A note on Bragg scattering of surface waves by sinusoidal bars

James T. Kirby
Center for Applied Coastal Research, Department of Civil Engineering, University of Delaware, Newark, Delaware 19716

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The reflection of linear surface waves by sinusoidal bottom undulations is considered in the case where the incident wave is not necessarily close to the resonant frequency. For finite detuning away from the resonant frequency, two previous solutions are shown to give results which are inconsistent with direct numerical solutions, especially when the results are extended to oblique incidence. The correction to the methods is given, and various consequences of the new results are examined.

I. INTRODUCTION

In recent years, the problem of the reflection of surface waves by undular bottom features has drawn considerable attention, owing primarily to the mechanism's possible relation to coastal morphology. Kirby[1] has derived an extension to the usual mild-slope equation[2] which provides a theoretical umbrella for most of the existing analytical results. Davies and Heathershaw[3] solved for the reflection coefficient for the case of weak reflection (far from resonance) and sinusoidal bars in otherwise constant depth. Their solution was recovered from Kirby's equation by Kirby and Anton,[4] who also provided the extension to the case of oblique incidence over the bar field. Close to the resonant condition, where the bar wave number is twice the wave-number component of the surface wave in the direction normal to bar crests, Mei[5] provided a solution using a multiple scales approach. His solution was extended to oblique incidence by Dalrymple and Kirby[6] and Mei *et al.*[7] (hereafter referred to as MHN). Kirby[1] showed that Mei's analysis could be obtained from the extended mild-slope formulation. Further explication of the problem (along with some additional solution techniques) may be found in Davies *et al.*,[8] Kirby,[9] and Benjamin *et al.*[10] As in these papers, we restrict our attention here to the case of perfectly sinusoidal bars.

The resonant reflection solution of Mei proceeds by assuming that the waves are described by carrier waves which are exactly in resonance with the bar field, together with amplitudes which vary slowly in both space and time, to account for both frequency-wave-number detuning and slow evolution over the bar field. The analysis results in a set of perfectly tuned, coupled evolution equations for the wave amplitudes; these may be solved for the reflection coefficient and for the wave heights over the bar in the case of time-periodic motions.

For the case of finite, arbitrary detuning, it is natural to extend the approach given by Mei to consider the frequency and wave-number detuning as part of the carrier wave; as a result, the coupled amplitude equations become explicitly detuned, but may still be easily solved for the case of periodic waves. Liu[11] and Yoon and Liu[12] (referred to as YL) took this approach in two related studies. Liu considered the resonant reflection of linear monochromatic waves in a channel with corrugated sides and bottom, while YL considered the scattering of incident cnoidal waves by a sinusoidal bar field in the context of Boussinesq theory. (The latter problem has also been addressed by Kirby and Vengayil.[13]) For the case of normal incidence on the bar field, Liu's solution gave "cutoff" conditions (demarcating the boundary between exponential and sinusoidal behavior of the incident and reflected wave envelopes) which differed markedly from the values given by Mei.[5] These differences were attributed to the effect of finite detuning.

In this paper, we formulate the problem of detuned scattering by a field of sinusoidal bars, and obtain a solution with different properties than would be found using the methods in Liu and YL. In Sec. II, a new formulation of the detuned-interaction theory is developed. This is followed in Sec. III by a discussion of previous results and of their breakdown in the oblique incidence case. The methods of Liu and YL are extended to the oblique incidence case in the Appendix. In Sec. IV, numerical results are used to establish the validity of the present formulation. We also show that the resonant interaction theory of Mei[5] is reasonably robust for the entire range of physically relevant cases. The effect of laminar bottom boundary-layer damping on the reflection process is briefly discussed in Sec. V.

II. SOLUTION OF THE GENERALIZED MILD-SLOPE EQUATION FOR DETUNED RESONANCE

In this section, we present a perturbation solution of the equation of Kirby[1] which differs from the previous detuned-interaction results given by Liu and YL. The derivation here follows the notation of MHN closely, although the governing modulation equations are arrived at by a different method. We consider the case of a patch of sinusoidal bars $\delta(x)$ given by

$$\delta(x)=D\sin(\lambda x),\quad \lambda=2\pi/L, \tag{1}$$

where L is the bar wavelength and D is the bar amplitude. A finite patch of n bars rests in the interval $0\leqslant x\leqslant nL$, superimposed on a uniform depth h. We generalize the previous results somewhat by considering the case of oblique incidence on the bar field, and take θ to represent the

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angle between x and the incident wave-number vector. The general two-dimensional form of the governing equation is then written as

$$\nabla \cdot (p\nabla\tilde{\phi})+k^2CC_g\tilde{\phi}=0, \tag{2}$$

where

$$p=CC_g-[g\delta(x)/\cosh^2 kh], \tag{3}$$

and where the angular frequency ω, wave number k, phase speed C, and group velocity C_g are determined based on the mean depth h and the actual (tuned or detuned) wave period. For obliquely incident waves, we take

$$\tilde{\phi}=\phi(x)e^{imy}, \quad m=k\sin\theta. \tag{4}$$

The governing equation is reduced to the second-order ordinary differential equation in x;

$$\phi_{xx}+(p_x/p)\phi_x+\gamma^2\phi=0, \tag{5}$$

where the factor γ^2 is given by

$$\gamma^2=k^2p^{-1}CC_g-m^2, \tag{6}$$

which, for the case of small bottom perturbations, may be approximated by

$$\gamma^2=l^2+\frac{gk^2\delta}{CC_g\cosh^2 kh}, \quad l=\sqrt{k^2-m^2}=k\cos\theta. \tag{7}$$

As in Kirby,[1] we seek coupled first-order equations of the form

$$\phi_x^+=i\gamma\phi^+ +F(\phi^+,\phi^-), \tag{8}$$

$$\phi_x^-=-i\gamma\phi^- -F(\phi^+,\phi^-), \tag{9}$$

where the total wave field is divided into incident and reflected components,

$$\phi=\phi^+ +\phi^- \tag{10}$$

and $+(-)$ denotes the incident (reflected) wave. Substituting (8)–(10) in (5) leads to the result

$$F=-[(\gamma p)_x/2\gamma p](\phi^+ -\phi^-). \tag{11}$$

To the leading order of approximation in powers of δ, the coefficients in (8) and (9) are given by

$$\gamma=l\left(1+\frac{g\delta}{2CC_g\cosh^2 kh\cos^2\theta}\right), \tag{12}$$

$$\frac{(\gamma p)_x}{2\gamma p}=-\frac{\alpha\Omega_0\delta_x}{kC_gD}, \tag{13}$$

where

$$\alpha=\cos 2\theta/\cos^2\theta \tag{14}$$

and

$$\Omega_0=gk^2D/4\omega\cosh^2 kh. \tag{15}$$

Note that Ω_0 here differs from the value given in Mei[5] in that it is evaluated at the detuned wave number rather than the resonant wave number. We also note that no approximations have been made in obtaining the basic form of the coupled equations (8) and (9); the splitting is reversible and no information has been suppressed. We introduce the explicit form of δ_x:

$$\delta_x=(\lambda D/2)(e^{i\lambda x}+e^{-i\lambda x}) \tag{16}$$

and seek a solution for slowly varying incident and reflected waves of the form

$$\phi^+=A(x)e^{ilx}=A(x)e^{ik\cos\theta x},$$
$$\phi^-=B(x)e^{-ilx}=B(x)e^{-ik\cos\theta x}, \tag{17}$$

Following Liu and YL, we construct an approximate detuned-interaction model by substituting (17) in (8) and (9) and retaining terms which come closest to satisfying resonance conditions. This leads to the coupled evolution equations

$$A_x=-\frac{\Omega_0\alpha\cos\theta}{C_g}\left(\frac{k_{\rm res}}{k}\right)Be^{i\beta x}, \tag{18}$$

$$B_x=-\frac{\Omega_0\alpha\cos\theta}{C_g}\left(\frac{k_{\rm res}}{k}\right)Ae^{-i\beta x}, \tag{19}$$

where

$$k_{\rm res}=\lambda/2\cos\theta \tag{20}$$

is the resonant wave number, and

$$\beta=\lambda-2k\cos\theta \tag{21}$$

is the detuning parameter, as introduced in Liu and YL. For $l=k\cos\theta=\lambda/2$, we recover the exactly resonant interaction equations following from MHN, Eq. (2.5). The detuned equations differ from the results in MHN by the inclusion of the factor $(k_{\rm res}/k)$, which introduces an asymmetry in the coupling coefficient. We see that waves with lower wave numbers (indicating relatively shallower water) become more strongly coupled, as may be expected on physical grounds.

We can increase the correspondence between the results of MHN and the present derivation by introducing a detuning wave number

$$K=k-k_{\rm res} \tag{22}$$

and a corresponding frequency parameter

$$\Omega=KC_g. \tag{23}$$

Note that Ω again differs from Mei's Ω since detuning is finite and the relation between the detuned wave number and the detuned frequency here is not linear, except in the limit of extremely shallow water. The detuning parameter β may now be written as

$$\beta=-2K\cos\theta. \tag{24}$$

We introduce the transformation

$$\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} \tilde{A}e^{-iK\cos\theta} \\ \tilde{B}e^{iK\cos\theta} \end{pmatrix} \tag{25}$$

and obtain the coupled equations

$$\tilde{A}_x=iK\cos\theta\,\tilde{A}-\frac{\Omega_0\alpha\cos\theta}{C_g}\left(\frac{k_{\rm res}}{k}\right)\tilde{B}, \tag{26}$$

$$\tilde{B}_x = -iK\cos\theta\,\tilde{B} - \frac{\Omega_0\alpha\cos\theta}{C_g}\left(\frac{k_{res}}{k}\right)\tilde{A}. \quad (27)$$

We may reduce these to two second-order equations for $\tilde{A}$ and $\tilde{B}$:

$$\begin{pmatrix}\tilde{A}\\ \tilde{B}\end{pmatrix}_{xx} + P^2\begin{pmatrix}\tilde{A}\\ \tilde{B}\end{pmatrix} = 0. \quad (28)$$

The parameter P is determined from

$$P = \frac{\Omega_0\cos\theta}{C_g}\left[\left(\frac{\Omega}{\Omega_0}\right)^2 - \alpha^2\left(\frac{k_{res}}{k}\right)^2\right]^{1/2} \quad (29)$$

and may be either real or imaginary depending on the term in square brackets. The expression for P here is also similar to the expression given in MHN, with the exception of the appearance of the k_{res}/k ratio. The cutoff condition in the solution corresponds to the value $P=0$.

Equations (28) are solved with the boundary conditions

$$\tilde{A}(0) = A_0, \quad \tilde{B}(nL) = 0, \quad (30)$$

which indicates an incident wave amplitude of A_0 at the start of the bar field and a reflection coefficient of zero at the downwave end. The solution is given by

$$A(x) = A_0 e^{-iK\cos\theta x} \times \frac{PC_g\cos P(nL-x) - i\Omega\cos\theta\sin P(nL-x)}{PC_g\cos nPL - i\Omega\cos\theta\sin nPL} \quad (31)$$

and

$$B(x) = A_0 e^{iK\cos\theta x}\frac{\Omega_0\alpha(k_{res}/k)\cos\theta\sin P(nL-x)}{PC_g\cos nPL - i\Omega\cos\theta\sin nPL}. \quad (32)$$

The reflection coefficient upwave of the bar field is given by

$$R = \left|\frac{B(0)}{A_0}\right| = \left|\frac{\Omega_0\alpha(k_{res}/k)\cos\theta\sin nPL}{PC_g\cos nPL - i\Omega\cos\theta\sin nPL}\right|. \quad (33)$$

We illustrate the present solution in comparison to the resonant solution of MHN by plotting contours of reflection coefficient as a function of θ and $2k/\lambda$, where resonance occurs at $2k\cos\theta/\lambda=1$. The solution of MHN is implemented exactly as given in their paper, with K specified based on the chosen wave-number parameter and Ω computed from $\Omega = KC_{g(res)}$.

To maintain correspondence with experimental results given in Davies and Heathershaw,[3] we choose $n=4$, $D/h=0.32$, $D=5$ cm, $L=1$ m, and $h=15.625$ cm. Solutions are computed for the parameter range $0.25 \leqslant 2k\cos\theta/\lambda \leqslant 1.75$, $0° \leqslant \theta \leqslant 85°$. Solutions for the resonant theory of MHN are shown in Fig. 1. The theoretical results are symmetric about the resonance condition $2k\cos\theta/\lambda=1$, and the solution dies out for large values of θ. Figure 2 shows reflection coefficient contours and the locus of the cutoff condition for the present analytic theory. The results here are markedly asymmetric about the resonance condition. Agreement between the two analytic solutions is good, both for reflection coefficients and cutoff conditions, up to

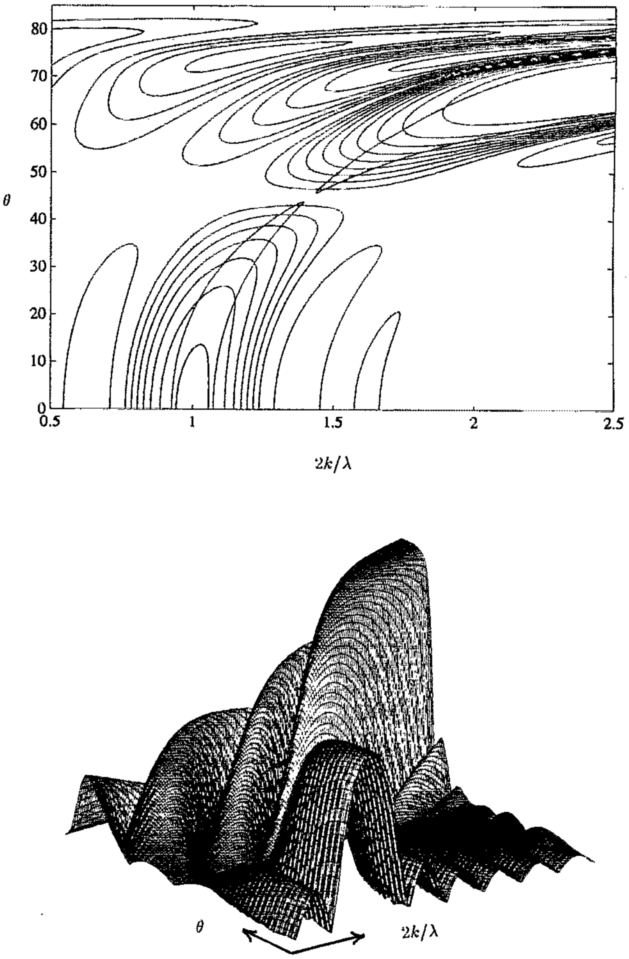
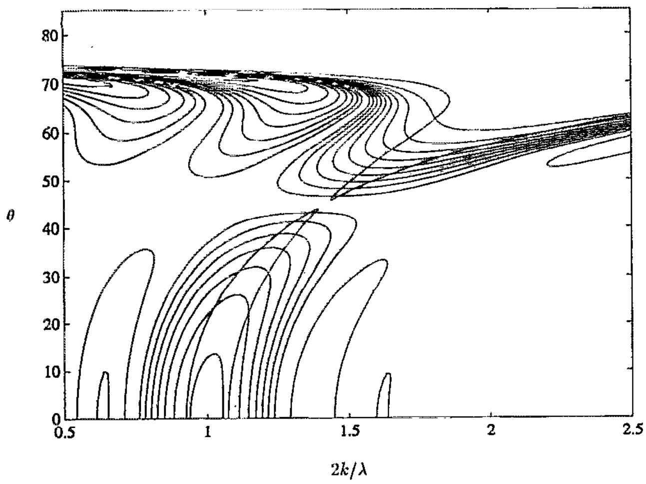


FIG. 1. Reflection coefficient and cutoff condition: solution of Mei *et al.*

$\theta=55°$. For larger θ's and for values of $2k\cos\theta/\lambda<0.5$ and >1.5, the asymmetry of the solution becomes marked and the two approximate solutions deviate.

The apparent agreement between the cutoff conditions predicted here and the conditions predicted by the Mei and MHN theory is in marked contrast to the large deviation between cutoffs seen in comparing the theories of Liu and Mei (see Sec. 4 in Liu). Direct numerical determination of the detuning β corresponding to the cutoff condition has been made for the bar geometry considered here, but with a range of D/h values. The cutoff condition on the high-frequency side of resonance is compared to Mei's theory in Fig. 3, where we plot the percent error in Mei's result relative to the present theory. The results typically agree to within 1% or so. In contrast, Liu shows deviations on the order of 50% between his and Mei's predicted cutoffs.

For angles less than about 60°, we will evaluate the disagreement between the approximate solutions here by comparison with numerical solutions for the full problem. We close this section by remarking that both of the approximate solutions studied here would be expected to break down far from resonance, since the effect of neglected nonresonant terms in the equations would become as important as the effects of the retained terms.



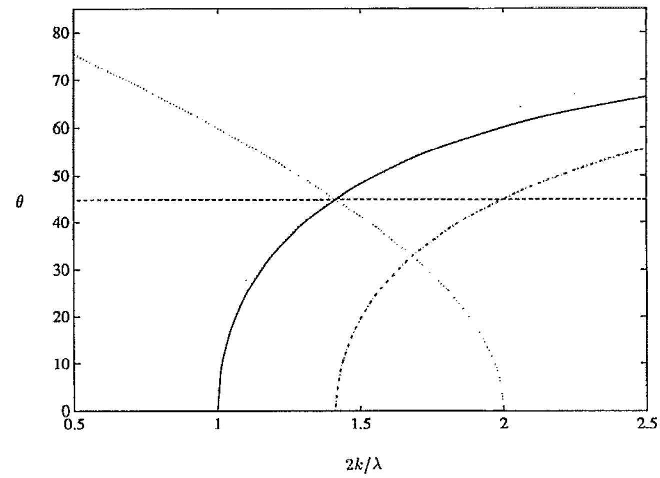


FIG. 4. Comparison of loci of zero reflection: ---, present solution; –.–, Liu solution; ···, YL solution.——, resonance curve.

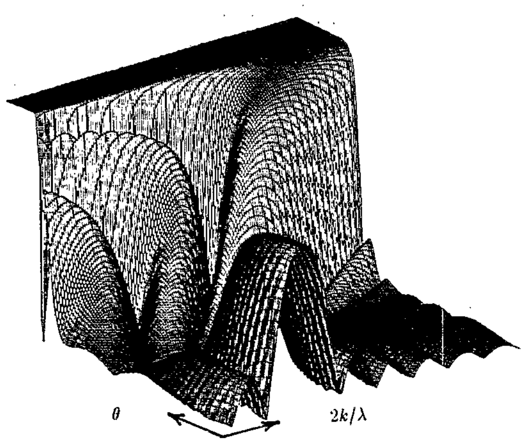


FIG. 2. Reflection coefficient and cutoff condition: present solution.

III. COMPARISON WITH PREVIOUS RESULTS

In this section, we compare the present results to the results obtained by extending the derivation techniques used in Liu and YL. Those derivations are described in the Appendix. The resulting derivations produce coupled equations with the same form as Eqs. (18) and (19), and hence we can consider the correspondence between the form of the coupling coefficients, the values of the factor P and loci of the cutoff conditions, and numerical values of predicted reflection coefficients. We limit the comparison here to an examination of the coupling coefficients and, in particular, to the determination of the zeros of those coefficients, where we see a decoupling between the wave and bar fields. Both the Liu and YL theories have the interesting feature of predicting decoupling for normally incident waves at a finite value of the parameter $2k/\lambda$. When extended to oblique incidence, both theories produce patterns of decoupling which are at odds with the present theory, with the MHN resonant theory, with the nonresonant theory of Kirby and Anton,[4] and with numerical results.

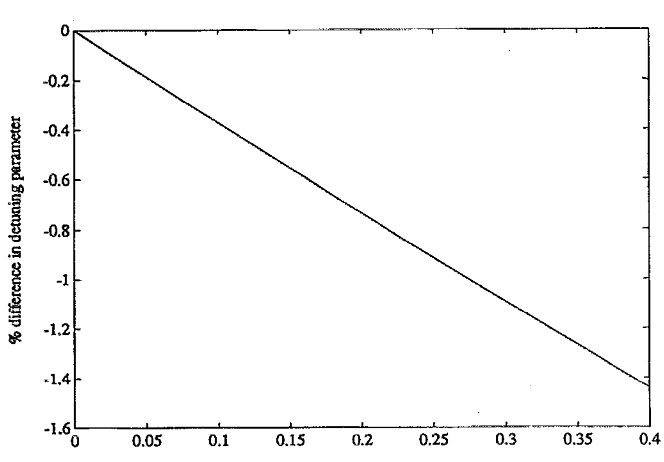


FIG. 3. Percent error in detuning parameter β based on Mei's theory and detuning parameter β based on present theory, for the cutoff condition.

The interaction coefficient for the present theory may be written as

$$\frac{\Omega_0 \cos 2\theta}{C_g \cos \theta} \left(\frac{k_{\text{res}}}{k}\right), \tag{34}$$

which has a zero at $\theta=45°$ for all incident wave frequencies. This result also was found by MHN for the resonant case, and is a feature of a wide range of nonresonant scattering theories for small obstacles (see Kirby and Anton for a review). The locus of the zero in the present coupling coefficient is shown in Fig. 4 along with the locus of the resonance curve $2k/\lambda=1/\cos \theta$.

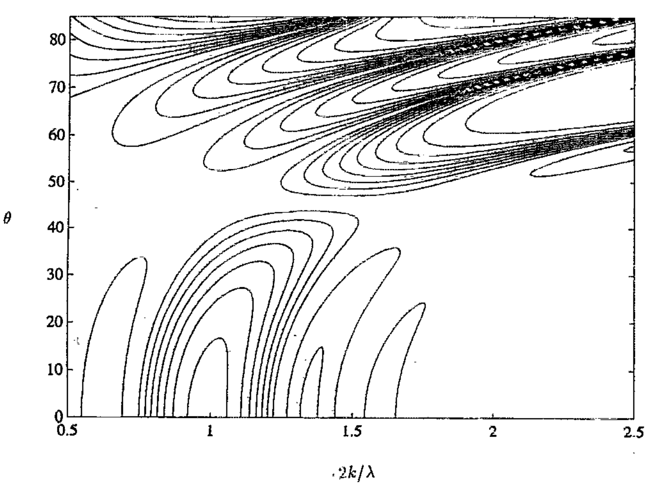
The coupling coefficient α_1 obtained from the YL derivation has zeros on the curve $2k/\lambda=2 \cos \theta$, shown in Fig. 4. The intersection of this curve with the resonance curve occurs at $\theta=45°$ and $2k/\lambda=\sqrt{2}$, as does the curve for the zeros of the present theory. Away from this intersection, the predicted zeros deviate markedly.

For the second theory of the Appendix, the interaction coefficient α_2 leads to decoupling on a curve $2k/\lambda=\sqrt{2}/\cos \theta$, which lies to the right of the resonant interaction curve in Fig. 4 and does not intersect it anywhere.

The two theories described in the Appendix clearly predict patterns of wave–bottom interaction which are at odds with the remaining body of theory on this problem. This result points out the sensitivity of the results here to the method of derivation; in particular, theories which neglect terms of the order of terms that appear eventually in the equations to be solved (A_{xx} and B_{xx} here) lead to incorrect results. The splitting method employed by Kirby[1] and Kirby and Vengayil[13] appears to be a robust method.

IV. NUMERICAL RESULTS

Numerical solutions are obtained for the parameter range $0.5\leqslant 2k/\lambda\leqslant 2.5$, $0°\leqslant\theta\leqslant 85°$, using second-order accurate centered finite differences applied to (5). Boundary conditions for the numerical problem are discussed in Kirby.[9] Results for the $n=4$ Davies and Heathershaw bar field are shown in Fig. 5. We see that the numerically predicted solution shows nearly total reflection at large angles of incidence, as well as very small reflection at $\theta=45°$, as predicted by the analytic theories developed here. The reflection coefficient does not drop identically to zero for $\theta=45°$ owing to the effect of nonresonant modes in the solution which are not accounted for in the analytic approaches.



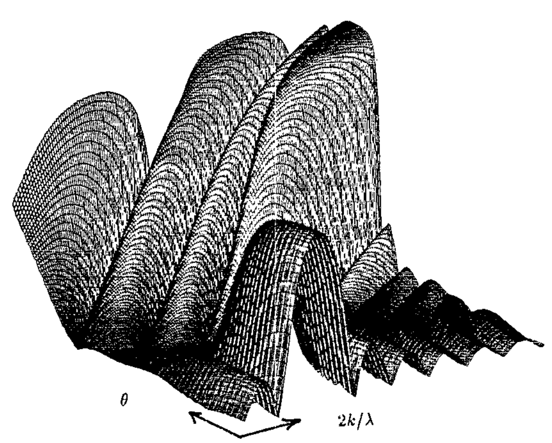


FIG. 5. Reflection coefficient computed numerically from (5).

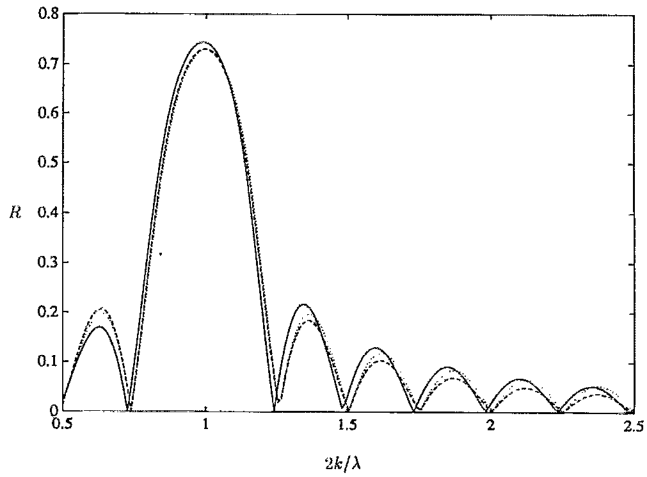


FIG. 6. Reflection coefficient at normal incidence. ——, numerical solution; ---, present analytical solution; · · ·, Mei *et al.*

In Fig. 6, reflection coefficients for normally incident waves obtained numerically, from Mei's solution, and from the present analytical solution are compared. The numerical solution (solid line) is seen to obtain a somewhat higher maximum than in the analytic solutions, and the maximum is shifted to a slightly lower wave number. The two analytic solutions agree at resonance but deviate slightly away from resonance. The numerical solution also indicates that reflection is slightly stronger for the first lobe on the short wave side of the main resonance, in contrast to the prediction of the present analytic theory. The solution of Mei is therefore a somewhat more accurate representation of the numerical solution, although the two analytic results agree more closely with each other than they do with the numerical result. Overall, the deviation between the numerical and analytical results would not be resolvable within the accuracy of the Davies and Heathershaw data set.

V. EFFECTS OF FRICTIONAL DAMPING

The propagation of waves over a bar field is affected by a number of mechanisms which are not included in the analysis presented above. YL and Kirby and Vengayil[13] have discussed some of the consequences of nonlinear interaction on the scattering process. In real applications, it is expected that a number of dissipative mechanisms could come into play. Bottom boundary-layer damping would affect the waves moderately, causing a small decrease in the transmitted and reflected waves predicted by the theory described above. For bars fronting a beach face in relatively shallow water, the protruding bar crests could also induce breaking of steep incident waves, leading to significant reduction of incident energy over the bar field.

Here, we consider the effect of laminar frictional damping on the predicted reflection coefficient. Following

Booij[14] and Kirby,[1] we may extend the generalized mild-slope equation to include damping effects according to

$$\tilde{\phi}_{tt}+W\tilde{\phi}_t-\nabla\cdot(p\nabla\tilde{\phi})+(\omega^2-k^2CC_g)\tilde{\phi}=0 \tag{35}$$

or, for time-harmonic waves,

$$\nabla\cdot(p\nabla\tilde{\phi})+k^2CC_g\tilde{\phi}+i\omega W\tilde{\phi}=0. \tag{36}$$

Here, W is a generalized complex damping term. Following the derivation in Sec. II leads to the coupled evolution equations

$$\tilde{A}_x=-\frac{W}{2C_g\cos\theta}\tilde{A}+iK\cos\theta\,\tilde{A}-\frac{\Omega_0\alpha\cos\theta}{C_g}\left(\frac{k_{\text{res}}}{k}\right)\tilde{B}, \tag{37}$$

$$\tilde{B}_x=\frac{W}{2C_g\cos\theta}\tilde{B}-iK\cos\theta\,\tilde{B}-\frac{\Omega_0\alpha\cos\theta}{C_g}\left(\frac{k_{\text{res}}}{k}\right)\tilde{A}, \tag{38}$$

where we have assumed that the dissipative effects act uniformly and equally on the incident and reflected waves. The new dissipative term may be absorbed in the previous derivation by defining a new complex detuning factor

$$\beta'=\beta-i\frac{W}{C_g\cos\theta}=-2K\cos\theta-i\frac{W}{C_g\cos\theta}. \tag{39}$$

From this point, the derivations in Sec. II follow through exactly, with the exception that the definition of P in (29) is replaced by

$$P=\frac{\Omega_0\cos\theta}{C_g}\left[\left(\frac{\Omega}{\Omega_0}\right)^2+i\frac{\Omega W}{\Omega_0^2\cos^2\theta}-\left(\frac{W}{2\Omega_0\cos^2\theta}\right)^2-\alpha^2\left(\frac{k_{\text{res}}}{k}\right)^2\right]^{1/2}. \tag{40}$$

For the case of laminar bottom boundary-layer damping, an expression for W is given by[15]

$$W=(1-i)\frac{gk^2}{\omega\cosh^2 kh}\sqrt{\frac{\nu}{2\omega}}. \tag{41}$$

For laminar damping alone, the effects on the results given above are quite small, owing to the relatively short length of the bar field. The reflection coefficients with damping can be either smaller or larger than the coefficients without damping, due to the effect that damping has on the wave speed and wavelength. It is not anticipated that local damping over the bar field has any significant effects in any of the available laboratory data sets.

VI. DISCUSSION

The results of this and previous studies have shown that numerical solutions of a generalized mild-slope equation may be used to model the reflection of waves from simple (nearly sinusoidal) bar fields. A body of analytic approximations are also now available for providing estimates in the same situation.

Several recent studies have indicated that bottom configurations having multiple Fourier components can lead to additional reflection peaks appearing at difference wave numbers resulting from the interaction of two or more individual bottom components.[16] This situation is not handled properly by the existing generalized mild-slope equation. The problem of these subharmonic resonances is being pursued further by the present author in the context of a generalization of the mild-slope equation to handle the presence of nonpropagated modes.

ACKNOWLEDGMENT

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APPENDIX: EXTENSION OF LIU AND YL RESULTS TO THE CASE OF OBLIQUELY INCIDENT WAVES

The problem of specifying the evolution equations for the detuned interaction between surface waves and the sinusoidal bar field has been considered previously by Liu and YL, as well as by Kirby and Vengayil.[13] The study of Liu was restricted to normal incidence of linear waves in a channel with both bottom and side-wall corrugations, while the theory of YL was for the same channel configuration but for weakly nonlinear, weakly dispersive waves obeying the Boussinesq equations. Here, we consider the case of normal or oblique incidence of linear dispersive waves over an infinitely wide bar field, as in Sec. II, and extend the theories of Liu and YL accordingly. Section III of the main text gives a comparison between various features of these extended results and the results in Sec. II.

The YL approach is conceptually simpler than the Liu approach, and so we follow it first. We start by rewriting the governing equation as

$$\nabla^2\tilde{\phi}+k^2\tilde{\phi}=\frac{gk}{\omega C_g\cosh^2 kh}(\delta\nabla^2\tilde{\phi}+\delta_x\tilde{\phi}_x), \tag{A1}$$

where we have used the fact that the mean depth is taken to be uniform and the bar field varies as $\delta(x)$ only. Employing the separation (4) in the main text reduces (A1) to

$$\phi_{xx}+l^2\phi=\frac{gk}{\omega C_g\cosh^2 kh}(\delta\phi_{xx}+\delta_x\phi_x-m^2\delta\phi). \tag{A2}$$

We now make the separation into incident and reflected waves with slowly varying amplitudes, as specified in (10) and (17). These are substituted directly into (A2) without a prior splitting. An apparent ordering is made which equates the importance of slow derivatives of A and B with terms that are first order in δ. Terms of $O(\delta^2)$ are then dropped. Then, substituting the expression (1) for δ and retaining terms closest to resonance leads to the coupled equations

$$A_x=-(\Omega_0/C_g)\alpha_1 Be^{i\beta x}, \tag{A3}$$

$$B_x=-(\Omega_0/C_g)\alpha_1 Ae^{-i\beta x}, \tag{A4}$$

where

$$\alpha_1=(\cos\theta)^{-1}\left[2\left(\frac{k_{\text{res}}}{k}\right)\cos^2\theta-1\right], \tag{A5}$$

and where all other notations are as in the main text. The resulting model is similar in form to (18) and (19) in the main text, except for a change in the interaction coefficient. We remark that the principal reason for the differences between the present model and the model in the main text arises due to neglect of terms involving the factors δA_x and δB_x. If these terms were retained by effectively dividing through by the quantity p as in the main text, the resulting model would be equivalent.

For the case of normal incidence and in the limit of long wave theory, the interaction coefficient becomes

$$\frac{\Omega_0}{C_g}\alpha_1 \rightarrow \frac{D}{4h}(\lambda - k) \tag{A6}$$

as in YL. At resonance, this reduces to $\lambda D/8h$ as in Kirby and Vengayil,[13] but it disagrees with their theory (again, based on a splitting procedure) away from resonance.

Turning to the work of Liu, we rewrite (A2) as

$$(p\phi_x)_x + l^2 CC_g \phi = 0. \tag{A7}$$

We then employ the Liouville transformation

$$\phi = p^{-1/2}\zeta \tag{A8}$$

and obtain

$$\zeta_{xx} - \frac{1}{2}p^{-1}p_{xx}\zeta + l^2 p^{-1} CC_g \zeta = 0. \tag{A9}$$

Retaining terms to $O(\delta)$, this becomes

$$\zeta_{xx} + l^2\zeta - \frac{4\Omega_0}{DkC_g}\left(\frac{\delta_{xx}}{2} + l^2\delta\right)\zeta = 0. \tag{A10}$$

We then employ the split into incident and reflected waves according to

$$\zeta = \psi^+(x)e^{ilx} + \psi^-(x)e^{-ilx}. \tag{A11}$$

Direct substitution and the neglect of second-order terms in the slow derivatives leads to the set of coupled equations

$$\psi_x^+ = -(\Omega_0/C_g)\alpha_2\psi^- e^{i\beta x}, \tag{A12}$$

$$\psi_x^- = -(\Omega_0/C_g)\alpha_2\psi^+ e^{-i\beta x}, \tag{A13}$$

where

$$\alpha_2 = \cos\theta\left[2\left(\frac{k_{\rm res}}{k}\right)^2 - 1\right]. \tag{A14}$$

For the case of normal incidence, the interaction coefficient in (A12) and (A13) reduces to

$$\frac{\Omega_0}{C_g}\alpha_2 \rightarrow \frac{\Omega_0}{k^2 C_g}\left(\frac{\lambda^2}{2} - k^2\right) \tag{A15}$$

as in Liu.

The main drawback in the derivation of (A12) and (A13) lies in the fact that the amplitudes ψ^+ and ψ^- are not properly decoupled. For example, compare the transformed incident wave to an untransformed form:

$$\psi^+ e^{ilx} = p^{1/2} A e^{ilx}. \tag{A16}$$

Taking x derivatives of both sides of (A16) leads to the expression

$$\psi_x^+ = p^{1/2}A_x + \tfrac{1}{2}p^{-1/2}p_x A. \tag{A17}$$

We see that the derivative of ψ^+ contains a term which would resonate with the reflected, B wave, indicating that the incident-reflected wave separation is incomplete in this method.

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