On the Theory of the Equilibrium Range in the Spectrum of Wind-Generated Gravity Waves

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(Manuscript received 23 November 1982, in final form 14 February 1983)

ABSTRACT

It is shown that an exact analog of Kolmogoroff's spectrum in a random field of weakly nonlinear surface gravity waves gives a spectral form for frequency spectra $S(\omega) \sim \omega^{-4}$ in close agreement with the results of recent observational studies. The proposed theory also indicates the existence of a "transitional" range of wavenumbers (frequencies) where the deviation from Kolmogoroff's equilibrium is due to gravitational instability (wave breaking). Because of this it is suggested that the equilibrium form for the spectrum of wind-generated waves has two asymptotic regimes: Kolmogoroff's *and* Phillips' type of equilibrium with a relatively rapid transition from the first to the second. The experimental data favor such an interpretation.

1. Introduction

Since Phillips' fundamental contribution to the concept of the equilibrium range in the spectrum of wind-generated waves (Phillips, 1958), numerous experimental studies have demonstrated quite clearly that a substantial portion of the wind-wave spectrum above the frequency of the spectral peak is saturated and in an equilibrium with the local wind. However, questions concerning the dynamical processes in the wind-wave field that are responsible for such equilibrium and what is the precise form of a spectrum in equilibrium conditions, are still the subject of serious controversies. The three latest observational studies (Forristall, 1981; Kahma, 1981a, Donelan *et al.*, 1982) have demonstrated very convincingly systematic deviations from Phillips' −5 power law in the range of frequencies just above the peak frequency of the spectrum $S(\omega)$. The frequency spectrum there has a wind-dependent form $S(\omega) \propto U_a^{5-n}\omega^{-n}$ (U_a is the wind speed) with values of n close to 4.0. The latter value leads to an equilibrium wind-dependent frequency spectrum $S(\omega) \propto gU_a\omega^{-4}$ (where g is gravity). This form of equilibrium in wind-wave spectra was first suggested by Kitaigorodskii (1962) as a possible consequence of Kolmogoroff's type of energy cascade from low to high frequencies, and was derived later on the basis of the calculation of the dynamical resonant interactions between weakly nonlinear surface gravity deep-water waves by Zaharoff and Filonenko (1966). The very fact that there can exist an exact analog of Kolmogoroff's spectra in a random wave field which gives a spectral form in qualitative agreement with observations indicates the importance of nonlinear interaction between wave components in the formation of the equilibrium interval of the steady-state wave spectra in it. Therefore, the concept of an equilibrium range based solely on a consideration of the process of wave breaking due to gravitational instability (Phillips, 1958) can be generalized by taking into account the fact that the growth of shorter waves on the rear face of the spectrum (limited by breaking) may not be due to direct energy input from wind, but rather is due to energy flux from the lower wavenumbers.

In the present paper, I attempt to incorporate this factor in determining the form of frequency spectra in equilibrium range by using an approach similar to the theory of locally-isotropic three-dimensional turbulence: namely by postulating the existence of Kolmogoroff's type of equilibrium in the spatial statistical characteristics of a random wave field and then determining the "internal" scale for surface gravity waves, at which deviations from the energy cascade mechanism can occur due to gravitational instability (wave breaking).

2. The concept of a Kolmogoroff's equilibrium range in wind-wave spectra

If the turbulent wind continues to blow for a sufficiently long time and if the fetch is also sufficiently large, the wave amplitudes continue to increase, so that ultimately nonlinear interactions among the Fourier components of the wave field become increasingly important. The pioneering calculations by Hasselmann *et al.* (1973) have demonstrated that the shape of the wind-wave spectrum is determined primarily by the nonlinear energy transfer (due to resonant wave-wave interactions) from the central region of the spectrum to *both* shorter and longer wave components. We will here be mostly interested in the evolution of spectral components at frequencies higher than those of the peak, on the rear face of the spectrum of wind waves. To find an analogy with Kolmogoroff's spectrum for a field of wind-generated surface gravity waves *we must assume* that the regions (in Fourier space) of wave "generation" by wind and "dissipation" are separated. Since in *such* situations we have equations for the evaluation of the spectral energy density of weakly nonlinear dispersive waves, derived from first principles, it is important to find out if it is possible for nonlinear interactions between different wave modes to generate an energy cascade through the spectrum as in Kolmogoroff's view of turbulence. In other words, the question is, can wave spectra in certain wavenumber intervals be determined by energy flux from low to high wavenumbers (frequencies) and if it can, then what form must wave spectra have in such a region? To obtain an answer we must consider the so-called wave kinetic equation (Hasselmann, 1968) and in particular its stationary solutions. It is convenient to write this equation in terms of spectral density of wave action per unit mass $N(\mathbf{k})$ defined as (Phillips, 1977)

$$N(\mathbf{k}) = \frac{gF(\mathbf{k})}{\sigma_\mathbf{k}}. \quad (1)$$

In (1) $\sigma_\mathbf{k}$ is frequency of free surface gravity waves [in deep water $\sigma_\mathbf{k} = \sigma(k) = (gk)^{1/2}$] and $F(\mathbf{k})$ nonsymmetrical wavenumber spectrum, normalized as

$$\int d\mathbf{k}F(\mathbf{k}) = \overline{\zeta^2(\mathbf{x}, t)}, \quad (2)$$

where $\zeta(\mathbf{x}, t)$ is surface displacement. The equation for $N(\mathbf{k})$ for weakly nonlinear surface gravity waves was originally derived by Hasselmann (1962), and without "generation" and "dissipation" terms, it can be written for a statistically homogeneous wave field as

$$\frac{\partial N(\mathbf{k})}{\partial t} = I, \quad (3)$$

where so-called collision integral I for *four* resonantly interacting gravity waves has the form (Phillips, 1977)

$$I = I_4(\mathbf{k}) = \iiint |Q^2_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}|\{[N(\mathbf{k}) + N(\mathbf{k}_1)]N(\mathbf{k}_2)N(\mathbf{k}_3) - [N(\mathbf{k}_2) + N(\mathbf{k}_3)]N(\mathbf{k}_1)N(\mathbf{k})\} \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(\sigma_\mathbf{k} + \sigma_{\mathbf{k}_1} - \sigma_{\mathbf{k}_2} - \sigma_{\mathbf{k}_3}) \times d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3, \quad (4)$$

where the coupling coefficient $Q_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ is a complicated homogeneous function of the wavenumbers $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and $\mathbf{k}$, respectively, of order $|\mathbf{k}|^3$ [for details see the derivation of (4) and the expressions for Q in Hasselmann (1968) and West (1981)].

The intriguing question in the analysis of (4) is whether the stationary distribution corresponding to Kolmogoroff's inertial subrange, i.e., the conditions for a predominantly *local* (in wavenumber space) energy transfer, exists for dispersive nonlinear surface gravity waves. The necessary (but not sufficient) condition for this is localness of $I[N(\mathbf{k})]$, such that the coupling coefficient $Q_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ rapidly diminishes for $k \gg k_1, k_2, k_3$ or $k \ll k_1, k_2, k_3$. If this is so, we can, following Kolmogoroff's idea, introduce the energy transfer rate through the energy spectrum $gF(\mathbf{k})$ and assume that there exists a certain range of wavenumbers $\mathbf{k}$ in the wind–wave spectrum where this energy flux is in essence constant (inertial subrange). It is convenient for further discussion to deal with average (over all directions of wave component propagation γ) of $F(\mathbf{k})$ and $N(\mathbf{k})$, defined as

$$F_k = \int_\gamma F(\mathbf{k})d\gamma = \int_{-\pi}^{+\pi} F(k, \gamma)d\gamma, \quad (5)$$

$$N_k = \int_\gamma N(\mathbf{k})d\gamma = \int_{-\pi}^{+\pi} N(k, \gamma)d\gamma. \quad (6)$$

For the isotropic dispersion relationship $\sigma_\mathbf{k} = \sigma(k)$

$$N_k = \frac{gF_k}{\sigma_k}. \quad (7)$$

Now F_k and N_k characterize the energy distribution as a function of wavenumber modulus $|\mathbf{k}| = k$. The condition of constant energy flux $\epsilon(k)$ through the spectra F_k can be written as

$$\epsilon(k) = gF_kk^2\tau_k^{-1} = N_k\sigma_kk^2\tau_k^{-1} = \text{constant} = \epsilon_0, \quad (8)$$

where τ_k is a characteristic time of the nonlinear interactions in a narrow interval of wavenumbers around $|\mathbf{k}| = k$. From Eq. (4) we can easily find an estimate of this quantity

$$\tau_k^{-1} = \frac{\int_\gamma I d\gamma}{N_k} = \frac{I_k}{N_k}. \quad (9)$$

For $\sigma_k = (gk)^{1/2}$ the contribution from δ-functions in I_k can be estimated and the scaling of I_k as $|Q^2| \sim k^6$ will give us

$$I_k \approx Q^2_{kkkk}N_k{}^3k^{7/2}g^{-1/2} = \frac{N_k{}^3k^{19/2}}{g^{1/2}}, \quad (10)$$

which will lead to the following estimate of τ_k^{-1}:

$$\tau_k^{-1} \approx \frac{k^{19/2}N_k{}^2}{g^{1/2}} = F_k{}^2g^{1/2}k^{17/2}. \quad (11)$$

Substituting (11) in (8) we finally receive the analog of Kolmogoroff's energy spectra for deep-water surface waves, i.e.,

$$gF_k \approx \epsilon_0^{1/3} g^{1/2} k^{-3,5} \tag{12}$$

or

$$F_k \approx \epsilon_0^{1/3} g^{-1/2} k^{-3,5}. \tag{13}$$

The corresponding frequency spectra $S(\omega)$ can be immediately found from F_k if the dispersive relationship is isotropic (Kitaigorodskii *et al.*, 1975):

$$S(\omega) = \left. \frac{kF_k}{\dfrac{\partial\sigma(k)}{\partial k}} \right|_{k=\omega^2/g}. \tag{14}$$

It follows from (13), (14) that

$$S(\omega) \approx \epsilon_0^{1/3} g \omega^{-4}. \tag{15}$$

It was shown by Zaharoff and Filonenko (1966) that frequency spectra $S(\omega) \sim \omega^{-4}$ for an *isotropic* field of weakly nonlinear waves really is an exact stationary solution of the kinetic equation (4) and the collision integral for $S(\omega) \sim \omega^{-4}$ converges both for $\omega \rightarrow \infty$ and $\omega \rightarrow 0$. The equations [(12), (13), (15)] correspond to the idealized situation when the energy input from the wind is concentrated at $\omega = k = 0$ and the energy flux ϵ_0 resulting from nonlinear interactions is toward the dissipation region at $\omega = k = \infty$. Consequently, Eqs. (12) and (15) can be considered as an exact analog of Kolmogoroff's spectra for so-called "weak turbulence" in a random field of weakly nonlinear surface gravity waves. However, an open question remains: what are the properties of the angular energy distribution; i.e., the anisotropy in the wavenumber spectrum $F(\mathbf{k})$ in the above formulation, when only *directionally averaged values* F_k *and* N_k *are used.*

The applicability of Kolmogoroff's energy spectra (12), (15) to the description of the statistical characteristics of a wind-generated wave field is not at all obvious. First of all, even if the energy cascade structure in two-dimensional wavenumber space of surface waves is similar to those of three-dimensional turbulence, the wave spectrum $F(\mathbf{k})$ is not only anisotropic, but the observed angular distribution is not quite **k**-independent [it broadens noticeably toward higher wavenumbers (frequencies)].

However, it is possible that because a k-independent form of the angular distribution may still be a good approximation for the energy-containing, rear face of wave spectra, the *deviations* from Kolmogoroff's equilibrium spectra (12), (15) of averaged (over all angles γ) values of $N(\mathbf{k})$ and $F(\mathbf{k})$, i.e., N_k and F_k, will be insignificant.

Another problem of applying (12), (15) to observed spectra is related to the legitimacy of the assumption about the separation of the energy input and dissipation regions in the wavenumber or frequency domain. In other words, the unresolved question, at least theoretically, is: what are realistic values for the low and high boundaries (in wavenumber and frequency) of Kolmogoroff's subrange (12), (15) in wind-wave spectra? From the observational view point we can only say that the low-frequency boundary of Kolmogoroff's equilibrium subrange must lie *above* the peak frequency ω_m. This, however, does not mean that the direct atmospheric forcing is important only in the region $\Delta\omega$ with $\Delta\omega \approx \omega_m$. Therefore a more realistic assumption about the dominance of the energy transfer rate ϵ_0 in a certain frequency range $\omega > \omega_m$ over the direct wind input must be based on an inequality of the type

$$I_k \gg S_k \quad \text{for} \quad k > k(\omega_m), \tag{16}$$

where S_k characterizes the spectral energy input from the wind (again averaging over all angles γ). If we assume that

$$S_k = (\tau_k^s)^{-1} N_k, \tag{17}$$

then using (9), we can rewrite (16) in the form

$$\tau_k^s \gg \tau_k, \tag{18}$$

where τ_k^s characterise the growth rate due only to atmospheric forcing. According to (18) the formation of an energy cascade due to the nonlinear interactions between Fourier-components of wave field take place only after their initial growth due to energy input from wind. Therefore, it seems to me that (16), or (18), is a reasonable condition for the dominance of the energy cascade over a certain range of wavenumbers $k > k(\omega_m)$. If we use the formulas (11), (13), the conditions (16), (18) can also be written in the form

$$\frac{\epsilon_0^{2/3} k^{3/2}}{g^{1/2}} \gg (\tau_k^s)^{-1}. \tag{19}$$

Any attempts to estimate τ_k^s lead us into a very speculative area of the theory of wind-wave generation. Therefore, let us assume for the moment that for large enough fetch and duration of wind, ϵ_0 does not vary with time (or fetch) significantly. That means that Kolmogoroff's type of equilibrium [(12), (15)] is time (or fetch) independent. Then the values of τ_k^s can be found from empirical growth rates of the spectral density $S(\omega)$ for given ω. Such data are usually presented in the form

$$(\tau_k^s \sigma_k)^{-1} = s\left(\frac{k^{1/2} U_a}{g^{1/2}} = \frac{U_a \omega}{g} = \frac{U_a}{c}\right),$$

where s is the dimensionless growth rate of wave components with different k, ω or phase velocity c. Then (19) can be rewritten, for example, as

$$\frac{\epsilon_0^{2/3} k}{g} \gg s\left(\frac{k^{1/2} U_a}{g^{1/2}}\right). \tag{20}$$

The compilation of the empirical data on the variability of s (see, for example, Kahma, 1981a) clearly shows that at least in the range of

$$\frac{k^{1/2}U_a}{g^{1/2}} = \frac{U_a}{c} \approx 1.0 \div 3.0$$

there are no notable variations in the values of s with k and so we take s = constant. This suggests that according to (20) with $s \approx$ constant we can always find a range of high enough wavenumbers or frequencies on the rear face of the spectrum ($\omega > \omega_m$), where the conditions (16), (18)–(20) are justified. In such a case we can simply neglect the effect of direct atmospheric forcing in our formation of the equilibrium form of the wave spectra at $\omega > \omega_m$. Of course, it is not true for all ranges of frequencies up to $\omega = \infty$, but we show later that it is not a probably unreasonable hypothesis if applied to the description of the spectral components of the wind-wave field whose phase velocities $c = g\sigma^{-1} \leqslant U_*$, where U_* is the friction velocity in the atmospheric surface layer. Evidently at frequencies $\omega U_* g^{-1} \geqslant 1$ the direct interaction with atmospheric boundary layer is very important. However, because $U_*/U_a \approx 1/30$ (where U_a is the mean wind speed), we still have broad enough range of nondimensional frequencies

$$\frac{\omega U_a}{g} = \left(\frac{\omega_m U_a}{g} < \frac{\omega U_a}{g} < 30\right),$$

where, it seems to me, the search for Kolmogoroff's type of saturation regime is well justified and the low wavenumber and frequency boundaries of the equilibrium forms of wave spectra (12), (15) can be determined *empirically*. Unfortunately, there is one important circumstance which can make the task difficult. Since the nonlinear four wave–wave interactions described by the wave kinetic equation (3, 4) conserve both energy and action [momentum conservation is nontrivial only for an anisotropic wave spectrum $F(\mathbf{k})$] one can apply Kolmogoroff's argument (8) also to the action flux. Such a situation then will have a certain analogy with two-dimensional turbulence, which conserves both energy and enstropy. If instead of $\epsilon(k)$ in (8), we introduce the action flux $\epsilon_N(k)$ defined as

$$\epsilon_N(k) = N_k k^2 \tau_k^{-1} = \text{constant} = \epsilon_N, \qquad (21)$$

then by using the scaling of the collision integral (10) we find

$$N_k \approx \epsilon_N^{1/3} g^{1/6} k^{-23/6}, \qquad (22)$$

which leads to

$$F_k \approx \epsilon_N^{1/3} g^{-1/3} k^{-10/3}. \qquad (23)$$

According to (14) this will give us the frequency spectrum $S(\omega)$

$$S(\omega) \approx \epsilon_N^{1/3} g \omega^{-11/3}, \qquad (24)$$

in striking similarity in the power law for $S(\omega)$ with Kolmogoroff's spectrum (15), based on an *energy* cascade [the subrange based on constant action flux we will not call a Kolmogoroff's type of equilibrium, leaving the latter name only for the constant energy flux spectrum (15)].

Eqs. (22)–(24), however, correspond to an idealized situation, completely different from one leading to the ω^{-4} law (15), because the energy input from the wind here must be concentrated at $\omega = k = \infty$ and the action flux ϵ_N due to nonlinear interactions is directed to $\omega = k = 0$, where supposedly some unknown low-frequency dissipation mechanism can occur. From our point of view it is much more difficult to identify a broad enough range of frequencies for this type of statistical equilibrium on the rear face of the spectra of wind-generated waves, than to find a Kolmogoroff subrange (15). First of all, it is worthwhile to mention that the conservation of wave action is valid only in the case of weak nonlinear effects involving sets of four wavenumbers and in surface gravity waves the processes which include five wave–wave interactions will not conserve wave action. Therefore the Kolmogoroff type of equilibrium, based on energy conservation is based on more general physical arguments than (21). This is especially important for the general theory of the equilibrium range in the spectrum of wind-generated waves, which we propose in the next section, because it will be demonstrated there that nonlinear effects for *high enough wavenumbers and frequencies are no longer weak* and the simple scaling (10) of the wave kinetic equation in the form (3), (4) can be irrelevant for the description of the high-frequency part of $S(\omega)$.

Second, in the framework of a constant action flux theory, it is really difficult to find a representative physical mechanism for the sink of wave action, if we neglect the effects of nonstationarity and inhomogeneity. The numerical calculations (Hasselman *et al.*, 1973) have demonstrated convincingly enough that the energy flux due to nonlinear interactions from the region of the peak toward the high frequencies plays a very important role in maintaining the relatively narrow one-peak frequency spectrum $S(\omega)$. That means that at least for fetch-limited wave growth the action flux can be basically also directed from the peak toward higher frequencies. Even though one can imagine a *very idealized* situation, when the direct energy input from the wind is concentrated somewhere in the middle of the rear face of the energy-containing part of $S(\omega)$, say at the region $\Delta\omega^s \sim \omega^s \gg \omega_m$, it is very unlikely to find two asymptotic forms of energy spectra [(24), (15)] to describe correspondingly the regions $\omega_m < \omega < \omega_s$ and $\omega > \omega_s$ in the rather narrow range of frequencies which covers the energy-containing part of the rear face of $S(\omega)$. Therefore, we concentrate our attention in this paper on those situations when the importance of the spectral energy influx from wind is certainly weighted toward the smaller values of k (or ω), and the only realistic cascade mechanism in the main part of the rear face of the spectrum $S(\omega)$ is associated with energy flux ϵ_0 toward higher frequencies. Then an interesting and a very important question is what are the typical *cut-off* wavenumbers and frequencies for Kolmogoroff's subrange (12), (15) in the spectra of wind waves. If we disregard viscous effects then the primary mechanism for "dissipation" of wave energy is gravitational instability, which ultimately leads to wave breaking. A characteristic property of such breaking is the occurrence of fairly sharp wave crests with intermittent patches of foaming, which seem to develop when the crests no longer maintain their attachment to the remainder of the water (Phillips, 1958). Therefore, instead of kinematic viscosity in Kolmogoroff's three-dimensional turbulence, the governing parameter describing the "dissipation" of wave energy in "weak turbulence" due to nonlinear resonant wave-interactions in two-dimensional wavenumber space must be gravity g. The process of breaking takes place according to Phillips (1958) when downward acceleration of particles can reach (or exceed) g. The typical acceleration $a_k = [\ddot{\zeta}_k dk]^{1/2}$ associated with an energy transfer rate ϵ_0 for a given range of wavenumbers $\Delta k \approx k$ can be estimated as

$$a_k = a_k(k_1\epsilon_0) \sim \epsilon_0^{2/3}k \qquad (25)$$

and the condition of "breaking" can be written as

$$\frac{a_k}{g} \sim \frac{\epsilon_0^{2/3}k}{g} \approx \text{constant} \approx 1. \qquad (26)$$

It follows from (25), (26) that cut-off wavenumber k_g for Kolmogoroff's subrange in wave field is

$$k_g \sim \frac{g}{\epsilon_0^{2/3}}. \qquad (27)$$

Thus the analog of Kolmogoroff's internal scale in three-dimensional turbulence for weakly nonlinear surface gravity waves will be

$$l_g \sim \frac{\epsilon_0^{2/3}}{g}. \qquad (28)$$

The corresponding cutoff frequency ω_g for deep-water gravity waves is

$$\omega_g = (gk_g)^{1/2} \sim \frac{g}{\epsilon_0^{1/3}}, \qquad (29)$$

and we can expect (12), (15) to be valid for wavenumbers and frequencies below k_g and ω_g, respectively, and above k_m and ω_m [where $k_m = \omega_m^2/g$, ω_m being the main peak frequency of the spectra $S(\omega)$]. The exact form of the inequalities $k_m < k < k_g$ as well as the proportionality constant in (25)–(29) can be found only empirically. If we assume that gravitational instability (wave breaking) is the only process modifying Kolmogoroff's cascade of energy in a field of nonlinear surface gravity waves we can now formulate a general similarity theory for equilibrium range in the spatial statistical characteristics of wind-generated gravity waves.

3. The general similarity hypotheses for equilibrium range in the spatial statistical characteristics of a wind-wave field

It is important to notice that among three statistical characteristics of a wave field, i.e., energy, momentum and action density, only the latter is not necessarily conserved in the process of nonlinear wave–wave interactions in dispersive wave motion. To underline this fact, and to emphasize that all statistical characteristics of a wave field (not just energy spectrum) are determined by energy flux ϵ_0 (but not by ϵ_N), we will apply the similarity arguments first of all to action density N_k. According to the discussion in the previous section we can now formulate the first similarity hypothesis as follows: if, in wavenumber space, we restrict our attention to wavenumbers well below those associated with capillary ripples and those directly influenced by viscosity so that

$$k \ll k_T = (\rho_\omega g T^{-1})^{1/2}, \quad k \ll k_\nu = g^{1/4}\nu^{-1/2}, \qquad (30)$$

where T-surface tension, ρ_ω-density and ν-kinematic viscosity are well above those associated with strong direct energy input from wind ($k \sim k_s > k_m$) so that

$$k \gg k_m = \omega_m^2 g^{-1}, \qquad (31)$$

then for sufficiently large fetch and duration of the wind the values of N_k are determined only by the values of parameters ϵ_0, g and k.

It follows from this hypothesis that

$$N_k = \epsilon_0^{1/3}k^{-4}\psi(k\lambda_g), \quad \lambda_g = \epsilon_0^{2/3}g^{-1}, \qquad (32)$$

where ψ is a nondimensional function of nondimensional wavenumber $k = k\lambda_g = k\epsilon_0^{2/3}g^{-1}$.

The second similarity hypothesis can be formulated for range of wavenumbers satisfying the inequality $k\lambda_g \ll 1$. In this range of scales, if it exists and is broad enough, gravitational instability is not important because wave components do not reach critical amplitudes to produce breaking, and N_k depends only on ϵ_0 and k, so that

$$\begin{cases} \psi(k\lambda_g) = \text{constant} = A & (33) \\ N(k) = A\epsilon_0^{1/3}k^{-4} & (34) \end{cases}, \quad \text{for} \quad k\lambda_g \ll 1,$$

where A is an absolute constant. *By analogy with Kolmogoroff's theory of locally isotropic turbulence we can expect here that constant A must be close to unity.* It is easy to see that according to (7), (14) this corresponds to the following expressions for wave spectra:

$$F_k = A\epsilon_0^{1/3}g^{-1/2}k^{-3.5}, \qquad (35)$$

$$S(\omega) = \alpha\epsilon_0^{1/3}g\omega^{-4}; \quad \alpha = 2A. \qquad (36)$$

Finally, the third similarity hypothesis can be formulated following Phillips (1958) theory, according to which we can assume that for the range of wavenumbers $k\lambda_g \gg 1$ the governing parameters are those that determine the continuity of the wave surface, and therefore asymptotically N_k becomes independent of ϵ_0 and will depend only on g and k, so that

$$\psi(k\lambda_g) = B(k\lambda_g)^{-1/2} \quad \text{for} \quad \left.\begin{matrix} k_\tau\lambda_g \\ k_\nu\lambda_g \end{matrix}\right\} \gg k\lambda_g \gg 1, \tag{37}$$

$$N_k = Bg^{1/2}k^{-9/2}, \tag{38}$$

where B is an absolute constant. It follows from (7), (14) that this corresponds to the expressions for wave spectrum, first suggested by Phillips (1958):

$$F_k = Bk^{-4}, \tag{39}$$

$$S(\omega) = \beta g^2\omega^{-5}, \quad \beta = 2B, \tag{40}$$

where β is called Phillips' constant.

There are a few remarks that should be made at this point. The first is that independence of N_k, F_k and $S(\omega)$ from ϵ_0 means that the geometry of the limiting wave configuration near the sharp crests is determined by the condition (26). An increase in ϵ_0 would have the effect of increasing the rate at which wave crests are passing through the transient limiting configuration, but should not influence the geometry of the sharp crests itself (Phillips, 1958). According to (27) it simply means that the increase in ϵ_0 will lead to growth of longer wave components up to this limiting configuration, so that in wavenumber space the boundary of Phillips' equilibrium range will move toward lower wavenumbers with increase of ϵ_0.

The other remark is that according to our third similarity hypothesis, we consider the asymptotic situation, corresponding to indefinitely large values of $k\lambda_g$ (indefinitely large values of k or indefinitely large values of ϵ_0), where the statistical characteristics of the wave field are determined solely by the process of wave breaking. Therefore the magnitude of the spectrum in Phillips' *subrange* represents an upper limit of F_k, dictated by the requirement of crest attachment. Generally we cannot in principle disregard the possibility (because of the very nature of asymptotic arguments) that for $\epsilon_0 \to \infty$ ($\lambda_g \to \infty$, $k\lambda_g \to \infty$) the values of N_k continue to depend, no matter how slightly, on ϵ_0, so that instead of (37) we have

$$N_k = A\epsilon_0^{1/3}k^{-4}(k\lambda_g)^{-p}, \tag{41}$$

where $\frac{1}{2} > p > 0$ is some small power exponent. However it seems that the assumption of independence of N_k from ϵ_0, which leads to Phillips' spectra (39), (40) is the most elegant and natural first-order approximation to describe asymptotic form of wind-wave spectra in the range of scales where breaking is primarily important in limiting the growth of wave components. In connection with the derivation of Phillips' laws (39), (40) it is probably useful to mention another possible interpretation for the transition from Kolmogoroff's to Phillips' subrange. The concept of weak turbulence when nonlinear wave–wave interactions in a random wave field are so weak that they do not influence the dispersive properties of waves, is fundamental in the derivation of the wave kinetic equation (4). However it is valid only when the ratio of the characteristic time of nonlinear interactions τ_k to the wave period $\tau \sim \sigma_k^{-1}$ is much larger than 1, or the quantity $\mu = (\sigma_k\tau_k)^{-1} \ll 1$. We can now estimate μ in Kolmogoroff's subrange using; (11), (12); the result is

$$\mu = (\sigma_k\tau_k)^{-1} = N_k^2k^{9.0}g^{-1} \sim \epsilon_0^{2/3}kg^{-1}. \tag{42}$$

It is now clear from (42) that there must exist wavenumber $k \sim k_\mu$ for which interactions are so fast, that $\mu > 1$ and the Kolmogoroff concept, based on scaling the collision integral (10), can no longer be valid. Naturally, criteria for the applicability of weak turbulence model (4), (10) for the description of Kolmogoroff subrange based on a critical value of $\mu < \mu_0 \sim$ constant ≈ 1 coincide with the definition of the internal scale for Kolmogoroff's subrange, given earlier in terms of the critical acceleration (26). It is interesting to notice that if we substitute (39) into (11), Phillips' equilibrium form of the spectra (39) leads to

$$\mu \approx \text{constant} \approx B, \tag{43}$$

and with the empirical value of Phillips' constant $B \approx 10^{-2}$ gives value of μ probably much smaller than the values of μ in the Kolmogoroff subrange (42). This means that the first (in time history) to develop due to nonlinear interactions must be Kolmogoroff's cascade mechanism, leading to formation of the inertial subrange (34), (35) and (36). The transition from this subrange to Phillips' asymototic regime, as well as structure of Phillips' subrange by itself, *cannot be described theoretically* on the basis of the wave kinetic equation (4), based solely on resonance four-wave interaction mechanisms. Therefore, to discover the very existence of Kolmogoroff's subrange as well as the transition between Kolmogoroff's and Phillips' subranges, we must look into experimental data. Fortunately, recent observational studies give enough material to justify such a search.

4. Experimental evidence for the transition from Kolmogoroff's to Phillips' equilibrium in the spectra of wind-generated deep water surface gravity waves

It was shown very recently (Forristall, 1981; Kahma, 1981a; Doneland *et al.*, 1982) that in the *energy-con-*

taining region of the frequency spectra $S(\omega)$, i.e., where spectral levels are greater than say 1% of the peak, the rear face of the spectrum is well described by the ω^{-4} power law. That, of course, can be considered as an indication of the existence of Kolmogoroff's subrange (15), (36) in the observed statistical characteristics of a wind-wave field. However if we want to use such data for determination of the universal form of the spectra (32), or the universal constant $A(\alpha)$ in (33)–(36), we must determine the energy flux ϵ_0, or relate it in some way or other to such measurable external parameters as wind speed, fetch and duration. If the range of frequencies from which energy is extracted from the wind is distinct from the range over which energy is lost, and if the internal (viscous) energy dissipation is negligible in the wave field, then ϵ_0 represents the difference between the rate of gain of energy from wind (W_a) and the rate of increase of energy of those components of the wave system that are still developing and have not yet attained a statistical equilibrium state [see (13), (26)]. In the absence of direct measurements of the atmospheric input we can only use the *general energy balance* to estimate ϵ_0, i.e., we assume that

$$\rho_w \epsilon_0 \approx W_a, \tag{44}$$

where W_a characterizes the energy flux per unit area from wind to waves. Because W_a is proportional to $\rho_a U_a^3$ we can also write

$$\epsilon_0 = m \frac{\rho_a}{\rho_w} U_a^3, \tag{45}$$

where m is an *a priori* unknown nondimensional coefficient in which the ratio $W_a/\rho_a U^3$ has been incorporated, as well as a proportionality factor in (44).

It follows from (45) that the Kolmogoroff type of equilibrium in the spectrum of wind-generated waves according to (35), (36) corresponds to a *wind-dependent saturation* of the form

$$F_k = A_u g^{-1/2} U_a k^{-3.5}, \tag{46}$$

$$S(\omega) = \alpha_u g U_a \omega^{-4}; \quad \alpha_u = 2A_u, \tag{47}$$

where A_u, α_u are nondimensional coefficients into which the absolute constant A, the coefficient m and the constant ratio ρ_a/ρ_w are incorporated according to

$$\alpha_u = 2A\left(\frac{\rho_a}{\rho_w} m\right)^{1/3}. \tag{48}$$

If in accordance with the similarity hypothesis (Kitaigorodskii, 1962, 1981), the spectra of wind waves in various stages of their development are assumed to have a similar form, we can write the expression for $S(\omega)$ as

$$\tilde{S}(\omega) = F(\tilde{\omega}, \tilde{x}, \tilde{t}), \tag{49}$$

where

$$\left.\begin{aligned} \tilde{S}(\omega) &= \frac{S(\omega) g^3}{U_a^5}, \quad \tilde{\omega} = \frac{\omega U_a}{g} \\ \tilde{x} &= \frac{gx}{U_a^2}, \quad \tilde{t} = \frac{gt}{U_a} \end{aligned}\right\}. \tag{50}$$

Within the framework of similarity theory [(49), (50)] the existence of an equilibrium range in the frequency spectrum of wind-generated waves means, for sufficiently large fetch and duration, that there are ranges of frequencies $\tilde{\omega} > \tilde{\omega}_m(\tilde{x}, \tilde{t})$ where the universal function F of nondimensional frequency $\tilde{\omega}$ must satisfy the conditions

$$F = \psi_1(\tilde{\omega}) = \alpha_u \tilde{\omega}^{-4}, \quad \tilde{\omega}_m < \tilde{\omega} \ll \tilde{\omega}_g, \tag{51}$$

$$F = \psi_2(\tilde{\omega}) = \beta \tilde{\omega}^{-5}, \quad \tilde{\omega} \gg \tilde{\omega}_g. \tag{52}$$

Here β is a universal constant and α_u, according to (48), can depend on $\tilde{X}$ (or $\tilde{t}$) only because $\rho_w \epsilon_0/\rho_a U^3 = m$ can be variable.

It is important to underline the fact that according to (52) the Phillips constant β can be accurately determined *only with the data in the range of frequencies* $\omega \gg \omega_g$. If the transition in frequency spectra between Kolmogoroff's (51) and Phillips' (52) equilibrium is very sharp in the frequency domain, we can determine the transitional frequency $\tilde{\omega}_g$ using (51), (52), i.e., assuming

$$\psi_1(\tilde{\omega}_g) \approx \psi_2(\tilde{\omega}_g), \tag{53}$$

which leads to

$$\tilde{\omega}_g \approx \frac{\beta}{\alpha_u}. \tag{54}$$

To illustrate the existence of two asymptotic regimes [(51), (52)] and the sharp transition between them [(53), (54)], we analyzed the data collected recently by Kahma (1981b). These data were based on waverider buoy measurements in Bothnian Bay and include the values of $S(\omega)$ in the narrow and fixed low-frequency interval between 0.35 and 0.5 Hz. To avoid the bias due to the over-shooting effect at the peak frequency only spectra $S(\omega)$ in which peak frequency ω_m was below 0.25 Hz were considered. The range of variation of 10 min average wind speed U_a for those data, was from 3–18 m s^{-1}. This new set of data was combined with previously observed spectra $S(\omega)$ under exceptionally steady wind conditions (Kahma, 1981a). Fig. 1 shows the experimentally determined values of the universal function $F(\tilde{\omega}, \tilde{x}, \tilde{t})$ [see (49)], presented in the form $\tilde{S}(\tilde{\omega})\tilde{\omega}^5 = F(\tilde{\omega})\tilde{\omega}^5$ versus $\tilde{\omega}$ and wind speed. The linear variation of $\tilde{S}(\omega)\tilde{\omega}^5$ with $\tilde{\omega}$ or with wind speed is evident up to the values $\tilde{\omega} \approx 4.0$, indicating that for $\tilde{\omega} < 4.0$ $F(\tilde{\omega}) = \psi_1(\tilde{\omega}) \sim \omega^{-4}$. When the average value of $F(\tilde{\omega})\tilde{\omega}^5$ was calculated the corresponding curve in Fig. 2 very closely follows a straight line, predicted

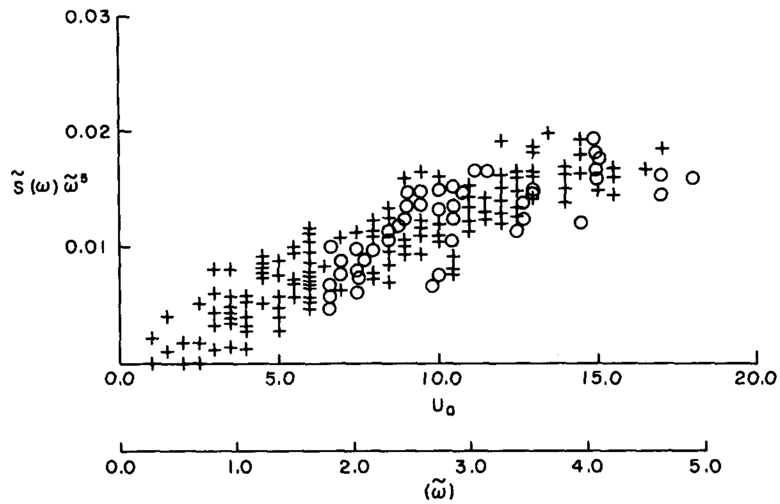

FIG. 1. The nondimensional function $\tilde{S}(\omega)\tilde{\omega}^5$ [Eqs. (49), (50)] according to observations (Kahma, 1981a,b). The plus marks show the values of $\tilde{S}(\omega)\tilde{\omega}^5$ determined from a fixed frequency interval 0.35–0.5 Hz as a function of wind speed U_a. The scale below ($\tilde{\omega}$) shows the dimensionless angular frequency $\tilde{\omega} = \omega U_a g^{-1}$ which corresponds to the frequency 0.425 Hz in the middle of this interval. The open circles correspond to the data reported by Kahma (1981a).

by Eq. (51) up to a wind speed of 14 m s^{-1} or in terms of nondimensional frequency $\omega U_a g^{-1}$ up to ~4.0. The important feature of Figs. 1 and 2 is the noticeable leveling of the function $F(\tilde{\omega})\tilde{\omega}^5$ at frequencies $\tilde{\omega} > 4.0$. It demonstrates that the behavior of $F(\tilde{\omega})$ here satisfies the relationship (52), when

$$F(\tilde{\omega})\tilde{\omega}^5 = \text{constant} = \beta. \quad (55)$$

It is important to point out that it follows from Figs. 1 and 2 that the constant value of $F(\tilde{\omega})\tilde{\omega}^5 = \beta$ for $\tilde{\omega} > 4.0$ is close to 1.5×10^{-2}. Such a value of Phillips constant is in agreement with the well known determination of β, based on experiments of Burling (1959) and Mitsuyasu (1977). One of the reasons for the variability of β reported by many authors (see for example, Hasselman *et al.*, 1973) is certainly due to the fact that the $\tilde{\omega}^{-5}$ law was applied to the region where $S(\omega) \sim \omega^{-4}$, i.e., for the range of frequencies $\tilde{\omega} < \tilde{\omega}_g$. Therefore among all the reported values of β only a few from our point of view can be trusted as really adequate for the description of the asymptotic Phillips law (52) in the range of frequencies $\tilde{\omega} > \tilde{\omega}_g$. One of the consequences of approximating the whole equilibrium range (51), (52) by a ω^{-5} law is the necessity to describe high-frequency portions of the spectra $S(\tilde{\omega})$ with a variable β; this was first suggested

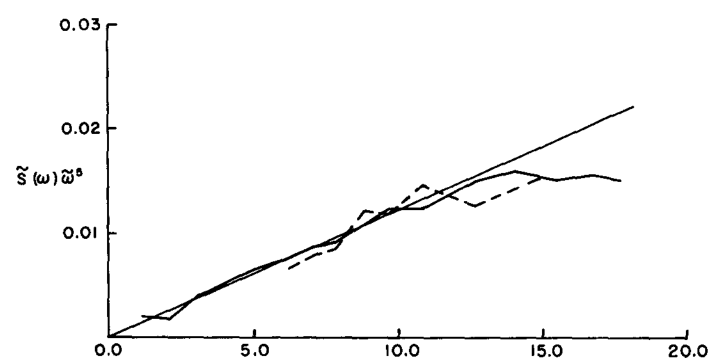

FIG. 2. The nondimensional function $\tilde{S}(\omega)\tilde{\omega}^5$ [Eqs. (49), (50)]. The irregular solid line is the average of the observations in Fig. 1. The straight solid line is calculated from Eq. (51) with $\alpha_u = 4.5 \times 10^{-3}$. The dashed line represents the observations in the exceptionally steady-state wind conditions reported by Kahma (1981a).

in the JONSWAP experiment (Hasselmann *et al.*, 1973) using the form $\beta = \beta(\tilde{x})$. However, it is possible to see that if in Kolmogoroff's subrange $\omega_m < \omega < \omega_g$, we determine the values of β, they will not necessarily show variation with fetch. To prove this one can calculate the values $\tilde{S}(\omega)\tilde{\omega}^5$ by using the data on $S(\omega)$ in the region $\omega < \omega_g$ (in the inertial subrange) at fixed frequencies over a very narrow range of wind speeds. Then if a relationship $\beta = \tilde{\beta}(x)$ was correct we would expect the ratio $S(\omega_i)/g^2\omega_i^{-5} = \beta_i$ to be fetch dependent (for constant wind). The experimentally derived values $S(\omega_i)/g^2\omega_i^{-5} = \beta_i$ for fixed frequency in the range of frequencies $\tilde{\omega} < 4.0$ for the data in Fig. 1 is shown in Fig. 3 for a very narrow range of wind speeds $U_a \approx 7$–9 m s^{-1}. There is no variation of β with fetch x, even though the values of $\beta \approx 1 \times 10^{-2}$ are different from those derived from the range of frequencies $\tilde{\omega} > \tilde{\omega}_g$ (where $\beta \approx 1.5 \times 10^{-2}$).

According to Figs. 1 and 2 the value of α_u in Kolmogoroff's spectra (51) is found to be 4.4×10^{-3}, which is practically equal to the values of α_u reported by Kahma (1981a) in his previous experiments, and also very close to the value of α_u calculated by Forristall (1981), where α_u was 4.5×10^{-3}. The results reported in Kahma (1981a,b) and Forristall (1981), as well as the data presented in Figs. 1–3 give, from our point of view, strong support to the idea that variability of high-frequency parts of the wave spectrum can be explained by *equilibrium range theory*, which includes the *existence of both an inertial subrange where* $\tilde{S}(\omega) = \alpha_u\tilde{\omega}^{-4}$ and *Phillips' subrange, where* $\tilde{S}(\omega) = \beta\tilde{\omega}^{-5}$. However, the existence of a relatively sharp transition between the two at the transitional frequency $\tilde{\omega}_g$ is still not well documented [according to Forristall (1981) and Kahma (1981a,b)] $\tilde{\omega}_g = \omega_g U_a g^{-1} \approx 4.0$–$5.0$). Very recently Donelan *et al.* (1982) also demonstrated that in the "energy-containing" region of the spectrum $S(\omega)$, i.e., for spectral levels greater than 1% of the peak, the rear face of the spectrum $S(\omega)$ is well described by ω^{-4} power law. As an illustration of this we present their results in Fig. 4, where the spectra $S(\omega)$ have been multiplied by ω^{-4} and normalized by the *average* level of the spectral estimates multiplied by ω^4 in the frequency region $1.5\omega_m < \omega > 3.0\omega_m$. It is clear that an ω^{-4} power law is a good description of the rear face of the spectrum in the energy-containing region (the comparison with ω^{-5} and ω^{-3} power laws are also shown in Fig. 4). These data cover wide ranges of the ratio $U_a/C_m = U_a\omega_m/g = \tilde{\omega}_m$ based on both laboratory and field experiments. It can be seen from Fig. 4 that for the highest frequency $\omega \approx 3\omega_m$ the field data correspond to the range of nondimensional frequencies $\tilde{\omega} = \omega U_a/g \approx 3.0 \div 15.0$ and we still don't observe the transition to the ω^{-5} region. However it must not be forgotten that the establishment of a power law appropriate to the rear face of the wave spectrum is often troubled by the possibility of Doppler shifting of the spectral estimates by currents and longwave components (Kitaigorodskii *et al.*, 1975). According to the theory presented in Section 3, there must be a transition from $k^{-3.5}$ to k^{-4} region in wave-number spectrum F_k. If such a transition really exists at high enough wavenumbers the behavior of F_k is k^{-4} rather than $k^{-3.5}$. *Without Doppler shifting* by the wind drift current (in the direction of the mean wind), we must expect for high enough values of ω, a frequency spectrum $S(\omega) \sim \omega^{-5}$. The spectra in Fig. 4 do not indicate the existence of the ω^{-5} region, which possibly means that due to Doppler shifting we will have a tendency for smaller (than ω^{-5}) slope of the rear face of $S(\omega)$, even though we are already in the region in wave-

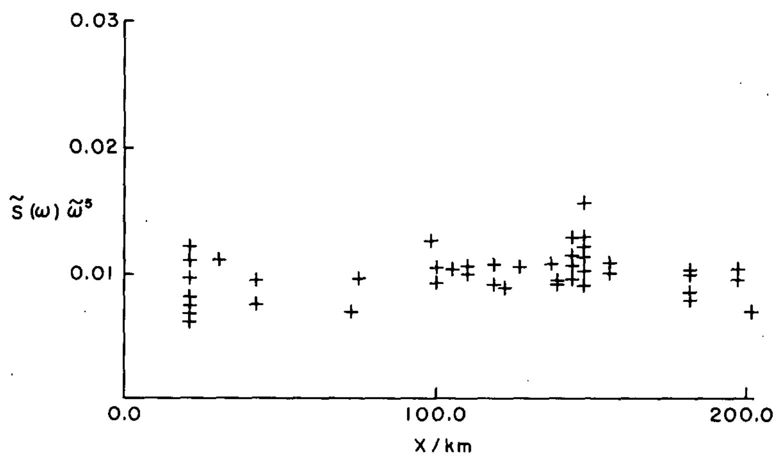

FIG. 3. The dependence of nondimensional function $\tilde{S}(\omega)\tilde{\omega}^5$ on the fetch x for fixed frequency $\tilde{\omega}_i < 4.0$ in a narrow range of wind speeds U_a (between 7 and 9 m s^{-1}).

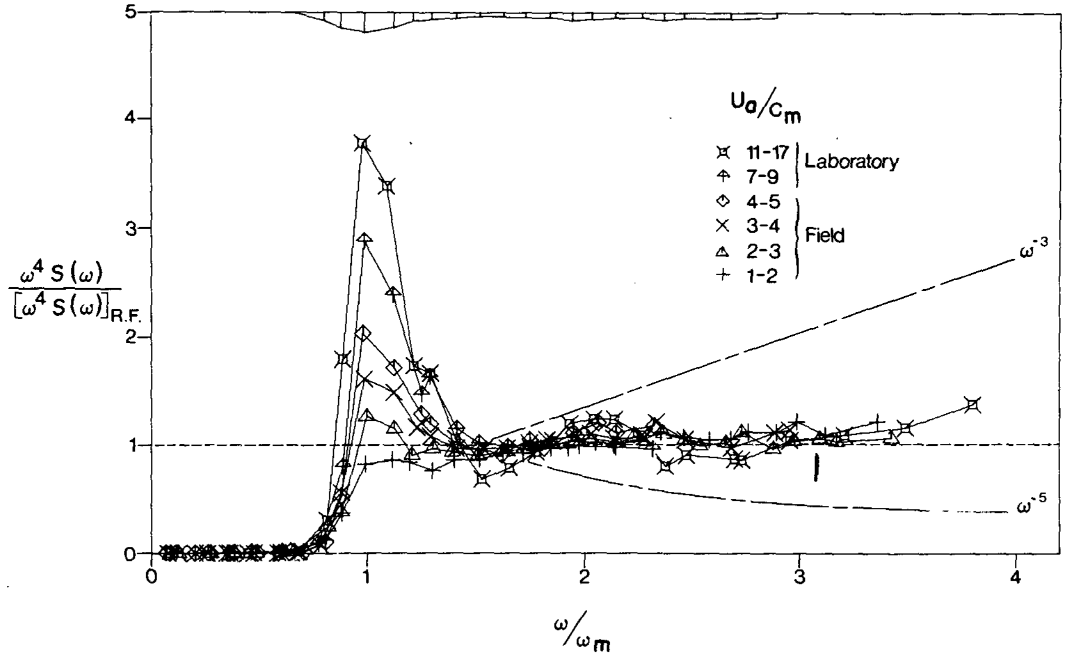

FIG. 4. Frequency spectra $\times\ \omega^4$ and normalized by the rear face $[\omega^4 S(\omega)]_{rf}$ which is the average of $\omega^4 S(\omega)$ in the region $1.5\ \omega_m < \omega < 3\omega_m$. The lines corresponding to ω^{-5} and ω^{-3} are also shown (from Donelan *et al.*, 1982). At the top of the figure the 95% confidence limits in each band of width $(\omega/10\omega_m)$ are indicated. The position of the vertical bar indicates the average value of (ω/ω_m) in each band. The spectra have been grouped into classes by the parameter U_a/C_m ($C_m = g/\omega_m$).

number space where the transition from $k^{-3.5}$ to k^{-4} had occurred. This is, of course, a very hypothetical explanation of the data presented in Fig. 4.

To avoid the problems related to kinematic effects due to Doppler shifting, it will be highly desirable to observe the transitional regime from the Kolmogoroff to the Phillips type of equilibrium in wavenumber spectrum F_k. The only data which I was able to find to demonstrate such a transition in wavenumber spectrum F_k was from Stereo-Wave Observation Project (reproduced in Phillips, 1977). The data are repeated in Fig. 5 because of their unique illustration of the very existence of the transition from the $k^{-3.5}$ to the k^{-4} law in the high wavenumber part of the spectra F_k. The value of peak wavenumber k_m in this case is $\sim 0.07 m^{-1}$ and it follows from this picture that transitions occur approximately at $k/k_m \approx 4.0$, which roughly corresponds to $\omega/\omega_m \approx 2.0$. This is not unreasonable compared with the Kahma (1981a,b) and Forristall (1981) results. Therefore, we can expect that in the region $\omega/\omega_m > 2.0$ Donelan's data (Fig. 4) were influenced by Doppler shifting (at least for the field data).

The parametrization of $S(\omega)$ proposed in Donelan *et al.* (1982) lead to the asymptotic form of high-frequency part of $S(\omega)$:

$$S(\omega) = \alpha_m g^2 \omega^{-4} \omega_m^{-1}, \quad (56)$$

where the nondimensional coefficient α_m was found to be variable with non-dimensional fetch. The empirical formula for α_m suggested in Donelan *et al.* (1982) is roughly

$$\alpha_m = 0.006\left(\frac{U_a}{C_m}\right)^{1/2}. \quad (57)$$

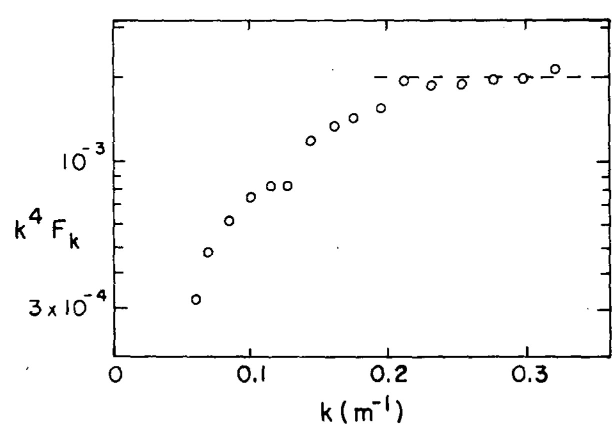

FIG. 5. The equilibrium range in the integrated wavenumber spectrum $F_k \times k^4$. Data from the Stereo-Wave Observation Project (reproduced from Phillips, 1977). The asymptotic behaviour of F_k for $k < 0.2\ m^{-1}$ and $k > 0.2\ m^{-1}$ roughly corresponds to $k^{-3.5}$ and k^{-4} power law (the value of k_m is approximately 0.07 m^{-1} in this case).

Using the linear dispersion relationship $C_m = g/\omega_m$, Eq. (56) is equivalent to (47) with

$$\alpha_u = 6 \times 10^{-3}\left(\frac{U_a}{C_m}\right)^{-1/2}. \quad (58)$$

For the field data in Fig. 4 with $U_a/C_m \approx$ 1–5 we determine an average value of $\alpha_u \approx 4.3 \times 10^{-3}$ again very close to the values of α_u reported by Forristall (1981) and Kahma (1981a,b). Therefore for very crude estimates of universal constants in an equilibrium range theory (Section 3), we can consider the coefficient α_u to be fetch independent with characteristic values $(4.3\text{–}4.5) \times 10^{-3}$.

5. Preliminary estimates of the values of universal nondimensional constants in the equilibrium range theory

According to the data analysis presented in the previous sections we now consider for simplicity the nondimensional coefficient α_u in Kolmogoroff's spectrum (47) as an *empirical constant*. This permits us to make attempts to determine (very crudely) the value of the *universal* constant $A(\alpha)$ in the inertial (Kolmogoroff's) subrange (35), (36). To do this we write the expression for W_a in (44) in the form

$$W_a = \gamma_a \tau_a \bar{c} = \gamma_a \rho_a C_f U_a^2 \bar{c}, \quad (59)$$

where $\bar{c} = g\bar{\sigma}^{-1}$ is the mean phase speed of the waves and $\bar{\sigma}$ is defined as

$$\bar{\sigma}^2 \int_0^\infty S(\omega)d\omega = \int_0^\infty \omega^2 S(\omega)d\omega. \quad (60)$$

The product $\gamma_a \tau_a$ in (59) characterizes the fraction of the momentum flux τ_a in atmosphere which goes to waves, and C_f in (59) is the drag coefficient. Comparing (59) with (44) and (45) we can find the expression for the nondimensional coefficient m in (45) as

$$m = \gamma_a C_f \frac{\bar{c}}{U_a}. \quad (61)$$

In general we must take into account the fact that both γ_a and C_f can vary with $\bar{c}U_a^{-1}$ (see, for example, Benilov *et al.*, 1978a,b) so that

$$m = \gamma_a\left(\frac{\bar{c}}{U_a}\right)C_f\left(\frac{\bar{c}}{U_a}\right)\frac{\bar{c}}{U_a}. \quad (62)$$

Taking $\gamma_a = 0.1\text{–}1.0$, $C_f \approx (1.0\text{–}1.3) \times 10^{-3}$ and $\bar{c}U_a^{-1} \approx 0.7\text{–}0.9$, we can reasonably well estimate the range of variations of m in (61) for active wind-wave generation conditions. We have

$$m = (0.07\text{–}1.17) \times 10^{-3}. \quad (63)$$

From expression (48) with $\rho_a/\rho_\omega \approx 1.2 \times 10^{-3}$, we have

$$A = \frac{5\alpha_u}{m^{1/3}}. \quad (64)$$

Experimentally derived average value of $\alpha_u \approx 4.5 \times 10^{-3}$ (Kahma, 1981b; Forristall, 1981; Donelan *et al.*, 1982) and the above values of m yield

$$A \approx 0.55\text{–}0.22. \quad (65)$$

Therefore, we expect the universal constant A in Kolmogoroff's spectrum (35), (36) to be of order 1 under any circumstance. This gives additional confirmation that in our similarity theory in Section 3 the choice of governing parameters was made correctly. It is much more difficult to make a reliable estimate for transitional wavenumber k_g or frequency ω_g. The expressions (27), (29) suggested by similarity theory in Section 3 include an *a priori* unknown numerical coefficient. We can try to find its approximate value on the basis of the following very crude estimate. If we consider the transitional regime in wavenumber space $\Delta k \sim k \approx dg$, where wave-breaking processes start to be important, the wave energy per unit area in this region $E_{\Delta k}$ is equal to $E_{\Delta k} = \frac{1}{2}\rho_w g a_{\Delta k}^2$, where $a_{\Delta k}$ is a mean square amplitude of waves in the region Δk. On the other hand, due to the continuous energy transfer rate ϵ_0 through the wave spectrum, $E_{\Delta k}$ can also be estimated as $E_{\Delta k} \approx \rho_\omega \epsilon_0 \tau_k$, where τ_k as before is the characteristic time of nonlinear interactions. However, in transitional regions where breaking processes are already very important, such interactions are very fast and according to (32), (33) we can use as an order of magnitude estimates for τ_k the relationship $\tau_k \simeq \sigma_k^{-1}$ (in the region $\Delta k \sim k_g$). Therefore we must have

$$\tfrac{1}{2} g a_{\Delta k}^2 \approx \epsilon_0 \sigma_k^{-1} \quad \text{at} \quad \Delta k \sim k \sim k_g. \quad (66)$$

For $\sigma_k = (gk)^{1/2}$, Eq. (66) can be rewritten as

$$k_g \approx \tfrac{1}{2}(a_{\Delta k} kg)^{4/3} g \epsilon_0^{2/3}, \quad (67)$$

and with $a_{\Delta k} k_g = \text{constant} \approx 10^{-1}$, we obtain

$$kg \approx 2.3 \times 10^{-2} \frac{g}{\epsilon_0^{2/3}}. \quad (68)$$

We now estimate the characteristic values of nondimensional transitional frequency $\tilde{\omega}_g = \omega_g U_a g^{-1}$ using Eq. (45) for ϵ_0 and the linear dispersion relationship $\omega_g = (gk_g)^{1/2}$. Using (68) with $\rho_a/\rho_w \approx 10^{-3}$ and expression (64) then yields

$$\tilde{\omega}_g \approx \frac{1.5}{m^{1/3}} = \frac{0.3A}{\alpha_u}. \quad (69)$$

Using the values $\alpha_u = 4.5 \times 10^{-3}$ and $A \approx 0.55$–0.22 [see (65)] then yields

$$\tilde{\omega}_g \approx 3.3\text{–}1.5. \tag{70}$$

These values of $\tilde{\omega}_g$ differ from those observed by Kahma (1981b) and Forristall (1981) only by a factor of 2. This suggests that the characteristic value of transitional wavenumber k_g [(27)] probably must include, according to (68), a proportionality factor of order 10^{-2}. However, we conclude that the accurate quantitative description of the transition between Kolmogoroff and Phillips type of equilibrium in the statistical characteristics of wind waves needs careful further investigation.

Acknowledgments. Comments on the first draft of the manuscript by Professors O. Phillips, K. Hasselmann and M. Longuet-Higgins are greatly appreciated.

This work was supported in part by the NASA-Ames Research Center under Grant MSG-2382 and in part by the Office of Naval Research, Physical Oceanography Program (Code 481).

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