Comments on "The Effects of a Jetlike Current on Gravity Waves in Shallow Water"

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ABSTRACT

Matching conditions used by Mei and Lo to study wave scattering at a vertical vortex sheet in the shallow water (long wave) limit are seen to be incorrect and, as a result, wave action is not conserved during scattering by a single vortex sheet. Here, we establish the correct kinematic condition for the more general case of a step discontinuity in depth as well as current, and modify Mei and Lo's results accordingly.

1. Introduction

Problems involving scattering of waves at vortex sheets, which represent surfaces of discontinuity between ambient streams of dissimilar velocity, have received increasing attention recently due to the general interest in wave–current interaction. In the general theory, the schematization of a shear layer as a discrete vortex sheet allows the flow to either side to be considered irrotational, thus allowing the introduction of a velocity potential for each region. Evans (1975) used such an approach to treat the scattering of a linear deepwater wave which was obliquely incident on a vortex sheet. Smith (1983) similarly treated the case of two adjacent sheets bounding jetlike, "top-hat" flow in deep water.

Recently, Mei and Lo (1984, hereafter referred to as ML) have used the shallow water theory in conjunction with jetlike flows in order to illustrate several correspondences between the quantum mechanics and water-wave scattering problems. ML present general forms for solutions for waves trapped on jets, and derive reflection and transmission coefficients for waves obliquely incident on a top-hat jet bounded by regions with no ambient current. The results in ML are interesting and useful; however, as will become evident below, the matching conditions used by ML are inappropriate, and their results are thus in quantitative error.

The purpose of this comment is to derive matching conditions at discontinuities which lead to scattering solutions that satisfy the appropriate condition of conservation of action. The general case of variation in both depth h and velocity V across the sheet is considered.

2. The linearized problem

The linear shallow water theory used here may be obtained from ML. Define $u(x, y, t)$, $v(x, y, t)$ as wave-induced velocities, $\{0, V(x)\}$ as the ambient current, $h(x)$ as the water depth, and $\eta(x, y, t)$ as the wave-induced free surface displacement. The linearized equations of motion and continuity are then given by

$$u_t + Vu_y = -g\eta_x \tag{2.1}$$

$$v_t + Vv_y + uV_x = -g\eta_y \tag{2.2}$$

$$\eta_t + V\eta_y + (hu)_x + hv_y = 0 \tag{2.3}$$

where subscripts denote differentiation. We assume motions which are uniformly periodic in time and in the y-direction, perpendicular to any variations in the ambient flow. Then

$$(u, v, \eta) = [u(x), v(x), \eta(x)]e^{i(my-\omega t)} \tag{2.4}$$

where ω is the angular frequency in stationary coordinates (x, y). Waves propagate in the $+y$-direction with wave number m; current $V(x)$ may be $\gtrless 0$, with no resulting loss of generality. Introducing (2.4) in (2.1)–(2.2) yields

$$u = \frac{-ig\eta_x}{\omega - mV} = \frac{-ig\eta_x}{\sigma} \tag{2.5}$$

$$v = -\left\{\frac{gV_x\eta_x}{(\omega - mV)^2} - \frac{gm\eta}{\omega - mV}\right\} = -\left\{\frac{gV_x\eta_x}{\sigma^2} - \frac{gm\eta}{\sigma}\right\} \tag{2.6}$$

where σ is the intrinsic frequency relative to the ambient current V. Substitution of (2.5) and (2.6) in (2.3) leads to the second-order ODE

$$\eta_{xx} + \left\{\frac{h_x}{h} + \frac{2mV_x}{\sigma}\right\}\eta_x + \left\{\frac{\sigma^2}{gh} - m^2\right\}\eta = 0. \tag{2.7}$$

In the following, we consider only regions which consist of subregions of constant depth h and current V, divided by step discontinuities. ML's treatment of continuous current and depth variation is correct as it stands. Thus, in any region (i), we have

$$\eta_{ixx} + \left\{\frac{\sigma_i^2}{gh_i} - m^2\right\}\eta_i = 0 \quad (2.8)$$

which has the solutions

$$\eta_i = ae^{il_ix} + be^{-il_ix}$$

$$l_i = \left(\frac{\sigma_i^2}{gh_i} - m^2\right)^{1/2}. \quad (2.9)$$

Depending on the parameters of the problem, l_i may be real, leading to freely propagating waves in region (i), or imaginary, leading to only forced waves in region (i). We consider here only the case of freely propagating waves everywhere, although our results are generally applicable to both cases.

3. Matching conditions at a step discontinuity in depth and current

Mei and Lo treated the case of a vortex sheet in water of constant depth, previously considered by Mollo-Christensen (1978). They chose as matching conditions that η and η_x be continuous across the discontinuity. Continuity of η implies the continuity of dynamic pressure and represents the shallow water limit of the condition that pressure be continuous over the vertical extent of the sheet in the general depth case (Smith, 1983; Miles, 1967). The second condition of η_x continuous does not have physical significance.

Mei and Lo show that the governing equation (2.7) may be transformed to the one-dimensional Schrodinger equation by introducing the quantity

$$\alpha = \frac{gh}{\sigma^2} = \frac{gh}{(\omega - mV)^2}. \quad (3.1)$$

Using this substitution reduces (2.7) to the form

$$(\alpha\eta_x)_x + (1 - m^2\alpha)\eta = 0. \quad (3.2)$$

[Note the correction of a typographic error appearing in ML's (5.1)]. Now consider the integral of (3.2) across a step discontinuity at $x = 0$, which gives

$$(\alpha\eta_x)_{\epsilon^+} - (\alpha\eta_x)_{\epsilon^-} = -\int_{\epsilon^-}^{\epsilon^+} (1 - m^2\alpha)\eta dx. \quad (3.3)$$

Assuming that α and η are suitably behaved in the vicinity of $x = 0$ (indeed, η is continuous while α has a single step discontinuity of finite value) and letting $\epsilon^+, \epsilon^- \longrightarrow 0$ gives

$$(\alpha\eta_x)_{0^-} = (\alpha\eta_x)_{0^+} \quad (3.4)$$

or, using (3.1),

$$\frac{h_1\eta_{1x}}{\sigma_1^2} = \frac{h_2\eta_{2x}}{\sigma_2^2}; \quad x = 0. \quad (3.5)$$

This condition replaces the condition used in ML when $h_1 = h_2$. The condition (3.5) is kinematic in nature and may be obtained by using a depth-averaged form of the general-depth kinematic condition used by Evans (1975) and Smith (1983). Evans defines the position of the vortex sheet as $\xi(y, z, t)$ and requires that particles in regions 1 and 2 which are on the vortex sheet remain on the vortex sheet. This leads to the kinematic requirements

$$u_1(z) = \xi_t + V_1(z)\xi_y \quad (3.6)$$

$$u_2(z) = \xi_t + V_2(z)\xi_y. \quad (3.7)$$

Assume $V_1(z)$, $V_2(z)$ = constant and take h_1 and $h_2 = h$. Performing the depth integration appropriate to shallow water theory gives

$$hu_1 = -i\sigma_1 \int_{-h}^{0} \xi dz \quad (3.8)$$

$$hu_2 = -i\sigma_2 \int_{-h}^{0} \xi dz \quad (3.9)$$

where

$$hu_1 = \int_{-h}^{0} u_1(z)dz; \quad hu_2 = \int_{-h}^{0} u_2(z)dz \quad (3.10)$$

and where we have used (2.5). Eliminating the integral of ξ from (3.8–9) gives

$$\frac{hu_1}{\sigma_1} = \frac{hu_2}{\sigma_2} \quad (3.11)$$

which, for uniform depth, implies

$$\sigma_1^{-2}\eta_{1x} = \sigma_2^{-2}\eta_{2x} \quad (3.12)$$

in place of the condition used in ML. Note that the integral of the vortex sheet displacement ξ from $x = 0$ leads to a quantity that represents the volume of displacement, per unit length of ξ, relative to the equilibrium position $x = 0$, due to wave motion. We denote this volume by $\mathcal{V}$ and state the condition that the volumes of displacement $\mathcal{V}_i$ caused by the flow in regions (i) on either side of a depth or current discontinuity must be equivalent.

For the case of nonequal depths, the resulting kinematic conditions are expressed by

$$h_1u_1 = \mathcal{V}_t + V_1\mathcal{V}_y; \quad x = 0^- \quad (3.13)$$

$$h_2u_2 = \mathcal{V}_t + V_2\mathcal{V}_y; \quad x = 0^+ \quad (3.14)$$

which gives

$$\frac{h_1u_1}{\sigma_1} = \frac{h_2u_2}{\sigma_2}. \quad (3.15)$$

Equation (3.15) reduces to the mass flux condition suggested by Lamb (1945) in the absence of currents and is equivalent to (3.5) after using (2.5).

We now consider the scattering of an incident wave from region 1 ($x < 0$) at a discontinuity of h and V at $x = 0$. The components of the wave train are given by

$$\left.\begin{array}{l}\eta_I = e^{il_1x} \\ \eta_R = \mathrm{R}e^{-il_1x}\end{array}\right\}, \quad l_1 = \left\{\frac{\omega^2}{gh_1} - m^2\right\}^{1/2} > 0; \; x < 0 \qquad \begin{array}{r}(3.16)\\(3.17)\end{array}$$

$$\eta_T = Te^{il_2x}; \; l_2 = \left\{\frac{(\omega - mV_2)^2}{gh_2} - m^2\right\}^{1/2}; \; x > 0 \quad (3.18)$$

where η_I, η_R, η_T are the incident, reflected, and transmitted wave, and where subscript 2 denotes quantities in region 2 ($x > 0$) with depth h_2 and velocity V_2.

Requiring continuity of η at $x = 0$ gives

$$1 + R = T. \quad (3.19)$$

Then, applying (3.5) at $x = 0$ gives

$$\frac{h_1 l_1}{\sigma_1^2}(1 - R) = \frac{h_2 l_2}{\sigma_2^2} T. \quad (3.20)$$

Solving for R and T then gives

$$R = \left(1 - \frac{h_2}{h_1}\frac{\sigma_1^2}{\sigma_2^2}\frac{l_2}{l_1}\right) \bigg/ \left(1 + \frac{h_2}{h_1}\frac{\sigma_1^2}{\sigma_2^2}\frac{l_2}{l_1}\right);$$

$$T = 2 \bigg/ \left(1 + \frac{h_2}{h_1}\frac{\sigma_1^2}{\sigma_2^2}\frac{l_2}{l_1}\right). \quad (3.21)$$

These results may be tested for consistency by investigating action conservation. Waves propagating on currents have wave action as a conserved integral quantity, with wave action defined as the ratio of the local energy density over the local intrinsic frequency σ. In the general two-dimensional, time-steady case, we have (Bretherton and Garrett, 1968)

$$\nabla_h \cdot \left\{\frac{E}{\sigma}(\mathbf{C}_g + \mathbf{U})\right\} = 0 \quad (3.22)$$

where $\mathbf{C}_g$ is the vector group velocity relative to $\mathbf{U}$, the ambient current. In the present shallow water case, with uniform conditions in the y-direction, (3.22) reduces to

$$\left[\frac{E}{\sigma}\sqrt{gh}\left(\frac{l}{k}\right)\right]_x = 0 \quad (3.23)$$

where l/k is the direction cosine, from

$$k^2 = \frac{\sigma^2}{gh} = l^2 + m^2. \quad (3.24)$$

Integrating across a step discontinuity at $x = 0$ implies

$$\left[\frac{E}{\sigma}\sqrt{gh}\left(\frac{l}{k}\right)\right]_{0-} = \left[\frac{E}{\sigma}\sqrt{gh}\left(\frac{l}{k}\right)\right]_{0+}. \quad (3.25)$$

It may be readily shown that (3.21) satisfies the condition (3.25).

4. Modification to ML results

Mei and Lo consider the scattering of waves by a top-hat current defined by

$$V = \begin{cases} 0, & |x| > a \\ V, & |x| \le a. \end{cases} \quad (4.1)$$

Denoting the jet region as (2) and the external regions as (1), we have

$$\eta = \begin{cases} e^{il_1(x+a)} + \mathrm{R}e^{-il_1(x+a)}, & x < -a \quad (4.2) \\ Ae^{il_2x} + Be^{-il_2x}, & |x| \le a \quad (4.3) \\ Te^{il_1(x-a)}, & x > a \quad (4.4) \end{cases}$$

where

$$l_1^2 = \frac{\omega^2}{gh} - m^2 = k^2 - m^2 \quad (4.5)$$

$$l_2^2 = \frac{(\omega - mV)^2}{gh} - m^2 > 0. \quad (4.6)$$

Using the matching conditions derived above, we obtain the reflection and transmission coefficients.

$$T = \frac{4b}{(1 + b)^2 e^{-2il_2a} - (1 - b)^2 e^{2il_2a}} \quad (4.7)$$

$$R = \frac{-(1 - b^2)[e^{-2il_2a} - e^{2il_2a}]}{(1 + b)^2 e^{-2il_2a} - (1 - b)^2 e^{2il_2a}} \quad (4.8)$$

which are equivalent to ML's results with the exception that b is here defined by

$$b = l_1\sigma_2^2/l_2\sigma_1^2 \quad (4.9)$$

rather than by l_1/l_2 as in ML. In ML, these results are expressed in terms of parameters

$$K = \omega/(m\sqrt{gh}) \quad (4.10)$$

$$F = V/\sqrt{gh}. \quad (4.11)$$

In terms of these quantities, b in (4.7–8) is given according to

$$b^2 = \frac{(1 - F/K)^4(K^2 - 1)}{(K - F)^2 - 1} \quad (4.12)$$

while their $\alpha_2 a$ (our $l_2 a$) remains unchanged. ML's other results may be adjusted accordingly.

REFERENCES

Bretherton, F. P., and Garrett, C. J. R., 1968: Wavetrains in inhomogeneous moving media, *Proc. Roy. Soc. London,* **A302,** 529–554.

Evans, D. V., 1975: The transmission of deep-water waves across a vortex sheet, *J. Fluid Mech.,* **68,** 389–401.

Lamb, H., 1945: *Hydrodynamics,* Dover, 738.

Mei, C. C., and Lo, E., 1984: The Effects of a jet-like current on gravity waves in shallow water. *J. Phys. Oceanogr.,* **14,** 471–477.

Miles, J. W., 1967: Surface-wave scattering matrix for a shelf, *J. Fluid Mech.,* **28,** 755–767.

Mollo-Christensen, E., 1978: Over-reflection of horizontally propagating gravity waves by a vertical shear layer, *Phys. Fluids,* **21,** 1908–1911.

Smith, J., 1983: On surface gravity waves crossing weak current jets, *J. Fluid Mech.,* **134,** 277–299.