On Phillips' Theory of Equilibrium Range in the Spectra of Wind-Generated Gravity Waves

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ABSTRACT

Phillips' hypothesis concerning the equilibrium range in the spectrum of wind-generated surface waves in deep water can be expressed either in the frequency form $S(\omega)\propto\omega^{-5}$ or the wavenumber form $\Psi(k)\propto k^{-4}$. If one adopts the wavenumber form as universal, it can then be shown that for shallow water $S(\omega)\propto\omega^{-3}$, a relation supported by observations.

It is noted that deviations from the $S(\omega)\propto\omega^{-5}$ relation can be introduced not only by finite depth but also by permanent currents, as well as by the presence of long-wave components in the sea wave spectrum. In the latter case determination of $S(\omega)$ has to include both the Doppler effect, due to orbital displacements of liquid particles in long waves, as well as the effect of additional vertical acceleration in short waves propagating along the surface of the long-wave field. The article also gives calculations that illustrate the significance of the foregoing effects on deviations from the Phillips -5 power law in the frequency spectrum $S(\omega)$.

1. Phillips' hypotheses on the equilibrium range in the spectra of wind-generated waves

According to Phillips (1958, 1966) space and time scale ranges can exist in the spectra of wave-generated gravity waves within which the energy distribution must depend only on physical parameters characterizing the formation of sharp crests in breaking waves. If we consider that region of space and time scales, in which the viscosity of the water and surface tension have no influence on wave motion, then the only significant parameter, according to Phillips (1958, 1966), is the acceleration of gravity g. Then on dimensional grounds one obtains the following expressions, first suggested by Phillips in 1958:

$$\Psi(\mathbf{k})=Bk^{-4}\varphi(\nu), \qquad (1.1)$$

$$S(\omega)=\beta g^2\omega^{-5}, \qquad (1.2)$$

where $\Psi(\mathbf{k})$ and $S(\omega)$ are, respectively, the wavenumber and frequency spectra, k is the modulus of the wavenumber vector $\mathbf{k}=(k\cos\nu,\ k\sin\nu)$, ω the frequency, ν the angle characterizing the direction of wave propagation, B and β are universal non-dimensional constants, and $\varphi(\nu)$ is a certain universal function describing the angular distribution of wave component energy within the equilibrium range at any value of ν and satisfying the standard normalization condition

$$\int_{-\pi}^{\pi}\varphi(\nu)d\nu=1.$$

It immediately follows from both Eq. (1.1) and this normalization condition that

$$\chi(k)=\int_{|\mathbf{k}|=k}\Psi(\mathbf{k})d\mathbf{k}=\int_{-\pi}^{\pi}\Psi(k,\nu)kd\nu=Bk^{-3}, \qquad (1.3)$$

where $\chi(k)$ is the spectrum of wavenumber moduli characterizing energy distribution over k regardless of the direction of wave component propagation [in Phillips' (1966) monograph $\chi(k)$ simply designates the integral

$$\int_{-\pi}^{\pi}\Psi(k,\nu)d\nu,$$

which obviously is not the spectrum of wavenumber moduli].

At present Eqs. (1.1)–(1.3) are used as basic and, actually, as the only constructive inferences about the shape of wind-generated gravity wave spectra at sufficiently large k and ω. It is exactly these formulas that are being intensively checked experimentally. In this connection it appears worthwhile to mention certain points which are usually left out of consideration when Phillips' laws (1.1) and (1.2) are deduced.

First of all, in the rigorous statement of the law [Eq. (1.1)] for the spatial spectrum $\Psi(\mathbf{k})$, we must speak about the existence of an equilibrium range not only with respect to k, but also to ν, i.e., about a certain region (k,ν), in which spectral components are saturated and the spectrum $\Psi(\mathbf{k})$ is determined only by the parameter g (and also by k and ν). As the boundaries of such a region are not known *a priori*, no unambiguous conclusions about the shape of the $\chi(k)$ spectrum can be drawn even if the function $\varphi(\nu)$ in (1.1) is known, unless one makes some additional assumptions not following from the similarity theory itself.[1] Thus, strictly speaking, the expression (1.3) does not necessarily follow from (1.1). On the other hand, in analogy to the theory of small-scale axisymmetric turbulence, we may set down certain expressions stemming from Phillips' hypotheses and directly describing spatial statistical wave motion characteristics already averaged over all directions of the vector $\mathbf{k}$, i.e., directly for the $\chi(k)$ spectrum. However, Eq. (1.1) will not then necessarily follow from (1.3). As (1.1) requires some specific form of angular energy distribution of the spectrum $\Psi(\mathbf{k})$, which is not a function[2] of k, then the latter of the foregoing approaches to deriving $\chi(k)$ seems to be preferable.

In addition, Phillips (1958, 1966) has derived the laws (1.1) and (1.2) independently and from identical assumptions, although it is obvious in advance that similarity hypotheses for spatial and temporal spectra are far from being of equivalent validity. One example of this is supplied by the theory of small-scale locally isotropic turbulence in which it is known that the $-5/3$ power law can be obtained on dimensional grounds only for a spatial spectrum, and that an expression for the corresponding frequency spectrum is found only via additional information of the type of a "frozen turbulence" hypothesis (see, for example, Monin and Yaglom, 1967).

In the present paper we attempt to incorporate the above-mentioned factors in determining the form of frequency spectra in the equilibrium range by using an approach typical in turbulence theory; namely, by postulating the existence of an equilibrium range of spatial statistical characteristics of waves. Initially, for the $\chi(k)$ spectrum, we determine corresponding frequency spectra by means of some sort of dispersion relations which are a wave analog to the concept of "frozen turbulence." Further on, it will be seen that more information concerning the spectrum $S(\omega)$ can be obtained in this way than from dimensional analysis alone.

[1] Such as, for instance, the one utilized by Phillips (1966) who assumed in (1.1) that

$$\varphi(\nu)=\begin{cases}1/\pi, & \text{at} \quad |\nu|\leqslant\nu_m=\pi/2\\ 0, & \text{at} \quad |\nu|>\nu_m\end{cases}$$

[2] To some extent the general form of Eq. (1.1) contradicts the data of the most detailed investigation of angular energy distribution in the wave spectrum, as obtained by Longuet-Higgins *et al.* (1963) and Ewing (1969). It seems that k-independence of the angular distribution (including isotropy at $|\nu|\leqslant\nu_m=\pi/2$) can be approximately satisfied only in the shortest-wave range of the spectrum $\Psi(\mathbf{k})$.

2. Dispersion relations and the relationships between spatial and temporal spectra of wind-generated waves

Usually, when wind-generated waves are investigated, the scope is restricted to using the dispersion relation of the linear theory of free surface waves. In the case of gravity waves this relation takes the form

$$\omega(k)=[gk\tanh(kh)]^{\frac{1}{2}}, \tag{2.1}$$

where h designates the depth. The dispersion relation (2.1) is based upon an assumption of free waves of small amplitude and thus should be modified for actual wind-generated waves. It appears that corrections to (2.1), describing the effects of interactions between surface waves and the atmospheric turbulent boundary layer, as well as nonlinearity corrections, are small (for details see Phillips, 1966). However, significant modification of (2.1) is produced by including the effects of large-scale movements caused by currents and long-wave components of surface seawaves.

A permanent (mean) flow $\mathbf{v}$ is taken into account by simply correcting (2.1) for the Doppler effect:

$$\omega(\mathbf{k})=[gk\tanh(kh)]^{\frac{1}{2}}+\mathbf{k}\cdot\mathbf{v}. \tag{2.2}$$

The use of deterministic dispersion relations of the type (2.1), (2.2) leads to a formulation of the relation between $\Psi(\mathbf{k})$ and $S(\omega)$. However, such a relation is simple only in the case when a solution $k=k(\omega,\nu)$ of the dispersion relation $\omega=\omega(\mathbf{k})=\omega(k,\nu)$ is a one-valued function of ω defined for positive ω only. Dispersion relation (2.1) is a one-valued function of ω defined for $\omega\geqslant0$, but (2.2) is a multiple function of ω defined for $\omega\geqslant0$ and $\omega\leqslant0$ when $\mathbf{k}\cdot\mathbf{v}<0$. In any case one may write by definition

$$\int_{-\infty}^{\infty}\Psi(\mathbf{k})d\mathbf{k}=\int_{-\pi}^{\pi}\int_{0}^{\infty}\Psi(k,\nu)kdkd\nu$$
$$=\int_{0}^{\infty}\chi(k)dk=\int_{0}^{\infty}S(\omega)d\omega=\langle\zeta^2\rangle, \tag{2.3}$$

where $\langle\zeta^2\rangle$ is the variance of the stochastic function $\zeta(\mathbf{x},t)$ which is assumed uniform over horizontal position $\mathbf{x}$ and time t, by means of which we simulate the sea surface. For a one-valued function $k(\omega,\nu)$, defined for $\omega\geqslant0$, the relation between $\Psi(\mathbf{k})$ and $S(\omega)$ can easily be found from (2.3) by a change of variables in the integrals:

$$S(\omega)=\int_{-\pi}^{\pi}\left[\Psi(k,\nu)\frac{k}{G(k,\nu)}\right]_{k=k(\omega,\nu)}d\nu, \tag{2.4}$$

where $G(k,\nu)=\partial\omega/\partial k$. The quantity G corresponds to group speed $|\nabla_{\mathbf{k}}\omega|$ only if ω is independent of ν. If $k(\omega,\nu)$ is multiple function of ω defined for positive and negative ω, then the relation between $\Psi(\mathbf{k})$ and $S(\omega)$ is more complex and special analysis is required for every particular dispersion relation. General analysis relevant to this question and its practical application are given in the Appendix.

If a dispersion relation is isotropic [$\omega=\omega(k)$], which corresponds to $\mathbf{k}\cdot\mathbf{v}=0$ in (2.2), then a simple algebraic relation between the frequency spectrum and the wavenumber moduli spectrum $\chi(k)$ follows from (2.4), namely

$$S(\omega)=\left[\frac{\chi(k)}{G(k)}\right]_{k=k(\omega)}. \tag{2.5}$$

Thus, for an isotropic dispersion relation it is sufficient to know only the wavenumber moduli spectrum $\chi(k)$ [and, obviously, the dispersion relation itself $\omega=\omega(k)$] to calculate the frequency spectrum $S(\omega)$. But when anisotropic dispersion relations of the type (2.2) are considered, the integral relation between $S(\omega)$ and $\Psi(\mathbf{k})$ of the type (2.4) has to be used to transform wavenumber spectra into frequency spectra. In this case, to calculate $S(\omega)$ it is not sufficient to specify the equilibrium range only in terms of wavenumber moduli spectra; in addition, it is necessary to include assumptions, both about the shape of the angular dependence $\varphi(\nu)$ in (1.1) and about the shape of the region (k,ν) of equilibrium amplitudes of spectral components.

Effects of large-scale components of surface waves lead to random perturbations of the dispersion relation for wind-generated short gravity waves. To determine their effect on the relation between the spatial and temporal spectra, it is necessary to state the problem in more detail. In the analysis of this relation, we shall assume a model of surface sea waves in which the long-wave end of the spectrum is separated from the short-wave end by the "transparency range" $[k^0,k_0]$ with $k^0\ll k_0$, where k^0 is the wavenumber of the spectral maximum of large-scale disturbances, and k_0 is the wavenumber of the spectral maximum of growing wind-generated sea waves and differs only slightly from the lower limit of the equilibrium range k_* ($k_*\gtrsim k_0$). Such a model of wind-generated sea waves growing against the background of the swell, although rather special, makes it possible to reproduce in a most refined way the picture of short steep waves propagating on the surface of much longer waves. In reality, the existence condition of a broad "transparency range" $[k^0,k_0]$ may be too stringent; instead, the weaker inequality $k^0\lesssim k_0$ may be quite acceptable. As double-peak spectra of this type are also characteristic for purely wind-generated sea waves (see Krilov *et al.*, 1973), the results can be applicable to this case.

If, for the sake of simplicity, the effect of finite sea depth and that of the current are neglected ($h=\infty$, $\mathbf{v}=0$), then the short-wave component dispersion relation ($k\geqslant k_*$) in the linear approximation takes the form

$$\omega=\omega(\mathbf{k})=[(g+\bar{a})k]^{\frac{1}{2}}+\mathbf{k}\cdot\bar{\mathbf{v}}, \tag{2.6}$$

where $a=\partial^2\bar{\zeta}/\partial t^2$ is the vertical acceleration of large-scale components $\bar{\zeta}$ of surface waves with $k\sim k^0$; $\bar{\mathbf{v}}$ is the horizontal drift velocity of the long waves, and $\bar{a}$ and $\bar{\mathbf{v}}$ (as well as $\bar{\zeta}$) are slowly varying random functions with scales $\lambda_0\sim(k_0)^{-1}$, $\tau_0\sim(\omega_0)^{-1}$, and without the approximation under discussion they depend on $\mathbf{x}$ and t only parametrically, in their various realizations.

It is evident that if we utilize dispersion relations of the type (2.6), then the general transformation formula (2.4) will give the frequency spectrum corresponding to a fixed spatial spectrum $\bar{S}(\omega)=\bar{S}(\omega,\bar{a},\bar{\mathbf{v}})$ at fixed values of the random parameters $\bar{a}$ and $\bar{\mathbf{v}}$ which enter the integrand in (2.4) via G and k. Thus, when spatial spectra of short waves, propagating on the surface of swell or long wind-generated waves, are transformed into corresponding frequency spectra, we face the necessity of considering conditional spectra $\bar{S}(\omega)$. The unconditional frequency spectrum, which is our ultimate goal, is found in this specific case by averaging $\bar{S}(\omega)$ over all possible values of $\bar{a}$ and $\bar{\mathbf{v}}$:

$$S(\omega)=\langle\bar{S}(\omega)\rangle. \tag{2.7}$$

The foregoing formulas relating $S(\omega)$ to $\Psi(\mathbf{k})$ will be applied below to determine the frequency spectrum of wind-generated sea waves in the equilibrium range when finite sea depth (Section 3), permanent currents (Section 4) and large-scale components of surface sea waves (Section 5) are taken into account.

3. Equilibrium range in frequency spectra of wind-generated waves in a finite-depth sea

In deriving (1.1) and (1.2), Phillips (1958, 1966) has not stipulated whether they are only valid for deep-sea wind-generated waves ($kh\gg1$) or whether they can be used as well in the case when an equilibrium range exists in the gravity wave spectrum of a sea with finite depth h (i.e., at arbitrary values of kh). On the other hand, Phillips (1958) himself has noted that (1.1) and (1.2) for spatial spectra agree with the appearance of sharp crests because the average square of a Fourier transform of a function, which has slope discontinuities, must be proportional to k^{-4}.

This latter consideration provides a reason to believe that shapes of equilibrium spatial spectra of sea waves at large k will be identical for both deep and shallow seas. If $\Psi(\mathbf{k})\sim k^{-4}$ [and, consequently, $\chi(k)\sim k^{-3}$], then the proportionality coefficient B in (1.1) and (1.3) may, in the general case of a finite-depth sea, depend both on g and h, and on wind speed v_a and fetch X. Thus, we cannot neglect a possible dependence of B on such non-dimensional parameters as gh/v_a^2 and gX/v_a^2. If, however, we assume in accordance with the initial concepts of the Phillips hypothesis (1958) that the upper limit of spectral density in the equilibrium range must not depend on wind-supplied energy inflow, then the coefficient B is found to depend only on g and h. As it is

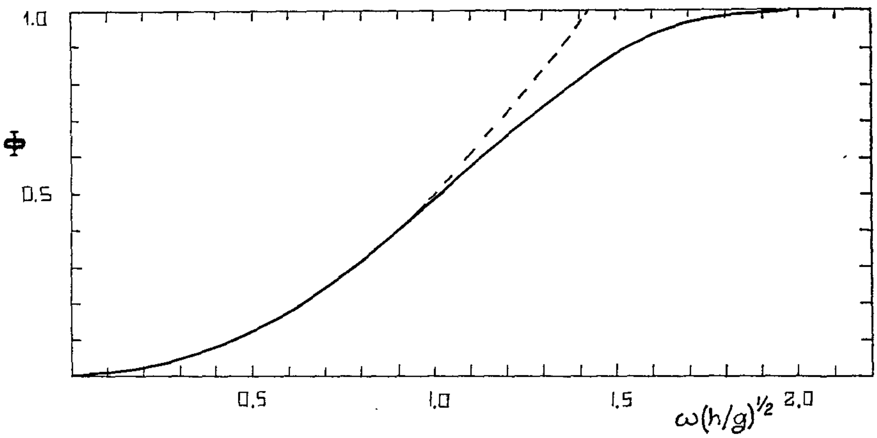

FIG. 1. The universal dimensionless function Φ from (3.1) (solid curve) and the function $\omega_h^2/2$ (dashed curve).

impossible to form a dimensionless combination of g and h, then for a finite-depth sea we can assume that the spatial spectrum of sea waves in the equilibrium range is determined as before by (1.1) and (1.3) with the universal constant B.

Thus we see that the spatial spectra (1.1) and (1.3) can describe extreme configurations of wave surface in the general case of wind-generated gravity waves in a finite-depth sea (i.e., at any value of kh). However, this is by no means true in the case of (1.2) for the $S(\omega)$ spectrum, because according to the transformation formula (2.5) and the simplest dispersion relation (2.1) the frequency spectrum may depend not only on g but also on h. In this case, instead of (1.2) we can, on dimensional grounds, put down for $S(\omega)$ a more general expression

$$S(\omega)=\beta g^2\omega^{-5}\Phi(\omega_h), \tag{3.1}$$

where $\omega_h=\omega h^{\frac{1}{2}}/g^{\frac{1}{2}}$ and $\Phi(\omega_h)$ is some universal nondimensional function. The function $\Phi(\omega_h)$ can be found from Eq. (1.3) for $\chi(k)$, from Eq. (2.5), and from the dispersion equation (2.1). The solution of (2.1) for k is of the form

$$k(\omega)=\frac{\omega^2}{g}\mathcal{H}(\omega_h), \tag{3.2}$$

where a universal function $\mathcal{H}(\omega_h)$ is found from

$$\mathcal{H}\tanh(\omega_h^2\mathcal{H})=1. \tag{3.3}$$

Calculating then the frequency spectrum $S(\omega)$ from (2.5) and comparing the result to (3.1), we find

$$\Phi(\omega_h)=\mathcal{H}^{-2}(\omega_h)\left\{1+\frac{2\omega_h^2\mathcal{H}(\omega_h)}{Sh[2\omega_h^2\mathcal{H}(\omega_h)]}\right\}^{-1}. \tag{3.4}$$

It is easily verified that $\Phi(\omega_h)\rightarrow 1$ when $\omega_h\rightarrow 0$. This asymptotic behavior gives the following relation between constants β in (1.2) and B in (1.3):

$$B=\beta/2, \tag{3.5}$$

which, in the first approximation, agrees with independent experimental data on β and B (Phillips, 1966).

In another extreme case at $\omega_h\rightarrow 0$, the function $\Phi(\omega_h)\rightarrow\omega_h^2/2$ and (3.1) leads to a new universal power law

$$S(\omega)=Bgh\omega^{-3}. \tag{3.6}$$

Eq. (3.6) assumes that the equilibrium range is determined by spectral components for which the long-wave approximation is valid $[\omega=(gh)^{\frac{1}{2}}k]$. However, in reality, the numerical calculation of the function $\Phi(\omega_h)$, plotted in Fig. 1, shows that the range of applicability of (3.6) can be much wider. As follows from Fig. 1, this range goes up to $\omega_h=1$, i.e., up to such values of ω_h as can be quite typically realized in the case of shallow-sea wind-generated waves.

The difficulty in comparing (3.6) to empirical data is caused by the fact that the latter must correspond to a wind-generated sea growing on a water area with horizontal bottom. Otherwise, wave spectrum transformations will be produced by refraction which may obscure deviations from the -5 power law described by $\Phi(\omega_h)$. The data, kindly placed at our disposal by Dreyer (1973) and shown in Fig. 2, are free from this shortcoming because they have been collected on a specially selected area of shallow sea where the depth h was 4 m and did not change over the whole expanse of wave initiation and growth.

Fig. 2 shows that the high-frequency parts of the $S(\omega)$ spectra are not described by the -5 power law but do fit fairly satisfactorily a $S(\omega)\propto\omega^{-3}$ law. Deviations from the -5 power law in wind-generated wave spectra within the littoral zone were also observed in other experiments (Druet *et al.*, 1969; Kakinuma, 1967) but is is unlikely that the condition of horizontal bottom relief was satisfied in any one of these.

It seems worthwhile to attempt by means of the $S(\omega)$ spectra (given in Fig. 2) and via Eq. (3.6) to calculate the constant β. Note that up to now β was deter-

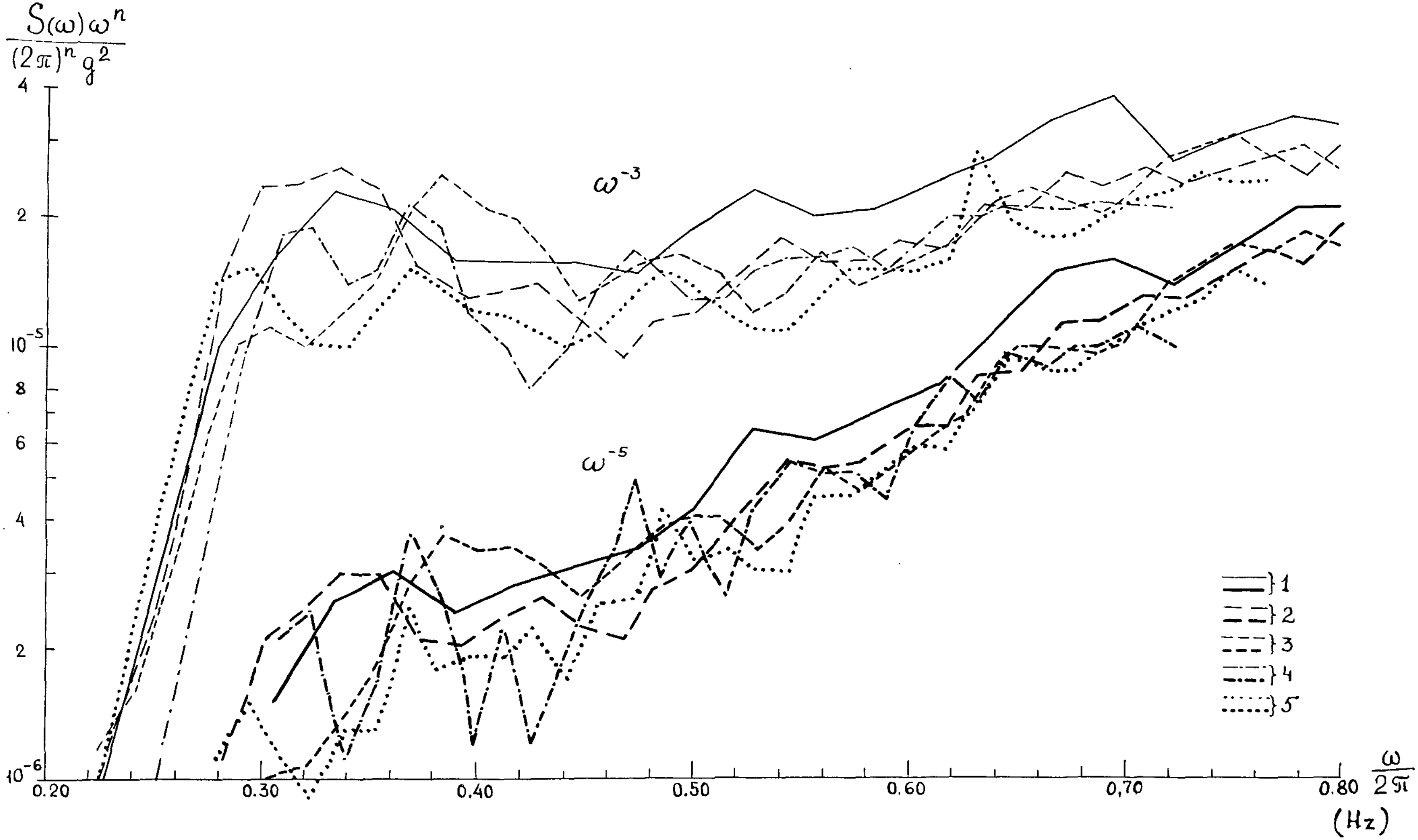

FIG. 2. High-frequency segments of frequency spectra in a shallow sea. Realizations were obtained at wind velocities of 13.2 m s^{-1} (curve 1), 10 m s^{-1} (curves 2 and 3), 13.5 m s^{-1} (curve 4) and 17 m s^{-1} (curve 5).

mined only from spectra with clearly pronounced −5 power law segments. Thus it is of considerable interest to compare the empirical values of β obtained by approximating observed frequency spectra by two different asymptotic formulas [namely (1.2) for a deep sea and (3.6) for a shallow sea].

According to the data plotted in Fig. 2, we can roughly estimate β from

$$(2\pi)^{-2}g^{-2}\omega^3 S(\omega)\approx(2\pm1)\times10^{-5}\ \mathrm{s}^2, \tag{3.7}$$

which, according to (3.6), results in $\beta=(4\pm2)\times10^{-3}$, not too different from β estimates obtained up to now from the data on the Phillips spectrum (1.2). Thus the transformation of the universal spatial spectrum (1.3) into frequency spectra $S(\omega)$ via (2.5) leads to results which agree with observations not only in the deep sea case ($kh\gg1$) but also in the other extreme case ($kh\ll1$) that corresponds to a shallow sea and to Eq. (3.6).

4. Equilibrium range in frequency spectra of wind-generated waves in the case of permanent currents

Perturbation of the relation $S(\omega)\propto\omega^{-5}$ in the high-frequency end of the wind-generated wave spectra may be caused not only by the finite-sea depth effect but also by an effect of sea currents. Indeed, if we assume that in the case of permanent currents the spatial spectrum of wind-generated sea still has in the equilibrium range the universal shape (1.1),[3] then the frequency spectrum must depend, aside from g, also on the modulus of the current velocity vector $\mathbf{v}$. This dependence can be established if, in the transformation of the equilibrium spatial spectrum into a frequency spectrum by means of the formulas given in Section 2, we utilize the dispersion relation (2.2) which includes the Doppler frequency shift caused by the action of permanent sea currents.

Calculating $S(\omega)$ through (1.1), (2.2) and (2.4), we shall restrict ourselves for the sake of simplicity to the case of sufficiently small $v=|\mathbf{v}|$ to be assured that the relation (2.4), which is an approximation in this case, is valid (see Appendix) and to the case of a deep sea where the dispersion relation (2.2) reduces to the equation

$$\omega=\omega(k,\nu)=(gk)^{\frac{1}{2}}+kv\cos(\nu-\gamma), \tag{4.1}$$

where γ is the angle between flow direction and mean wind velocity; the angle ν is measured in relation to this latter vector ($\nu=0$ coincides with the wind direction). The solution of (4.1) for k has the form

$$k=k(\omega,\nu)=\frac{\omega^2}{g}\mathcal{K}_v(y),$$

where

$$y=\omega_v\cos(\nu-\gamma), \tag{4.2}$$

$$\omega_v=v\omega/g,\quad \mathcal{K}_v(y)=\left[\frac{(1+4y)^{\frac{1}{2}}-1}{2y}\right]^2.$$

The function $\mathcal{K}_v(y)$ is defined for $y\geqslant-\frac{1}{4}$. At $y=-\frac{1}{4}$ the "group velocity" $G=\partial\omega/\partial k$ of spectral components is zero (energy does not propagate upstream), while the phase velocity $c=\omega/k$ is directed upstream and equal to $-v\cos(\nu-\gamma)$. To apply the transformation formula (2.4) it is necessary to calculate from (4.1) the "group velocity" which is then

$$G(\omega,\nu)=\frac{1}{2}\frac{g}{\omega}f(y), \tag{4.3}$$

where $f(y)=\mathcal{K}_v^{-\frac{1}{2}}(y)+2y$. The following expressions are obtained for the frequency spectrum when (4.2) and (4.3) are substituted into (2.4) and by expressing $\Psi(\mathbf{k})$ by means of (1.1):

$$S(\omega)=\beta g^2\omega^{-5}J(\omega_v,\gamma), \tag{4.4}$$

where

$$J(\omega_v,\gamma)=\int_{-\pi}^{\pi}r(y)\varphi(\nu)d\nu, \tag{4.5}$$

$$r(y)=\mathcal{K}_v^{-3}(y)f^{-1}(y). \tag{4.6}$$

It is easily seen that $J(\omega_v,\gamma)\to1$ as $\omega_v\to0$. Thus it follows from (4.4) that if the main contribution in the equilibrium range is made by those spectral components for which the phase velocity modulus satisfy $c=g/\omega\gg v$, then the Phillips law (1.2) still holds. Note that the function $J(\omega_v,\gamma)$ has some definition domain determined by the condition $y\geqslant-\frac{1}{4}$ and using the approximate relation (2.4). As a result, the non-dimensional frequency ω_v can vary from zero to some value, determined by γ. We shall omit the relevant analysis but note that as a rule the phase velocity of gravity waves in the equilibrium range in practice satisfies $c\gg v$, and thus $\omega_v=v/c\ll1$. From this latter inequality it follows that for $0\leqslant\gamma\leqslant\pi$ [the function $J(\omega_v,\gamma)$ is even with respect to γ if $\varphi(\nu)$ is even with respect to ν] and $0\leqslant\omega_v\ll1$ the variables ω_v and γ do not leave the definition domain of the function $J(\omega_v,\gamma)$.

Figs. 3a and 3b show the results of numerical calculations of the correction function $J(\omega_v,\gamma)$ for three model angular distribution functions $\varphi(\nu)$:

$$\left.\begin{aligned}
&\varphi_1(\nu)=\delta(\nu)\\
&\varphi_2(\nu)=\begin{cases}1/\pi, & \text{at } |\nu|\leqslant\pi/2\\ 0, & \text{at } |\nu|>\pi/2\end{cases}\\
&\varphi_3(\nu)=\begin{cases}2/\pi\cos^2\nu, & \text{at } |\nu|\leqslant\pi/2\\ 0, & \text{at } |\nu|>\pi/2\end{cases}
\end{aligned}\right\}. \tag{4.7}$$

[3] This appears natural because, as has been mentioned above, arguments in favor of the k^{-4} law can be associated only with geometric properties of extremal configurations of the wind-generated sea surface.

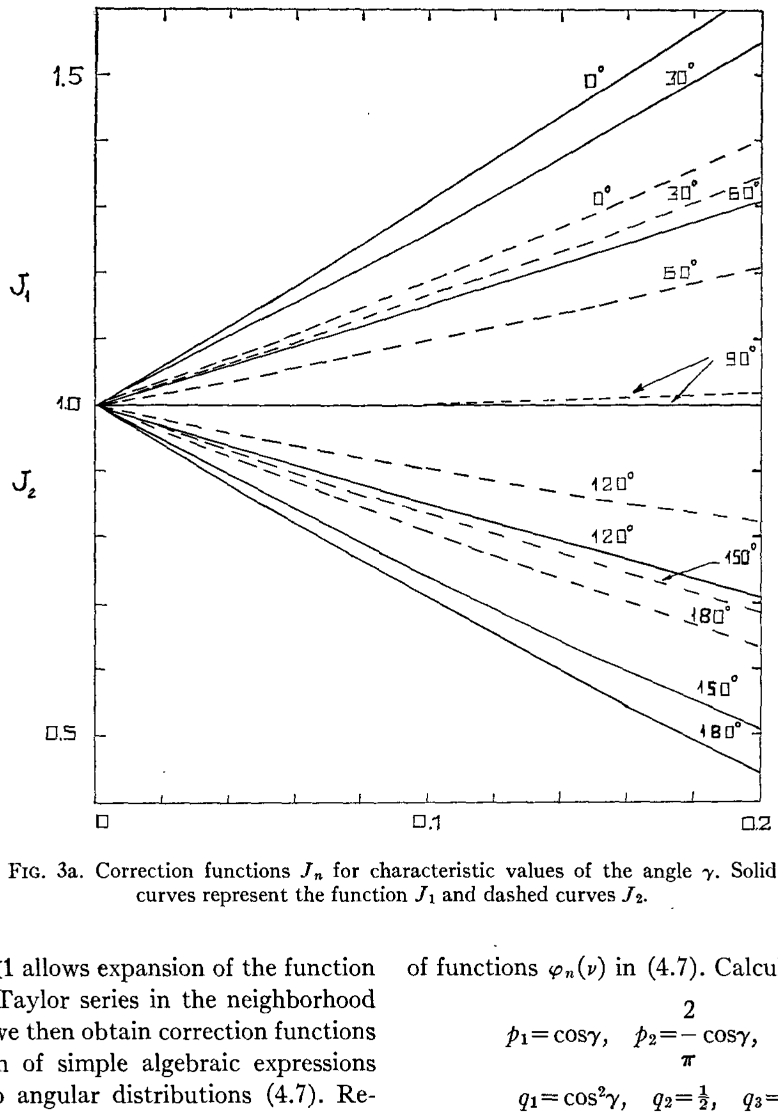

FIG. 3a. Correction functions J_n for characteristic values of the angle γ. Solid curves represent the function J_1 and dashed curves J_2.

The condition $\omega_v \ll 1$ allows expansion of the function $r(y)$ in (4.5) into a Taylor series in the neighborhood of $y=0$. As a result, we then obtain correction functions $J_n(\omega_v,\gamma)$ in the form of simple algebraic expressions which correspond to angular distributions (4.7). Restricting the expressions to terms quadratic in y, we have

$$r(y)=1+3y+y^2. \tag{4.8}$$

Substituting (4.8) into (4.5) we obtain

$$J_n(\omega_v,\gamma)=1+3p_n\omega_v+q_n\omega_v^2, \tag{4.9}$$

where

$$p_n=\int_{-\pi}^{\pi}\cos(\nu-\gamma)\varphi_n(\nu)d\nu, \tag{4.10}$$

$$q_n=\int_{-\pi}^{\pi}\cos^2(\nu-\gamma)\varphi_n(\nu)d\nu, \tag{4.11}$$

and the subscript $n=1$, 2, 3 corresponds to subscripts of functions $\varphi_n(\nu)$ in (4.7). Calculations give

$$p_1=\cos\gamma,\quad p_2=\frac{2}{\pi}\cos\gamma,\quad p_3=\frac{8}{3\pi}\cos\gamma, \tag{4.12}$$

$$q_1=\cos^2\gamma,\quad q_2=\tfrac{1}{2},\quad q_3=\tfrac{1}{2}\cos^2\gamma+\tfrac{1}{4}. \tag{4.13}$$

Eq. (4.9) with coefficients (4.12) and (4.13) corresponds well enough to the results of direct numerical calculations through formulas (4.5)–(4.7); these results are illustrated in Fig. 3.

As follows from the curves of Fig. 3, the sea current effect is enhanced when the angular distribution narrows. When $\cos\gamma>0$, a trend is seen in the (4.4) spectrum toward an apparent increase of the exponent $(n>-5)$, and when $\cos\gamma<0$ there is a decrease $(n<-5)$. Actually the equilibrium frequency spectrum in the case of currents is not, according to (4.4), described by the power law.

It should be noted that the role of onshore-offshore currents in a shallow sea can be quite important and deviations from (1.2) will be associated with the combined influence of finite depth and currents.

Thus, in the extreme case of a very shallow sea, if we assume instead of (4.1) that $\omega=\omega(k,\nu)=(gh)^{\frac{1}{2}}k+kv\times\cos(\nu-\gamma)$, then for the case $v<(gk)^{\frac{1}{2}}$ instead of (4.4) we get

$$S(\omega)=Bgh\omega^{-3}J_h\left[\frac{v}{(gh)^{\frac{1}{2}}},\gamma\right], \quad (4.14)$$

where

$$J_h\left[\frac{v}{(gh)^{\frac{1}{2}}},\gamma\right]=\int_{-\pi}^{\pi}(1+x)^2\varphi(\nu)d\nu, \quad (4.15)$$

and $x=v(gh)^{-\frac{1}{2}}\cos(\nu-\gamma)$. For the angular distribution functions (4.7) we can calculate integrals (4.15) analytically. We omit the calculations associated with it and only note that J_h does not depend on frequency and thus the spectrum (4.14) is a power function and is proportional to ω^{-3}. At $v=0$ the spectrum (4.14) transforms into (3.6).

5. Equilibrium range in frequency spectra of wind-generated sea waves in the case of random dispersion relation parameters

In this section, the discussion centers on the effects of large-scale components of surface waves on the form of spectra in the equilibrium range. Considering the shape of the equilibrium frequency spectrum $S(\omega)$ and taking into account the random nature of the parameters $\bar{a}$ and $\bar{\mathbf{v}}$ in the dispersion relation (2.6), we limit the discussion here by assuming that the spatial spectrum of wind-generated sea in the equilibrium range is still described by (1.1).

It is evident that the random parameters $\bar{a}$ and $\bar{\mathbf{v}}$ in (2.6) are not independent because they are related to one another by a corresponding system of dynamic equations for large-scale components of surface sea waves. A simple approximation of the relations $\bar{a}(\bar{\zeta})$ and $\bar{\mathbf{v}}(\bar{\zeta})$, acceptable within the present analysis, can be obtained easily if we use the empirical fact that the large-scale components' spectrum is narrow. Indeed, due to this feature we can assume an approximation $\bar{\zeta}=A^0\cos(\mathbf{k}^0\mathbf{x}-\omega^0t+\psi^0)$, where $A^0=A^0(\mathbf{x},t)$ and $\psi=\psi^0(\mathbf{x},t)$ are slowly varying random functions with scales $\lambda^0=2\pi/k^0$ and $\tau^0=2\pi/\omega^0$. Neglecting then the dependence of these slowly-varying variables on $\mathbf{x}$ and t in the set of linearized hydrodynamical equations for $\bar{\zeta}$, we obtain

$$\bar{a}(\bar{\zeta})=-(\omega^0)^2\bar{\zeta},\quad \bar{\mathbf{v}}(\bar{\zeta})=\omega^0\frac{\mathbf{k}^0}{k^0}\bar{\zeta}, \quad (5.1)$$

where in the selected approximation the direction of the vector $\mathbf{k}^0$ is constant.

Substituting $\bar{\mathbf{v}}$ from (5.1) into (2.6) we obtain

$$\omega=\omega(k,\nu)=[(g+\bar{a})k]^{\frac{1}{2}}+k\bar{W}\cos(\nu-\gamma^0), \quad (5.2)$$

where γ^0 is a fixed angle between the propagation direction of a long-wave component of surface waves and that of the mean wind velocity, and $\bar{W}=\omega^0\bar{\zeta}$. Comparison of (5.2) and (4.1) demonstrates that the expression for the conditional frequency spectrum $\bar{S}(\omega)$ in (2.7) can be derived by substituting $g+\bar{a}$ for g, $\bar{W}$ for v, and

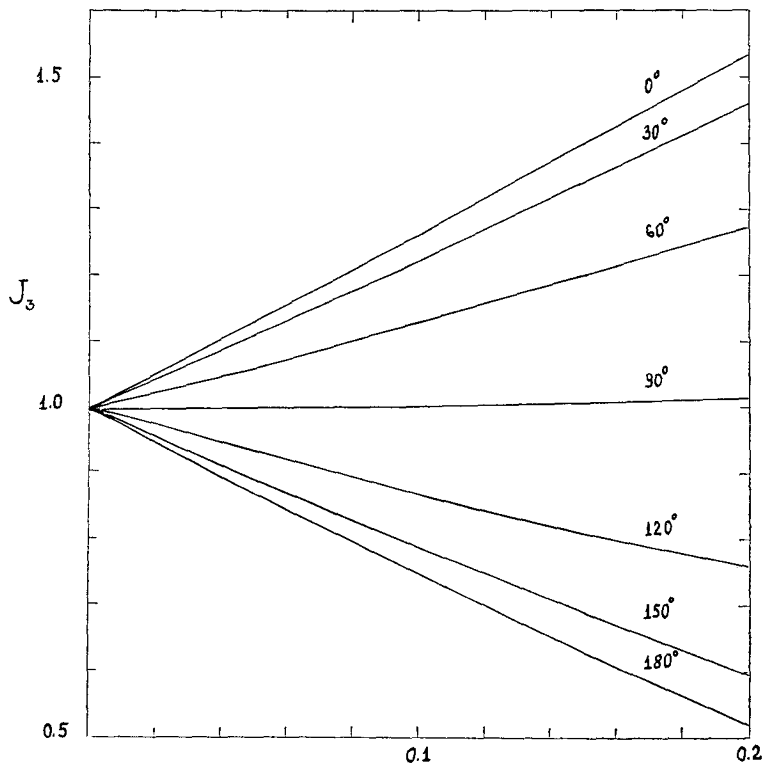

Fig. 3b. Correction function J_3 for characteristic values of the angle γ.

γ^0 for γ in (4.4). Thus

$$\bar{S}(\omega)=\beta g^2\omega^{-5}\bar{J}(\bar{\omega},\gamma^0), \tag{5.3}$$

where

$$\bar{J}(\bar{\omega},\gamma^0)=\left(1+\frac{\bar{a}}{g}\right)^2 J(\bar{\omega},\gamma^0), \tag{5.4}$$

$$\bar{\omega}=\frac{\omega\bar{W}}{g+\bar{a}}=\frac{\omega}{\omega^0}\frac{\bar{\eta}}{1-\bar{\eta}}. \tag{5.5}$$

Here the right side of (5.5) is obtained by utilizing (5.1), and $\bar{\eta}=-\bar{a}/g=k^0\bar{\zeta}$. Substituting (5.3) into (2.7) we arrive at

$$S(\omega)=\beta g^2\omega^{-5}J^0(\omega,\gamma^0), \tag{5.6}$$

where

$$J^0(\omega,\gamma^0)=\langle\bar{J}(\bar{\omega},\gamma^0)\rangle. \tag{5.7}$$

The function $J^0(\omega,\gamma^0)$ can be espressed through moments of the random function $\bar{\zeta}$ if $\bar{J}(\bar{\omega},\gamma^0)$ is expanded into a power series in $\bar{\eta}$. An expression for $J^0(\omega,\gamma^0)$ to the accuracy of the second moment can be derived if in (5.4) we use the approximate expression (4.9) for $J(\bar{\omega},\gamma^0)$. Substitution of (4.9) into (5.4) gives

$$\bar{J}_n(\bar{\omega},\gamma^0)=1+p_n{}^*k^0\bar{\zeta}+q_n{}^*(k^0\bar{\zeta})^2, \tag{5.8}$$

where

$$p_n{}^*=3p_n\frac{\omega}{\omega^0}-2,\quad q_n{}^*=1-3p_n\frac{\omega}{\omega^0}+q_n\left(\frac{\omega}{\omega^0}\right)^2. \tag{5.9}$$

As earlier, p_n and q_n are given by (4.10) and (4.11) where γ is replaced by γ^0 and the subscript n corresponds to the subscripts of the angular distribution functions $\varphi_n(\nu)$ in (4.7). Averaging (5.8) we obtain

$$J^0(\omega,\gamma^0)=1+q_n{}^*\frac{\langle\bar{a}^2\rangle}{g^2}, \tag{5.10}$$

where

$$\frac{\langle\bar{a}^2\rangle}{g^2}=\langle\bar{\eta}^2\rangle=(k^0)^2\langle\bar{\zeta}^2\rangle=\frac{(A^0k^0)^2}{2}. \tag{5.11}$$

The function $q_n{}^*=q_n{}^*(\omega/\omega^0,\gamma^0)$ in (5.10) becomes equal to unity in the particular case in which short waves are two-dimensional ($n=1$) and the propagation direction of long-wave perturbations is normal to that of short waves ($\gamma^0=\pi/2$). In this case it indeed follows from (4.12) and (4.13) that $p_1=q_1=0$; hence, $q_1{}^*=1$. Then the frequency spectrum in the equilibrium range is described by[4]

$$S(\omega)=\beta g^2\omega^{-5}\left(1+\frac{\langle\bar{a}^2\rangle}{g^2}\right). \tag{5.12}$$

Eq. (5.12) may also be considered as a consequence of dispersion relations (2.6) or (5.2) when the Doppler effect is omitted from calculations. It is evident that in the case under discussion large-scale components of surface sea waves do not change the functional relation of the frequency spectrum to frequency in the equilibrium range [$S(\omega)\propto\omega^{-5}$] but cancel universality of the proportionality coefficient in the Phillips law (1.2). However, since the ratio $\langle\bar{a}^2\rangle/g^2$, according to (5.11), is proportional to the square of the slope of large-scale perturbation, this correction is practically always very small.

In other cases this correction may be more significant. For example, according to (5.9), (4.12) and (4.13), variation of the ratio ω/ω^0 within the range $\sim$(1–10) results in variation of $q_n{}^*$ within the range $\sim(-1$–$150)$; with $(A^0k^0)\sim0.1$, it leads to variation of $J^0(\omega,\gamma^0)$ within the range $\sim$(1–1.75).

6. Conclusions

In the last two Sections the presence of the basic current and large-scale components of surface waves have been accounted for only in terms of the dispersion relation for short gravity waves, whose spatial spectrum was described by a universal law [Eq. (1.1)]. Actually, as was shown recently by Banner and Phillips (1974) and Phillips and Banner (1974), when long waves move across the surface there is a nonlinear augmentation of the surface drift (with shear) near the long-wave crests so that short waves, superimposed on the longer ones, can attain a reduced value of maximum amplitude at the point of incipient breaking. It is obvious that this dynamical effect is in clear contradiction with the initial statement of Phillips' hypothesis on spatial equilibrium spectra of the type (1.1), because it assumes independence of $\Psi(\mathbf{k})$ from large-scale components of surface sea waves and wind drift. This fact, together with the effect of the strongly intermittent nature of short-wave components (see Zaslavskii and Kitaigorodskii, 1972), must be taken into account if one wishes to obtain a more complete and detailed description of the high-frequency parts of the wind-generated gravity wave spectrum. We hope to discuss some of the above-mentioned questions in the another paper.

APPENDIX

The Relationships between $\Psi(k)$ and $S(\omega)$

The general structure of the relationships between $\Psi(\mathbf{k})$ and the frequency-directional spectrum $\hat{F}(\omega,\nu)$ defined for positive and negative frequencies ω can be derived as follows:

$$\begin{aligned}\hat{F}(\omega,\nu)&=\int_0^\infty \Psi(k,\nu)k\delta[\omega-\omega(k,\nu)]dk\\&=\int_0^\infty \Psi(k,\nu)k\sum_{i=1}^N\left[\frac{\partial\omega(k,\nu)}{\partial k}\Big|_{k=k_i(\omega,\nu)}\right]^{-1}\\&\qquad\times\delta[k-k_i(\omega,\nu)]dk\\&=\sum_{i=1}^N\left[\Psi(k,\nu)\frac{k}{G(k,\nu)}\right]_{k=k_i(\omega,\nu)},\end{aligned} \tag{A1}$$

[4] Eq. (5.12) has been derived earlier by Zaslavskii and Kitaigorodskii (1971) by directly stating a more precise definition of Phillips' statement for the frequency spectrum $S(\omega)$, by incorporating fluctuations of the basic parameter $\bar{g}=g+\bar{a}$.

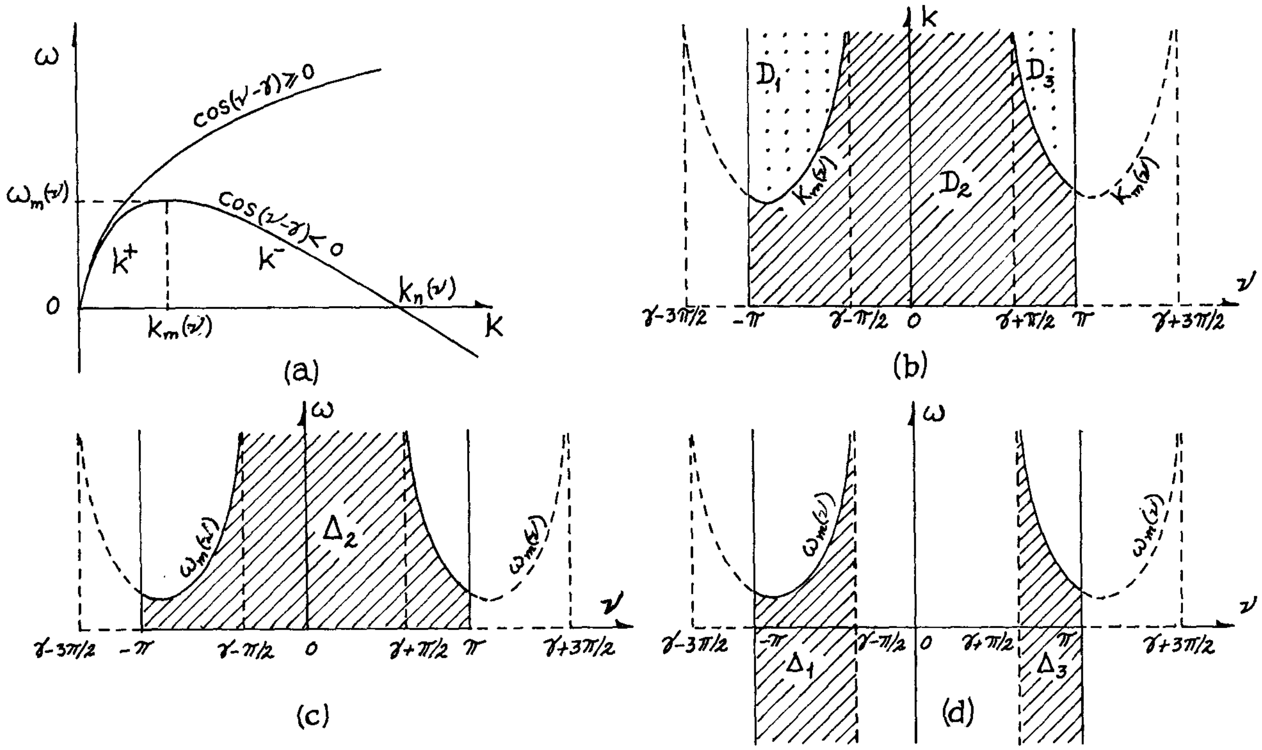

FIG. 4. Sketch of the dispersion curve (4.1), a, and domains of integrations in the case of $0\leqslant\gamma\leqslant\pi/2$, b–d.

where $k=k_i(\omega,\nu)$ are roots of equation $\omega=\omega(k,\nu)$ and δ is the Dirac delta function. Note that the number of roots N, and hence the number of terms of the sums in (A1), depends on a given fixed combination of ω and ν. As a consequence, each term in (A1) has some domain of definition Δ_i in the (ω,ν) plane and must be considered as zero for $(\omega,\nu)\notin\Delta_i$. The domains Δ_i depend upon the particular type of dispersion relation.

The frequency spectrum defined for positive and negative frequencies ω can be defined then by the relation

$$\hat{S}(\omega)=\int_{-\pi}^{\pi}\hat{F}(\omega,\nu)d\nu=S^+(\omega)+S^-(-\omega), \qquad \text{(A2)}$$

with the normalization condition

$$\int_{\omega\gtrless 0}\hat{S}(\omega)d\omega=\langle\zeta^2\rangle,$$

where $S^+(\omega)=0$ for $\omega<0$ and $S^-(\omega)=0$ for $\omega>0$. Then the frequency spectrum defined only for positive frequencies $(0\leqslant\omega<\infty)$ is

$$S(\omega)=S^+(\omega)+S^-(\omega), \qquad \text{(A3)}$$

with normalization conditions as in (2.3). In the case of one-valued $k(\omega,\nu)$ defined for $0\leqslant\omega<\infty$, we have $N=1$, $S^-(\omega)=0$, and relations (A1)–(A3) give (2.4).

The procedure of finding $S(\omega)$ depends on the particular type of dispersion relation under consideration. As an important example we shall consider the dispersion relation (4.1). It is easy to see from (4.1) that the $\cos(\nu-\gamma)<0$ the function $k(\omega,\nu)$ is a multiple function of ω and has two branches $k^+(\omega,\nu)$ and $k^-(\omega,\nu)$:

$$k^{\pm}(\omega,\nu)=\frac{\omega^2}{g}\mathcal{K}_v^{\mp}(y),\quad \mathcal{K}_v^{\pm}(y)=\left[\frac{(1+4y)^{\frac{1}{2}}\mp 1}{2y}\right]^2, \qquad \text{(A4)}$$

where y and ω_v have the same meanings as in (4.2). The dispersion curve (4.1) is illustrated qualitatively in Fig. 4a. If $\cos(\nu-\gamma)<0$ then the function $\omega(k,\nu)$ has a maximum $\omega_m(\nu)=-g/[4v\cos(\nu-\gamma)]$ for $k=k_m(\nu)\equiv g/[4v^2\cos^2(\nu-\gamma)]$. The left-hand part (relative to the maximum of the curve corresponds to k^+ $(0\leqslant k\leqslant k_m$, $0\leqslant\omega\leqslant\omega_m)$ and right-hand part to $k^-(k_m\leqslant k<\infty$, $\omega_m\leqslant\omega<-\infty)$. Transition to negative frequencies occurs at $k_n(\nu)=g/[v^2\cos^2(\nu-\gamma)]$. Both functions k^+ and k^- are defined for $y\geqslant -\frac{1}{4}$.

Because Eq. (4.1) has two roots k^+ and k^- for the case $\cos(\nu-\gamma)<0$, corresponding to $N=2$ in (A1), then

$$\hat{F}(\omega,\nu)=\hat{F}^+(\omega,\nu)+\hat{F}^-(\omega,\nu), \qquad \text{(A5)}$$

where

$$\hat{F}^{\pm}(\omega,\nu)=[\Psi(k,\nu)k/G(k,\nu)]_{k=k^{\pm}(\omega,\nu)}.$$

As was pointed out, each term in (A1) has its domain of definition in the (ω,ν) plane. To find these domains it must be remembered that

$$\int_D\int\Psi(k,\nu)kdkd\nu=\int_\Delta\int\hat{F}(\omega,\nu)d\omega d\nu, \qquad \text{(A6)}$$

where $D=\{-\pi\leqslant\nu\leqslant\pi,\ 0\leqslant k<\infty\}$. The domain D is divided by the curve $k_m(\nu)$ into a number of domains D_i which, in the (ω,ν) plane, correspond to domains of definition Δ_i of the functions k^+ and k^-. This division, and hence the transformation, of D in Δ depends on the value of the angle γ. It is convenient to consider separately two cases: 1) $0\leqslant\gamma\leqslant\pi/2$ and 2) $\pi/2\leqslant\gamma\leqslant\pi$.

For $0\leqslant\gamma\leqslant\pi/2$ the domain D can be divided into domains D_1, D_2 and D_3 (Fig. 4b) and each of those domains in the (k,ν) plane corresponds to the domains Δ_1, Δ_2 and Δ_3 in the (ω,ν) plane presented in Figs. 4c. and 4d. Such transformation corresponds to $\hat{F}^+$ in (A5) defined in Δ_2 and zero for $(\omega,\nu)\notin\Delta_2$, and $\hat{F}^-$ defined in Δ_1 and Δ_3, while zero for $(\omega,\nu)\notin\Delta_1$, Δ_3.

For sake of simplicity of the analysis and to derive a simple analytical formula for $S(\omega)$, we suppose that the velocity v in (4.1) is so small that $\omega_m\sim v^{-1}$ is larger than upper limit ω^* of equilibrium range. Then $k_m\sim v^{-2}$ is larger than upper limit k^* in the spectrum $\Psi(k,\nu)$. But because $\hat{F}^-$ is determined by the values $k>k_m$ (see Fig. 4) and taking into account that $\Psi\sim k^{-4}$ we can neglect the term $\hat{F}^-$ in the sum (A5). Thus $\hat{F}(\omega,\nu)\approx\hat{F}^+(\omega,\nu)$ for $-\pi\leqslant\nu\leqslant\pi$ and $\omega_*\leqslant\omega\leqslant\omega^*$, where ω_* is the lower limit of the equilibrium range. This means that relation (2.4) is approximately valid with $k=k^+(\omega,\nu)$. The case $\pi/2\leqslant\gamma\leqslant\pi$ can be treated in a similar way.

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