. The principal features of this solution are an increase in the value of Q in the uppermost 110 km and the introduction of a material with an η/μ value in the viscous layer 110 to 250 km of $\frac{1}{8}$ second. According to the data of Born and equation 20, this corresponds to $\frac{1}{4}\%$ molten basalt with a viscosity of 2.5×10^4 cgs. The interpretation is not unique owing to the introduction of additional degrees of freedom: the viscosity, the degree of melting, and the dimensions of the low velocity region.

The basic interpretation, even on the assumption of Q independent of frequency, is not unique because of the question of the mechanism as treated above and the experimental inaccuracy. These, coupled with the possibility of decrease of Q with increasing frequency according to the experimental results of Zemanek and Rudnick, show that the details of the distribution of Q in the upper mantle cannot be given with precision. The presence of an intermediate layer of high attenuation is indicated by the data presently known and under the assumptions made; verification will come with further refinements in taking the data and by comparing with the calculation of Q in the Rayleigh-spheroidal mode of excitation.

The interpretation of the data for Rayleigh waves and the spheroidal modes will not be attempted here. This interpretation depends crucially on the viscosity of molten material in the core of the earth. Not enough is known about the nature of the core at this time to evaluate carefully the influence of this term.

In summary, if the earth is solid it is highly likely that Q can be considered as independent of the frequency. If Q is independent of frequency, the apparent attenuation of surface waves and of the free modes of vibration can be interpreted in terms of an intrinsic Q in the interior of the earth. With the evidence now available a high degree of resolution is not possible. It is sufficient to say that, to the best estimate at the present time, a constant Q in shear in the upper mantle of about 110 is present; deviation from this value depends on the amount of absorption due to partial melting in the low velocity layer. Further resolution of this into a fine structure indicates the presence of an intermediate layer with high attenuation; this layer is discerned under a literal interpretation of the data and the assumptions. The value of Q in the lower mantle cannot be given with accuracy, but it is certain that, if the values of the spectral line widths in the free modes are accepted, the value of Q in this region is much higher than that in the upper mantle.

7. Q IN COMPRESSION AND BULK

Of interest in the listing in Table 5 is the high value of Q_T in the radial mode ${}_0S_0$ reported by *Ness et al.* [1961] and *Smith* [1961] for the Chilean earthquake of 1960. G. J. F. MacDonald (personal communication) finds a Q_T in excess of 10^4 for the same mode for the Alaskan earthquake of 1964. It now appears that this high value is not spurious, and the value reported by Smith may even possibly be too low. This result suggests that some additional constraints may be placed on the values of Q obtained in the earth and laboratory.

Let Q_K, Q_P, and Q_S be the values of Q obtained in experiments in which the bulk, compression, and shear properties of matter are excited. For the mode ${}_0S_0$, Q_K has been observed. For a Poisson solid, it is elementary to show that

$$5/Q_K = (9/Q_P) - (4/Q_S) \tag{31}$$

Let us assume that all the attenuation takes place in the outermost 1/10 of the earth's radius. In the ${}_0S_0$ mode, the amplitude distribution is substantially flat as a function of radius and hence the attenuation takes place in a volume 3/10 that of the earth. Thus, if Q_K for the ${}_0S_0$ mode is observed to be 10,000, then in the upper $0.1R_e$, $Q_K = 3000$. If we take Q_S as 110 in this region, we obtain from (31) $Q_P = 230$. If Smith's value of $Q_K = 900$ is taken, $Q_P = 165$. In other words, for such a large value of Q_K, $Q_S < Q_P$ for the earth. In passing, we note that, if $Q_K = \infty$, then $Q_S = 4/9\ Q_P$ and $Q_S = 6/5\ Q_Y$, where Q_Y is the Q measured in long rods.

From Tables 1, 2, and 3, it can be seen that the data of Wegel and Walther for copper, Mason and McSkimin for aluminum, Peselnick and Zietz for limestone, and Birch and Bancroft for granite are inconsistent with the conclusion

$Q_S < Q_P$. The data of Wegel and Walther for lead and soda-lime glass show the predicted effect only marginally.

One way this discrepancy can be accommodated is to say that (31) holds for solid materials; hence, if partial melting is present in the upper mantle, (31) will have to be modified. This presents us with the uncomfortable result that the unusually high Q in ${}_0S_0$ is a fortuitous consequence of a certain amount of partial melting in the upper mantle. An alternative resolution suggests that the relation $Q_P < Q_S$ observed in the laboratory is a result valid at low pressures and that this condition may be reversed at higher pressures.

Does an independent estimate exist for Q_P in the upper mantle? If we assume the result of the interpretation obtained here for Q_S from Love waves extends to Q_R from Rayleigh waves as well, i.e. that all the pertinent attenuation is observed in the outer $0.1R_e$ for waves in the period range $T < 300$ seconds, then an estimate of Q_P is possible. The computed value of $Q_S = 110$ for the upper mantle is consistent with the curve of Q_T for Love waves (Figure 12). We can infer, by inspection of Q_T for Rayleigh waves (Figure 12), that a reasonable model for Q_R is $Q_R = 150$ in the upper mantle and $Q_R = \infty$ in the lower mantle. *Knopoff* [1959] has shown that, for a homogeneous half-space with Poisson's ratio 1/4, the relation

$$1/Q_R = (0.067/Q_P) + (0.433/Q_S) \tag{32}$$

can be written. With Q_R and Q_S as quoted above, we compute for the upper mantle $Q_P = 25$. Here we have a result which may be too far in the opposite direction. But if dispersion is taken into account, (32) will have to be modified. The influence of dispersion on (32) is now being computed.

At the present time, it seems plausible that the laboratory result $Q_P < Q_S$ is consistent with observations and that the result that Q_K is large is perhaps a fortuitous result obtained because of partial melting in the upper mantle.

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