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SURFACE WAVES ON VERTICALLY SHEARED FLOWS: APPROXIMATE DISPERSION RELATIONS

James T. Kirby[1] and Tsung-Muh Chen[2]

Coastal and Oceanographic Engineering Department, University of Florida, Gainesville

Abstract. Assuming linear wave theory for waves riding on a weak current of $O(\epsilon)$ compared to the wave phase speed, an approximate dispersion relation is developed to $O(\epsilon^2)$ for arbitrary current $U(z)$ in water of finite depth. The $O(\epsilon^2)$ approximation is shown to be a significant improvement over the $O(\epsilon)$ result, in comparison with numerical and analytic results for the full problem. Various current profiles in the full range of water depths are considered. Comments on approximate action conservation and application to depth-averaged wave models are included.

1. Introduction

The problem of describing wave propagation through regions containing tidal, ocean, or discharge currents is fundamental to the description of the nearshore wave climate. Great strides have been made in extending wave propagation models to include the effect of irrotational, large currents (assumed to be uniform over water depth). However, currents typically do not possess so simple a form but instead have variations over depth and associated vorticity, which renders the assumption of irrotationality invalid. The resulting problem for wave motion on arbitrarily varying currents remains unsolved, even in the linearized, uniform domain extreme.

The purpose of this study is to describe a perturbation method for the special case of $U(z)/c \ll 1$, where c is the absolute phase speed, which allows for the solution of the linearized problem for arbitrarily varying current $U(z)$. This case is restrictive in the general context of the study of wave-current interaction, where U/c is taken to have no restriction on size. However, in the context of coastal wave propagation, where we typically consider waves propagating from a generation region into regions of varying current, current fields of practical interest typically satisfy the scaling restriction considered here except possibly in special situations such as strong flows in inlets.

We proceed in section 2 by establishing the problem for a linear wave in a uniform domain with arbitrary $U(z)$. We then outline a perturbation expansion based on small parameter $\epsilon = O(U/c) \ll 1$, following the method employed by Stewart and Joy [1974] for deep water. We then obtain solutions to the general problem to $O(\epsilon)$ (reproducing the result of Skop [1987]) and to $O(\epsilon^2)$. In section 3 we apply the method to linear, cosine, and 1/7-power current variations and compare results to analytically or numerically obtained exact solutions.

The results of the analysis show that the solutions are valid in the regime $(\max|U-\bar{U}|/\bar{U}) \ll 1$, where $\bar{U}$ is the depth-averaged current, leading to the conjecture that the expansions are valid for arbitrarily large currents having weak vorticity. In section 4 we outline an expansion for weak vorticity and obtain the results for a linear shear current, proving equivalence of the expansions for this restricted case.

In section 5, various results for the second-order approximation in the deepwater limit are described.

In section 6 we end with some analysis of action flux formulations and some cautionary notes on the direct use of the $O(\epsilon)$ depth-averaged velocity in wave propagation models.

2. Theory and Approximate Expressions for the Phase Speed

We consider here the inviscid motion of a linear wave propagating on a stream of velocity $U(z)$, where water depth and current speed are assumed to be uniform in the x direction (Figure 1). Associated with the wave-induced motion is a stream function ψ of the form

$$\psi(x,z,t) = f(z)e^{ik(x-ct)} \tag{1}$$

After eliminating pressure from the Euler equations and linearized free-surface boundary conditions, and using the continuity equation, the boundary value problem for $f(z)$ is given by the Rayleigh or inviscid Orr-Sommerfeld equation

$$[c-U(z)](f'' - k^2f) + U''(z)\, f = 0 \qquad -h < z < 0 \tag{2}$$

together with the boundary conditions

$$(U-c)^2f' = [g + U'(U-c)]\, f \qquad z = 0 \tag{3}$$

$$f = 0 \qquad z = -h \tag{4}$$

Here, primes denote differentiation with respect to z, g is the gravitational constant, h is the water depth in the absence of waves and k is the wave number given by $2\pi/\lambda$, where λ is the physical wavelength. This model has been used in a number of studies of waves on arbitrary or particular current distributions; reference may be made to

[1]Now at the Department of Civil Engineering, University of Delaware, Newark.
[2]Now at Department of Civil Engineering and Operations Research, Princeton University, Princeton.

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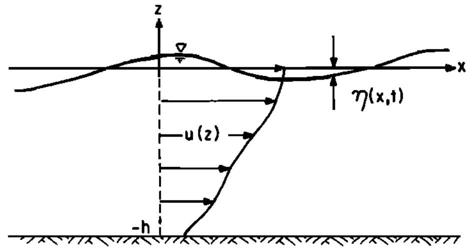

Fig. 1. Definition sketch.

Peregrine [1976] or Peregrine and Jonsson [1983] for a review of existing results. The goal of the present analysis is to obtain approximate expressions for c in the dispersion relation

$$\omega = kc \tag{5}$$

where ω is the wave frequency with respect to a stationary observer. The approximations are based on the assumption of U(z) arbitrary but $|U(z)| \ll c$; we will thus assume the current to be weak and then evaluate the extent of this restriction after obtaining the solutions.

2.1. Perturbation Method

A perturbation expansion for weak currents $\{U/c < O(1)\}$ is employed, following the analysis by Stewart and Joy [1974] for deep water. We take

$$U(z) \sim \varepsilon U(z) \tag{6}$$

where we introduce ε as an apparent ordering and retain dimensional variables. We then introduce the expansions

$$c = \sum_{n=0}^{\infty} \varepsilon^n c_n \tag{7}$$

$$f = \sum_{n=0}^{\infty} \varepsilon^n f_n \tag{8}$$

Equations (6)-(8) are substituted in (2)-(4), and the resulting expansions are ordered in powers of ε, which gives

$$f_n'' - k^2 f_n = F_n \qquad -h < z < 0 \tag{9}$$

$$c_0^2 f_n' - g f_n = S_n \qquad z = 0 \tag{10}$$

$$f_n = 0 \qquad z = -h \qquad n=0,1,\ldots \tag{11}$$

where the F^n and S^n are inhomogeneous forcing terms involving information at lower order than n. For n>1, the homogeneous solution of (9)-(11) is the lowest-order solution $f_0(z)$, as derived below, and it is necessary to construct a solvability condition according to the Fredholm alternative. Using Green's formula on the quantities f_0 and f_n leads to the condition

$$c_0^2 \int_{-h}^{0} f_0 F_n dz = f_0(0)\, S_n \qquad n > 1 \tag{12}$$

This relation is used below to solve for the phase speed corrections c_1 and c_2 due to the presence of a weak, arbitrary current profile.

2.2. Solutions to $O(\varepsilon^2)$ for Arbitrary U(z)

The perturbation problems obtained above are now solved in sequence.

O(1). We have

$$F_0 = S_0 = 0 \tag{13}$$

and the homogeneous solution (with amplitude arbitrarily taken as 1) is

$$f_0(z) = \sinh k(h+z) \tag{14}$$

with

$$c_0^2 = \frac{g}{k} \tanh kh \tag{15}$$

This is the usual result for linear waves on a stationary domain.

O(ε). At this order we have

$$F_1 = -U''(z)\, f_0(z)/c_0 = -U''\sinh k(h+z)/c_0 \tag{16}$$

$$S_1 = 2c_0(U(0) - c_1)\, f_0'(0) - c_0 U'(0)\, f_0(0)$$
$$= 2kc_0(U(0) - c_1)\cosh kh - c_0 U'(0)\sinh kh \tag{17}$$

Substituting (16)-(17) in (12) gives

$$c_1 = \frac{2k}{\sinh 2kh} \int_{-h}^{0} U(z) \cosh 2k(h+z) dz \equiv \tilde{U} \tag{18}$$

This is the finite-depth extension of the result of Stewart and Joy [1974], who obtained the result

$$\tilde{U} = 2k \int_{-\infty}^{0} U(z) e^{2kz} dz \tag{19}$$

in the limit $kh \to \infty$. The result (equation (18)) has also been obtained by Skop [1987]. To O(ε), the dispersion relation is given by

$$c = c_0 + \tilde{U} \tag{20}$$

or, equivalently,

$$\omega = \tilde{\sigma} + k\tilde{U} \tag{21}$$

where ω is the absolute frequency in stationary coordinates and $\tilde{\sigma}$ is the frequency relative to a frame moving with velocity $\tilde{U}$, the weighted mean current:

$$\tilde{\sigma}^2 = gk \tanh kh \tag{22}$$

The particular solution $f_{1p}(z)$ may be obtained by the method of variation of parameters; the entire solution $f_1(z)$ is then

$$f_1(z) = \left\{A_1 - \frac{1}{2kc_0}\int_{-h}^{z} U''(\xi)\sinh 2k(h+\xi)d\xi\right\}\sinh k(h+z) + \left\{B_1 + \frac{1}{2kc_0}\int_{-h}^{z} U''(\xi)[\cosh 2k(h+\xi)-1]d\xi\right\}\cosh k(h+z) \tag{23}$$

where $B_1=0$ in order to satisfy the bottom boundary condition and A_1 may be set to zero arbitrarily. Note that $f_1(z)$ is identically zero for a flow with constant or zero vorticity ($U''=0$). For flows for which U'' is known, $f_1(z)$ may be evaluated directly from (23). For general $U(z)$, repeated integration by parts is applied to (23) to obtain the expression

$$f_1(z) = -\left\{\frac{[U(z) + U(-h)]}{c_0} + \frac{2kI_1(z)}{c_0}\right\} f_0(z) + \frac{2I_2(z)}{c_0} f_0'(z) \tag{24}$$

where

$$I_1(z) = \int_{-h}^{z} U(\xi)\sinh 2k(h+\xi)d\xi \tag{25}$$

$$I_2(z) = \int_{-h}^{z} U(\xi)\cosh 2k(h+\xi)d\xi \tag{26}$$

$O(\varepsilon^2)$. We obtain

$$F_2 = -\frac{U''(z)f_1(z)}{c_0} + \frac{U''(z)(\tilde{U}-U)}{c_0^2} f_0(z) \tag{27}$$

$$S_2 = 2c_0(U(0) - \tilde{U})f_1'(0) - c_0U'(0)f_1(0) - [2c_0c_2 + [U(0) - \tilde{U}]^2]f_0'(0) + U'(0)[U(0) - \tilde{U}]f_0(0) \tag{28}$$

Substituting (27)-(28) in (12) gives an expression for c_2:

$$\frac{c_2}{c_0} = \left\{\frac{U'(0)[U(0) - \tilde{U}]}{2g} - \frac{[U(0) - \tilde{U}]^2}{2c_0^2}\right\} + \frac{(U(0) - \tilde{U})}{c_0}\frac{f_1'(0)}{f_0'(0)} - \frac{U'(0)}{2c_0}\frac{f_1(0)}{f_0'(0)} + \frac{1}{2gf_0^2(0)}\int_{-h}^{0} U''(z)\left\{c_0f_0(z)f_1(z) + (U(z) - \tilde{U})f_0^2(z)\right\}dz \tag{29}$$

where we have used the fact that $f_0(0)/f_0'(0) = c_0^2/g$. As above, several options are possible here. For flows with zero or constant vorticity, we have $f_1(z) = U''(z) = 0$, and we obtain directly

$$c_2 = \frac{U'(0)[U(0) - \tilde{U}]c_0}{2g} - \frac{[U(0) - \tilde{U}]^2}{2c_0} \tag{30}$$

For flows where U'' is known, we may evaluate $f_1(z)$ from (23) and substitute for $U''(z)$ in (29) and then solve for c_2. For general $U(z)$ (especially for tabulated $U(z)$ where higher derivatives are not known), we use (24) in (29) and integrate by parts to obtain the expression

$$\frac{c_2}{c_0} = \frac{\tilde{U}}{2c_0^2}\left[4kI_1(0) - (1 + 2\cosh 2kh)\tilde{U}\right] + \frac{k^2}{2gf_0^2(0)}\int_{-h}^{0} U^2(z)(1 + 2\cosh^2 k(h+z))dz + \frac{2k^3}{gf_0^2(0)}\int_{-h}^{0}\left[I_2(z)I_1'(z) - I_1(z)I_2'(z)\right]dz \tag{31}$$

The remaining integrals are expressed completely in terms of $U(z)$ and may not be tractable for a general $U(z)$; numerical approximation will then be required.

3. Examples Using Known Velocity Distributions

We now investigate the accuracy of the approximations derived above, considering the results both to $O(\varepsilon)$ and $O(\varepsilon^2)$. We consider three examples: (1) linear shear current

$$U(z) = U_s\left(1 + \alpha\frac{z}{h}\right) \tag{32}$$

where α is the normalized form of the constant vorticity ω_0 given by

$$\alpha = \omega_0 h/U_s \tag{33}$$

(2) cosine profile

$$U(z) = U_s \cos\left(\alpha \frac{z}{h}\right) \qquad (34)$$

(3) power law profile

$$U(z) = U_s \left(1 + \frac{z}{h}\right)^{1/7} \qquad (35)$$

These examples represent a hierarchy in increasing difficulty. For the linear shear profile, analytic solutions may be obtained for both the full problem and the perturbation solutions. (In fact, the perturbation may be carried to any order with no difficulty, as will be seen below.) For the cosine profile, analytic results may be obtained in the case of $c=0$ (i.e., a stationary wave). For $c\neq 0$, the problem must be solved numerically. In this case, however, the approximate results $\tilde{U}$ and c_2 are obtained explicitly. For the third case of a power law profile, analytic results again have been obtained only for the case $c=0$ [Lighthill, 1953]. Fenton [1973] has provided a numerical scheme for cases with $c\neq 0$ using a shooting method; this scheme is utilized below. For the case of a power law profile, evaluation of $\tilde{U}$ and c_2 also requires numerical approximation or approximation of the analytic result by means of truncated series expansions.

3.1. Linear Shear Current

For the case of a current with uniform vorticity, the stability problem posed in (2)-(4) reduces to

$$f'' - k^2 f = 0 \qquad -h < z < 0 \qquad (36)$$

$$(U_s - c)^2 f' = [g + \omega_0 (U_s - c)] f \qquad z = 0 \qquad (37)$$

$$f = 0 \qquad z = -h \qquad (38)$$

The solution to (36) and (38) is simply

$$f(z) = \sinh k(h+z) \qquad (39)$$

Substituting (39) in (37) then gives directly

$$c_E = U_s - \frac{\omega_0 \tanh kh}{2k} + \left(\frac{g \tanh kh}{k}\right)^{1/2} \left\{1 + \frac{\omega_0^2 \tanh kh}{4gk}\right\}^{1/2} \qquad (40)$$

which is the exact solution which we will denote by subscript E. Turning to the approximate solutions, we substitute (32) in (18) and obtain

$$\tilde{U} = U_s - \frac{\omega_0 \tanh kh}{2k} \qquad (41)$$

The exact solution may then be written as

$$c_E = \tilde{U} + c_0 \left\{1 + \frac{\omega_0^2 \tanh kh}{4gk}\right\}^{1/2} \qquad (42)$$

where c_0 is taken from (15). Evaluating the $O(\varepsilon^2)$ correction c_2 from (30) gives

$$c_2 = c_0 \left\{\frac{\omega_0^2 \tanh kh}{8gk}\right\} \qquad (43)$$

and the phase speeds to $O(\varepsilon)$ and $O(\varepsilon^2)$ are given by

$$c(\varepsilon) = c_0 + \tilde{U} \qquad (44)$$

$$c(\varepsilon^2) = c_0 \left\{1 + \frac{\omega_0^2 \tanh kh}{8gk}\right\} + \tilde{U} \qquad (45)$$

Comparing (42), (44), and (45), it is apparent that the $O(\varepsilon)$ approximation is fairly weak unless normalized ω_0^2 is in some sense $\ll 1$. The approximation to $O(\varepsilon^2)$ adds the first small term in the binomial expansion of the square root appearing in the exact solution. Referring to (30), it is apparent that c_2 also provides the leading-order correction for nonzero current shear.

The approximations obtained above are least accurate in shallow water. To inspect this limit, we normalize the phase speeds by $(gh)^{1/2}$ and define

$$F = U_s/(gh)^{1/2} \qquad c_0' = \left(\frac{\tanh kh}{kh}\right)^{1/2} = 1 + O(kh)^2$$

Further, the weighted mean current $\tilde{U}$ is given by

$$\tilde{U} = U_s - \frac{\omega_0 h}{2} + O(kh)^2 = \bar{U} + O(kh)^2 \qquad (46)$$

where $\bar{U}$ is the unweighted, depth-averaged current. ($\tilde{U}$ and $\bar{U}$ tend to converge as $kh \to 0$ for all velocity profiles as the wave motion loses its vertical structure; this may be verified by inspection of (18).) Introducing α defined in (33) and taking the limit $kh \to 0$, we may express the phase speeds relative to $\tilde{U}/(gh)^{1/2}$ or $\bar{U}/(gh)^{1/2}$ as

$$\text{Exact} \qquad c_{RE} = \left\{1 + \frac{\alpha^2 F^2}{4}\right\}^{1/2} \qquad (47a)$$

$$O(\varepsilon) \qquad c_{R1} = 1 \qquad (47b)$$

$$O(\varepsilon)^2 \qquad c_{R2} = 1 + \frac{\alpha^2 F^2}{8} \qquad (47c)$$

We may take $\alpha > 0$, where $\alpha = 1$ reduces the velocity to zero at the bed and $\alpha > 1$ indicates reversed flow at the bed relative to the surface. Ignoring the latter possibility, the worst case is for $\alpha = 1$, and we require $F \ll 2$ to employ the truncated binomial expansion implied by (47c). The approximation to $O(\varepsilon)^2$ is thus quite good for any subcritical flow. Plots of results for various choices of α and F are given in Figure 2 for $c_0' = 1$.

We remark that higher-order approximations to the problem posed by (36)-(38) may be obtained

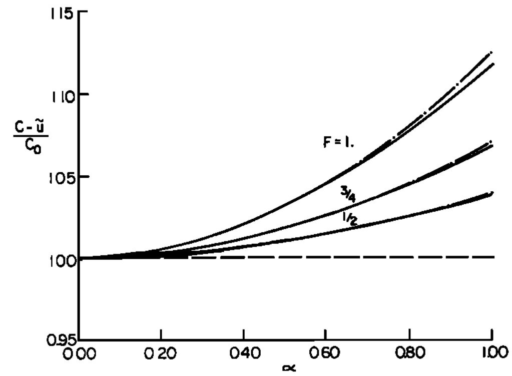

Fig. 2. Phase speed corrections $(c - \tilde{U})/c_0$ for long waves on a linear shear current. Solid lines show the exact solution, the dashed line is the $O(\varepsilon)$ approximation, and the dashed-dotted line is the $O(\varepsilon^2)$ approximation.

quite simply using a modification of the technique of section 2. Substituting the expansions for c in (37) and using (39) gives

$$(\varepsilon U_s - \sum_{n=0}^{\infty} \varepsilon^n c_n)^2 = \{g + \varepsilon\omega_0[\varepsilon U_s - \sum_{n=0}^{\infty} \varepsilon^n c_n]\}\frac{\tanh kh}{k}$$

$$= c_0^2\ \{1 + \frac{\varepsilon\omega_0}{g}\ [\varepsilon U_s - \sum_{n=0}^{\infty} \varepsilon^n c_n]\} \tag{48}$$

The next several approximations beyond the level attained in section 2 are

$$c_3 = 0 \qquad c_4 = -\frac{1}{8}\ \{\frac{\omega_0^2 \tanh kh}{4gk}\}^2\ c_0$$
$$c_5 = 0 \qquad c_6 = \frac{1}{16}\ \{\frac{\omega_0^2 \tanh kh}{4gk}\}^3\ c_0 \tag{49}$$

c_4 and c_6 are consistent with the next two smaller terms in the binomial expansion of (42), and it is apparent that the perturbation solution will eventually converge to the exact solution when ω_0^2/gk is suitably small. (Note that in the limit of $\omega_0 \to 0$, the solution to $O(\varepsilon)$ is exact for all current velocities, including stationary waves for which $U/c = \infty$.)

3.2 Cosine Profile

We now consider the cosine profile (equation (34)). Values of $0<\alpha<\pi/2$ produce unidirectional flow, with a uniform velocity for $\alpha=0$ and velocity reduced to zero at the bed for $\alpha=\pi/2$. Using the results of section 2.2, we obtain at $O(\varepsilon)$

$$\tilde{U} = U_s\ \frac{(1 + \beta\beta^*)}{(1 + \beta^2)} \tag{50}$$

where

$$\beta = \frac{\alpha}{2kh} \qquad \beta^* = \frac{\sin \alpha}{\sinh 2kh} \tag{51}$$

The particular solution $f_1(z)$ is given by

$$f_1(z) = \frac{U_s}{c_0(1 + \beta^2)}\cdot\{\beta[\sin \alpha + \sin(\frac{\alpha z}{h})]\cosh k(h+z)$$
$$- \beta^2[\cos \alpha + \cos(\frac{\alpha z}{h})]\ \sinh k(h+z)\} \tag{52}$$

Also, the depth-mean velocity $\bar{U}$ is given by

$$\bar{U} = U_s\ \sin \alpha/\alpha \tag{53}$$

and $\tilde{U}\to\bar{U}$ as $kh\to 0$ for arbitrary α. At $O(\varepsilon^2)$, we obtain

$$c_2 = \frac{U_s^2}{c_0}\ \{\frac{\beta^2}{1 + \beta^2}\frac{kh}{\sinh 2kh}$$
$$+ \beta\beta^*\cos \alpha[\frac{1}{1 + 4\beta^2} + \frac{\beta^2 - 1}{2(1 + \beta^2)^2}]$$
$$- (\frac{\beta\beta^*}{1 + \beta^2})^2\ \cosh 2kh$$
$$+ [\frac{1 + 2\beta^2}{2\ (1 + 4\beta^2)} - \frac{(1 + \beta\beta^*)^2}{2(1 + \beta^2)^2}]\} \tag{54}$$

The asymptotic validity of the approximate solution in the limit of weak vorticity ($\beta,\beta^* \ll 1$) may be investigated by comparison with the exact solution for the special case of $c = 0$ (stationary waves). Using this condition in (2)-(3) and then solving (2)-(4) for the cosine profile (equation (34)) leads to the results

$$f(z) = \sinh[kh(1 - 4\beta^2)^{1/2}] \tag{55}$$

$$U_s^2 = \frac{g \tanh[kh(1 - 4\beta^2)^{1/2}]}{k(1 - 4\beta^2)^{1/2}} \tag{56}$$

Expanding (56) for the case $\beta^2 \ll 1$ leads to the expression

$$U_s^2 = \frac{g \tanh kh}{k}\,\frac{(1 - 2\beta\beta^*)}{(1 - 2\beta^2)}$$
$$= \frac{g \tanh kh}{k}\,\frac{(1 + \beta^2)^2}{(1 + \beta\beta^*)^2} + O(\beta^4) \tag{57}$$

The second form of the right hand side is equivalent to the result of the approximate solution to $O(\varepsilon)$, given by

$$c_0 + \tilde{U} = 0 \quad \Rightarrow \quad \tilde{U}^2 = c_0^2 \tag{58}$$

We see again that the approximate solution (for weak U_s) converges to the exact solution under the condition of weak vorticity with no restriction on U_s, as in the constant vorticity case. In particular, $U/c \to \infty$ for this case, and the perturbation method is seemingly inapplicable.

For cases where $c \neq 0$, the full solution must be obtained numerically. We have obtained solutions here using a modification of Fenton's [1973] shooting method. Referring back to (2)-(4), we define a Froude number F according to

$$F = \frac{U(z) - c}{(gh)^{1/2}} \tag{59}$$

(Note the difference from F in section 3.1.) We nondimensionalize z according to $z' = z/h$ and define the variable transformation

$$q(z') = f(z)/\left(hf'(z)\right) \tag{60}$$

The problem is then reduced to a Riccati equation

$$q' = 1 - \gamma^2 q^2 \qquad -1 < z' < 0 \tag{61}$$

where

$$\gamma^2 = (kh)^2 \left\{\frac{(1 - 4\beta^2)\ \delta \cos(\alpha z') - 1}{\delta \cos(\alpha z') - 1}\right\} \tag{62}$$

$$\delta = U_s/c \tag{63}$$

Equation (61) is solved over the interval $-1 < z' < 0$ with initial condition

$$q(-1) = 0 \tag{64}$$

and with kh, α (and hence β), and δ specified. The surface boundary condition becomes

$$\frac{c^2}{gh} = q(0)/(\delta - 1)^2 \tag{65}$$

which determines c. U_s is then determined using (63). The numerical scheme may be tested for accuracy against the long wave result ($kh \to 0$), which is determined directly from the expression [Burns, 1953]

$$1 = g \int_{-h}^{0} \frac{dz}{(U(z) - c)^2} \tag{66}$$

and gives

$$\frac{c^2}{gh} = \frac{\delta \sin \alpha}{\alpha(1-\delta^2)(1-\delta \cos \alpha)} + \frac{2}{\alpha(1-\delta^2)^{3/2}} \tan^{-1}\left\{\frac{(1+\delta)\tan\left(\frac{\alpha}{2}\right)}{(1-\delta^2)^{1/2}}\right\} \qquad \delta<1 \tag{67a}$$

$$\frac{c^2}{gh} = \frac{\delta \sin \alpha}{\alpha(1-\delta^2)(1-\delta \cos \alpha)} - \frac{1}{\alpha(\delta^2-1)^{1/2}} \ell n \left| \frac{(1+\delta)\tan\left(\frac{\alpha}{2}\right) - (\delta^2-1)^{1/2}}{(1+\delta)\tan\left(\frac{\alpha}{2}\right) + (\delta^2-1)^{1/2}} \right| \qquad \delta>1 \tag{67b}$$

which is implicit in c and where δ is given by (63). Five decimal place accuracy was achieved straightforwardly in the numerical solution.

In Figure 3 we show the numerically determined exact dispersion relation for the cosine profile for the case $\alpha=\pi/2$ (velocity = 0 at bottom). The Froude number chosen is based on the surface current speed. Consideration is restricted to subcritical mean flow conditions in the long wave limit. In Figure 4 we display the absolute value of the absolute error $c_n - c_E$ for the long-wave limit and a range of α values, with subscript n denoting the order of approximation and c_n representing the perturbation solution. Figures 4a and 4b display results for the first- and second-order approximations, respectively. The second-order approximation is seen to provide an order of magnitude reduction in error in comparison to the first-order approximation (indeed, for the range of parameters considered it proves inconvenient to plot the two results on equivalent scales).

Figures 5a and 5b show first- and second-order results, respectively, for $|c_n - c_E|$ for the case of $\alpha = \pi/2$ and a range of kh values. Here, the comparison is between perturbation solutions and numerical solutions. The dramatic increase in accuracy given by the second-order approximation is again apparent.

3.3 1/7-Power Law Profile

We now turn to the 1/7-power profile given by (35). This profile differs from the previous examples both in analytic complexity and in the fact that a weak-vorticity range is not available through choice of parameters, owing to the form of the profile near the bed. Using the results of section 2.2, the expression $\tilde{U}$ at $O(\varepsilon)$ is given by

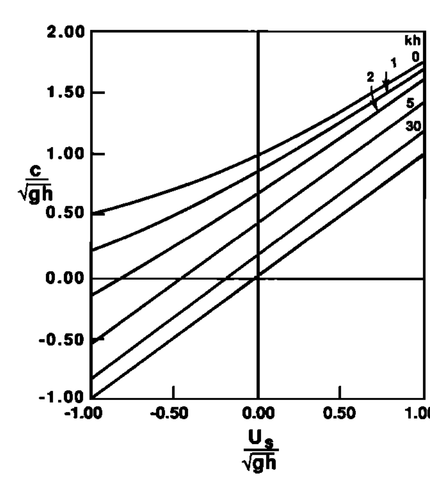

Fig. 3. Cosine profile; $\alpha=\pi/2$, numerical results for dispersion relation.

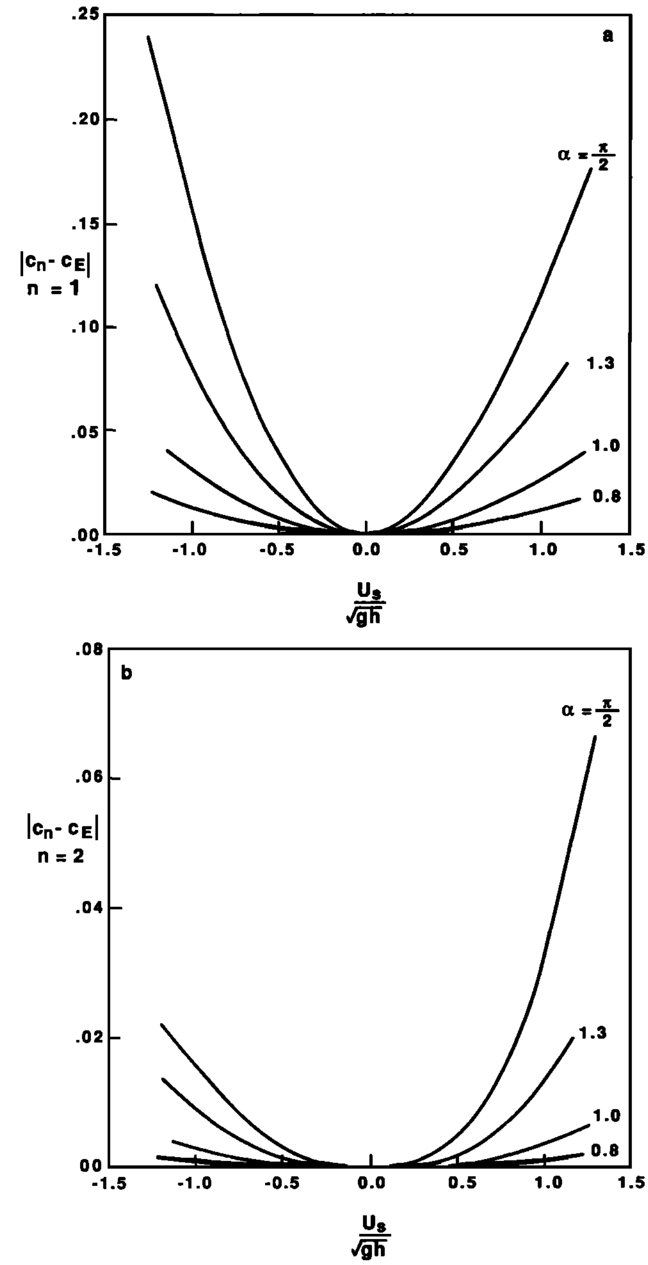

Fig. 4. Absolute error $|c_n - c_E|$ for cosine profile; kh=0, for (a) n=1, first-order approximation, and (b) n=2, second-order approximation.

$$\tilde{U} = \frac{2kU_s h^{-1/7}}{\sinh 2kh} \int_{-h}^{0} (h+z)^{1/7} \cosh 2k(h+z)dz \qquad (68)$$

This expression is of little direct use. Two integrations by parts yield

$$\tilde{U} = U_s\left[1 - \frac{\tanh kh}{14kh} - \frac{6}{7kh \sinh 2kh} \int_0^1 t^{-7} \sinh^2 kht^7 dt\right] \qquad (69)$$

where $t = (1 + z/h)^{1/7}$. This expression has been given previously by Hunt [1955]. The remaining integral in (69) vanishes in the limit $kh \to \infty$ but contributes significantly at finite values of kh (see below). Approximation is thus required in order to evaluate the integral. Numerical experiments with quadrature indicated that 32 weighting points were required in order to obtain three decimal place accuracy for kh = 1, with the number increasing for increasing kh. For this reason, we chose to treat the integral by taking the Taylor series expansion for cosh 2k(h+z) about h+z = 0

$$\cosh 2k(h+z) = \sum_{n=0}^{\infty} \frac{(2k)^{2n}(h+z)^{2n}}{(2n)!} \qquad (70)$$

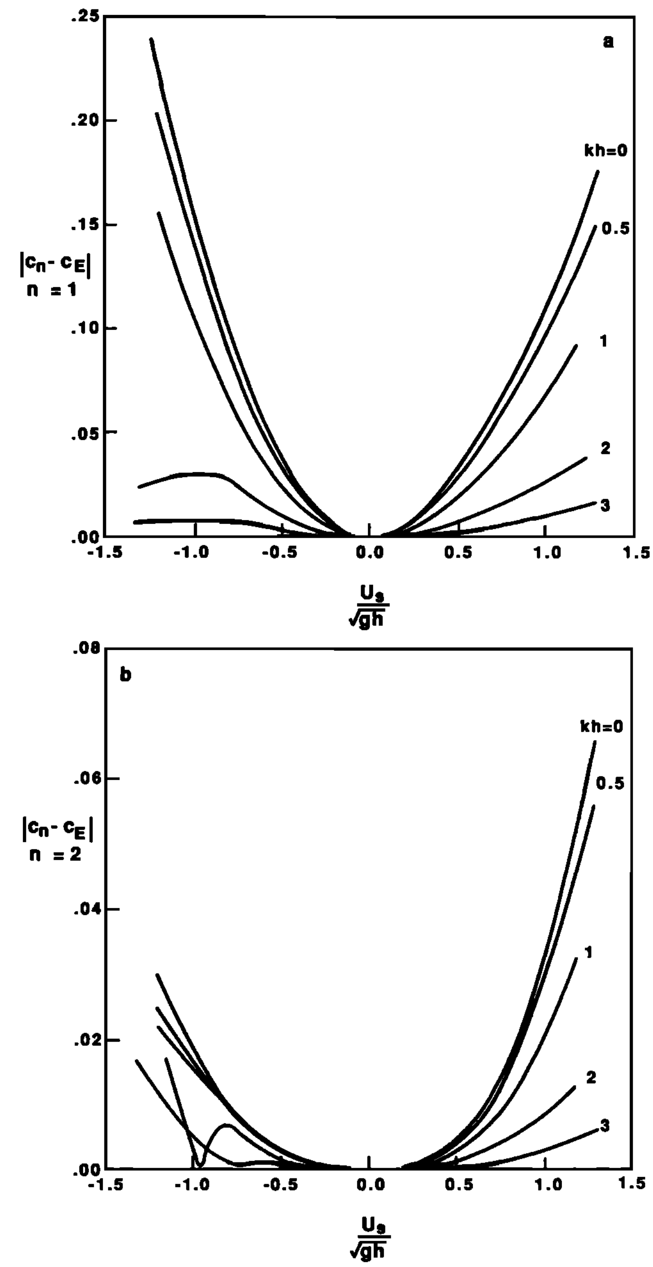

Fig. 5. Absolute error $|c_n - c_E|$ for cosine profile, $\alpha=\pi/2$, for (a) n=1, first-order approximation, and (b) n=2, second-order approximation.

and then summing the resulting expression for the integral up to the required number of terms. The resulting expression is given by

$$\tilde{U} = \frac{14kh\,U_s}{\sinh 2kh} \sum_{n=0}^{N} \frac{(2kh)^{2n}}{(2n)!(14n + 8)} \tag{71}$$

Values of N required to obtain three decimal place accuracy are given for various kh in Table 1. The expansion procedure is most appropriate for shallow water, with convergence being obtained more slowly as depth increases.

As $kh \to 0$, we obtain the shallow water limit

$$\tilde{U} \to \frac{7}{8} U_s = \bar{U} \tag{72}$$

Exact numerical results indicate that higher-order effects are not as important in this case as in previous examples, and so the solution is not carried to $O(\varepsilon^2)$ here. This result is due to the weak vorticity of the current profile near the surface; only the longest waves are strongly affected by the current shear. Reference may be made to the results of Thomas [1981], who conducted experiments on nearly deepwater waves over a turbulent shear flow of nearly 1/7-power form and found the waves to respond only to the surface current speed.

Numerical results were obtained following Fenton [1973] and the procedure outlined in section 3.2. Using δ defined by (63), and $q(z')$ defined by (60), we obtain the problem (as per Fenton [1973])

$$q' = 1 - \gamma^2(z')q^2 \qquad -1 < z < 0 \tag{73}$$

$$q(-1) = 0 \tag{74}$$

where now

$$\gamma^2 = (kh)^2 + \frac{6\delta(1 + z')^{-13/7}}{49(1 - \delta(1 + z')^{1/7})} \tag{75}$$

The surface boundary condition yields the relation

$$\frac{F(0)^2}{[1 + F(0)F'(0)]} = q(0) \tag{76}$$

where $F = (U-c)/(gh)^{1/2}$. This result leads to the expression

TABLE 1. Rate of Convergence of $\tilde{U}$ (Equation (71)) for Varying kh

kh	N
0.5	3
1.0	4
2.5	8
5.0	14

N is the number of terms required in the series expansion in order to obtain three-place accuracy.

$$\frac{c^2}{gh} = \frac{q(0)}{(\delta - 1)[(\delta - 1) - \delta q(0)/7]} \tag{77}$$

Equations (73)-(74) are solved using specified values of kh and δ. We then use (77) to determine c and then (63) to determine U_s. Numerical results were checked against plots given by Fenton and also against the long-wave analytic result, given by

$$\frac{U_s^2}{gh} = 7\Big[\frac{1}{5} + \frac{1}{2}\Big(\frac{c}{U_s}\Big) + \Big(\frac{c}{U_s}\Big)^2 + 2\Big(\frac{c}{U_s}\Big)^3 + 5\Big(\frac{c}{U_s}\Big)^4 + 6\Big(\frac{c}{U_s}\Big)^5 \ln\Big(\frac{c - U_s}{c}\Big) + \Big(\frac{c}{U_s}\Big)^4\Big(\frac{c}{c - U_s}\Big)\Big] \tag{78}$$

Note that (78) corrects an error appearing in Fenton's unnumbered expression in the logarithmic term.

Plots of normalized phase speed versus normalized U_s are given in Figure 6 as solid curves for kh values ranging from 0 to ∞ (long to short waves). Also plotted in Figure 6 are results of the $O(\varepsilon)$ perturbation solution given as dashed lines. The perturbation solution is seen to be in close agreement with the full solution except for kh small and $|U_s/(gh)^{1/2}|$ large, representing long waves on strong currents.

The expression (equation (71)) used to determine $\tilde{U}$ was found to be useful at all water depths tested but converges very slowly for large values of kh. An alternate expansion procedure for (68) could be based on using the binomial expansion

$$\Big(1 + \frac{z}{h}\Big)^{1/7} = 1 + \frac{1}{7}\frac{z}{h} - \frac{3}{49}\Big(\frac{z}{h}\Big)^2 + \frac{13}{343}\Big(\frac{z}{h}\Big)^3 + \ldots \tag{79}$$

Substituting (79) in (68) leads to an expression for $\tilde{U}$ which increases in accuracy as more terms in the expansion are retained. The first several expressions obtained in this manner are given by

Two terms

$$\tilde{U} = U_s\Big(1 - \frac{\tanh kh}{14kh}\Big) \tag{80a}$$

Three terms

$$\tilde{U} = U_s\Big(1 - \frac{\tanh kh}{14kh} + \frac{3}{49kh}\Big\{\frac{1}{\sinh 2kh} - \frac{1}{2kh}\Big\}\Big) \tag{80b}$$

Four terms

$$\tilde{U} = U_s\Big(1 - \frac{\tanh kh}{14kh} + \frac{81}{686kh \sinh 2kh} - \frac{3}{98(kh)^2} - \frac{78\tanh kh}{2744(kh)^3}\Big) \tag{80c}$$

These expressions represent an ascending series in powers of $(kh)^{-1}$ as $kh \to \infty$, and convergence is

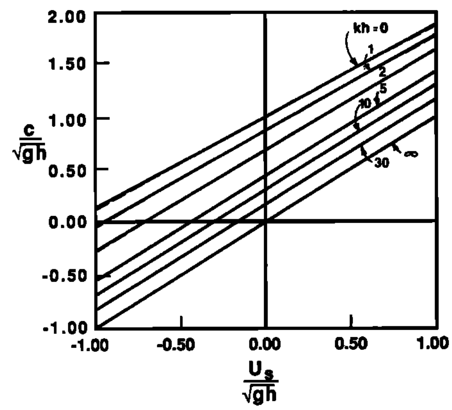

Fig. 6. Dispersion relation for 1/7-power profile. Solid lines indicate numerical results; dashed lines indicate first-order approximation.

rapid. As $kh \to 0$, however, all terms beyond the first two in a truncated expansion cancel identically, and the value of $\tilde{U} \to 0.929\ U_s$. Note that the third term in (69) contributes all the higher-order terms in the expansion; however, the $O(1)$ contribution as $kh \to 0$ is not obtained in a truncated expansion because the expression $(1 + z/h)^{1/7}$ is not differentiable at $z=-h$. Convergence of the truncated series is thus limited to a range excluding the neighborhood of $kh=0$.

Figure 7 shows several expressions for $\tilde{U}$. The solid curve represents the series (equation (71)) with a sufficient number of terms retained to obtain convergence. Each of the three truncated expansions (equations (80a)-(80c)) is also included, and shows the increase in accuracy with each included term as well as the failure of the series at $kh=0$. Finally, a plot of (71) truncated to 10 terms is included and shows the extreme sensitivity of the series to the number of retained terms as kh increases. It is noted also that the expression for $\tilde{U}$ given by (80a) has a maximum relative error of only 5.8% in the shallow water limit.

4. Expansion for Weak Vorticity

The results of sections 3.1 and 3.2 have indicated that the approximate solutions for the regime of weak currents with arbitrary vorticity are seemingly valid for the regime of arbitrarily strong currents having weak vorticity. This result may be expected in hindsight owing to the form of the approximate solutions. In particular, the weighted mean current $\tilde{U}$ deviates from the true mean current $\bar{U}$ by an amount proportional to the first power of some vorticity parameter, while the expressions for c_2/c_0 are proportional to some (current parameter x vorticity parameter)^2. Thus in the limit of small vorticity, $\tilde{U} \to \bar{U}$, and c_2/c_0 represents a consistently small correction for arbitrarily large $\bar{U}$ values. In order to further support these results and claims, we provide the schematic for an expansion for weak vorticity and examine the case of a linear shear current.

We proceed by defining a reference current U^* according to

$$U(z) = U^* + \hat{U}(z) \tag{81}$$

where $|\hat{U}/U^*| \ll 1$ owing to the weak vorticity assumption. From the results of section 3, a natural choice for U^* would be $\tilde{U}$; however, $\tilde{U}$ must be regarded as undetermined in the present context since it was found for the case $|U^*/c| \ll 1$, which does not hold here. We thus take $U^* = \bar{U}$. For the case of a linear shear current, we obtain

$$\bar{U} = U_s - \frac{\omega_0 h}{2} \qquad \hat{U}(z) = \omega_0\left(z + \frac{h}{2}\right) \tag{82}$$

from (32). Equations (2)-(4) may then be solved to obtain

$$f(z) = \sinh k(h+z) \tag{83}$$

subject to the condition

$$(\bar{U} + \epsilon\hat{U}(0) - c)^2 =$$
$$[g + \hat{U}'(0)(\bar{U} + \epsilon\hat{U}(0) - c)]\ \frac{\tanh kh}{k} \tag{84}$$

where $\epsilon \ll 1$. Introducing the expansion

$$c = \sum_{n=0}^{\infty} \epsilon^n c_n \tag{85}$$

and solving sequentially for the c_n then gives (to $O(\epsilon^2)$)

$$c_0 = \left(\frac{g \tanh kh}{k}\right)^{1/2} + \bar{U} \tag{86a}$$

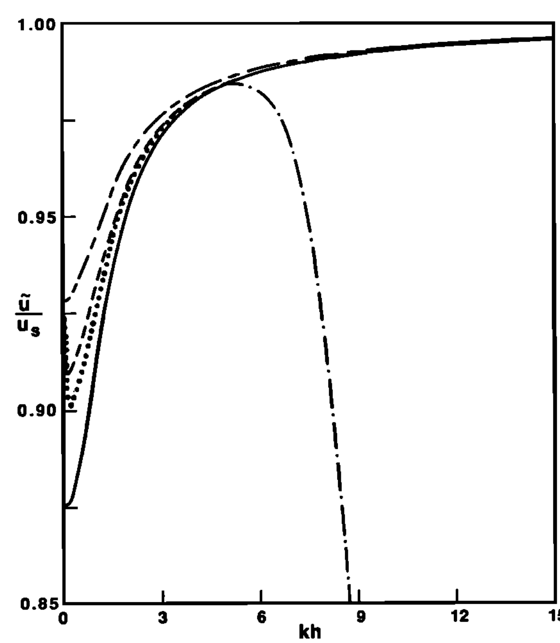

Fig. 7. Various estimates of $\tilde{U}$ for the 1/7-power profile: solid line, full series (equation (71)); dotted line, (80a); short-dashed line, (80b); long-dashed line, (80c); and dashed-dotted line, 10-term expansion (equation (71)).

$$c_1 = \frac{\omega_0 h}{2} - \frac{\omega_0 \tanh kh}{2k} \tag{86b}$$

$$c_2 = \left(\frac{g \tanh kh}{k}\right)^{1/2} \left(\frac{\omega_0^2 \tanh kh}{8gk}\right) \tag{86c}$$

or

$$c = \left(\frac{g \tanh kh}{k}\right)^{1/2} \left\{1 + \frac{\omega_0^2 \tanh kh}{8gk}\right\}$$

$$+ \bar{U} + \frac{\omega_0 h}{2} - \frac{\omega_0 \tanh kh}{2k} + O(\varepsilon^3) \tag{87}$$

Using (82) and (41), (87) may be written as

$$c = \left(\frac{g \tanh kh}{k}\right)^{1/2} \left\{1 + \frac{\omega_0^2 \tanh kh}{8gk}\right\} + \tilde{U} \tag{88}$$

where $\tilde{U}$ is given by (41). Equation (88) is identical to (45), and the two expansions are seen to be equivalent.

5. Results for Deepwater Waves

Skop [1987] has investigated the application of the first-order approximation of Stewart and Joy [1974] in the deepwater limit to several cases for which analytic results exist, including the general case for depth-limited current profiles of zero and constant vorticity [Taylor, 1955] and the case of stationary waves on these profiles as well as cosine and exponential profiles (tabulated by Peregrine and Smith [1975]). Skop has shown that the first-order approximation provides generally good estimates of the wave parameters. However, the second-order approximation for deepwater waves is particularly simple in form, and we include it here for comparison.

The deepwater formulation follows from (2)-(4) by replacing the bottom boundary condition with a boundedness condition on f; in the remainder of the formulation, lower limits of integration become $-\infty$. The results for the wave phase speed become

$$c = c_0 + c_1 + c_2 \tag{89}$$

where

$$c_0 = (g/k)^{1/2} \tag{90}$$

$$c_1 = \tilde{U} = 2k \int_{-\infty}^{0} U(z) e^{2kz}\, dz \tag{91}$$

$$c_2 = \frac{k}{c_0} \int_{-\infty}^{0} U^2(z) e^{2kz}\, dz - c_1^2/2c_0 \tag{92}$$

where c_1 was given by Stewart and Joy [1974] and c_2 is new here. These results may also be obtained by taking the appropriate limits of (15), (18) and (31) directly.

5.1. Comparison With Analytic Results

We consider first the case of waves propagating on opposing depth-limited currents. The case of a uniform current is given by

$$\begin{aligned} U(z) &= -U_s \qquad -d < z < 0 \\ U(z) &= 0 \qquad z < -d \end{aligned} \tag{93}$$

In order to facilitate comparison with Skop's results, we introduce dimensionless variables according to

$$\Omega = \frac{kdc}{U_s} \qquad S = \frac{(gd)^{1/2}}{U_s} \qquad \kappa = kd \tag{94}$$

where Ω is dimensionless frequency and S is an inverse Froude number. From Taylor [1955], the exact solution gives the dispersion relation

$$[(\Omega_E + \kappa)^4 - \Omega_E^2 S^2 \kappa] \tanh \kappa$$

$$+ (\Omega_E + \kappa)^2 (\Omega_E^2 - S^2 \kappa) = 0 \tag{95}$$

where subscript E denotes the exact value of Ω.

The first-order approximation is given by [Skop, 1987]

$$\Omega_1 = \kappa^{1/2} S + \kappa(1 - e^{-2\kappa}) \tag{96}$$

while the second-order approximation is given by

$$\Omega_2 = \kappa^{1/2} S + \kappa(1 - e^{-2\kappa})\left(1 + \frac{\kappa^{1/2}}{2S} e^{-2\kappa}\right) \tag{97}$$

Skop has presented plots of Ω_E and Ω_1 versus κ for a range of values of S. In Figure 8 we provide plots of absolute error $\Omega_n - \Omega_E (n = 1,2)$ versus κ for a range of S values. For $S < 2$, the trend of increased accuracy for the second-order approximation breaks down, indicating divergence of the expansion for the case of strong currents in deep water.

For the case of stationary waves on a current, $\Omega = 0$ and we determine the value of $S(\kappa)$. The exact result is given by

$$S_E = (\kappa \tanh \kappa)^{1/2} \tag{98}$$

and the two approximations are given by

$$S_1 = \kappa^{1/2} (1 - e^{-2\kappa}) \tag{99}$$

$$S_2 = \frac{S_1}{2} \left\{1 + \left(1 - \frac{2(\kappa^{1/2} - S_1)}{S_1}\right)^{1/2}\right\} \tag{100}$$

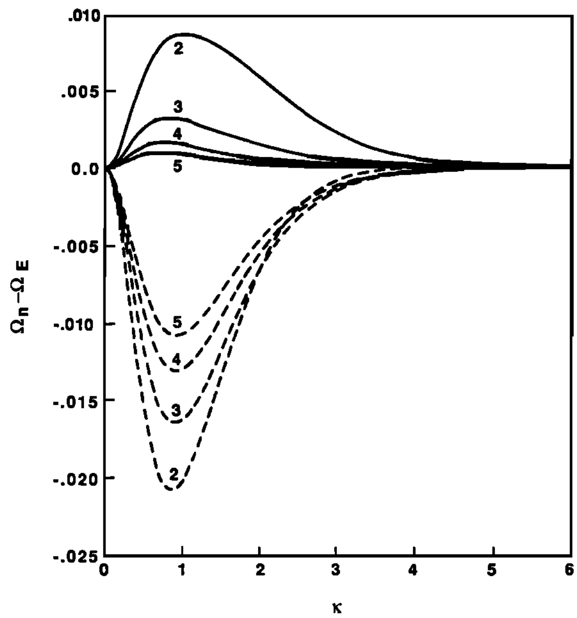

Fig. 8. Absolute frequency error $\Omega_n - \Omega_E$ for depth-limited, opposing uniform current in deep water. Solid lines indicate $O(\varepsilon^2)$ approximation (n=2); dashed lines indicate $O(\varepsilon)$ approximation (n=1). Curve labels are values of S.

Each form is asymptotic to the limit $S = \kappa^{1/2}$ as $\kappa \to \infty$. The results here also mirror problems with the expansion for strong currents, in that the term inside the square root in (100) has a zero at positive κ given by $\kappa_{CR} = -1/2\ \ell n(1/3) = 0.549$, for which the corresponding exact stopping inverse Froude number $S_E = 0.524$. For $\kappa < \kappa_{CR}$, the second order solution cannot predict the value of the stopping current. In contrast, the first-order solution predicts the value reasonably well for the entire range of κ, with increasing error as $\kappa \to 0$. Note that

$$\lim_{\kappa \to 0} S_E = \kappa \qquad (101)$$

$$\lim_{\kappa \to 0} S_1 = 2\kappa^{3/2} \qquad (102)$$

A plot of absolute error for $S_n - S_E$; n=1,2 is given in Figure 9. The first-order approximation is essentially accurate for $\kappa > 2$ ($S_E > 1.39$), while accuracy in the second-order approximation is deferred to $\kappa > 5$ ($S_E > 2.24$). This range of validity is likely to still be representative of relevant field conditions (note that for a surface current of speed 1 m s^{-1}, S = 2.25 implies a depth of flow of 51.7 cm; increased depth of flow further increases S and strengthens the validity of the second-order expansion).

A second case for which analytic results are available is the case of a linearly sheared jet

$$U(z) = -U_s\left(1 + \frac{z}{d}\right) \qquad -d < z < 0$$
$$U(z) = 0 \qquad z < -d \qquad (103)$$

The exact dispersion relation [Taylor, 1955] is given by

$$(\Omega_E + \kappa)\{1 - e^{-2\kappa} + (\Omega_E - \kappa) - (\Omega_E + \kappa)[2\Omega_E + e^{-2\kappa}]\}$$
$$+ \kappa S^2[2\Omega_E + 1 - e^{-2\kappa}] = 0 \qquad (104)$$

and first and second approximations are given by

$$\Omega_1 = \kappa^{1/2} S - \kappa\{1 - \frac{(1 - e^{-2\kappa})}{2\kappa}\} \qquad (105)$$

$$\Omega_2 = \Omega_1 + \frac{\kappa^{1/2}}{2S}\{\frac{1}{4\kappa}(1 - e^{-4\kappa}) - e^{-2\kappa}\} \qquad (106)$$

Plots of absolute error $\Omega_E - \Omega_n$; n = 1,2 are given in Figure 10 for $0 < \kappa < 6$ and $2.5 < S < 4$. For this case, the improvement afforded by the second-order approximation is dramatic. This result is most likely due to enhanced representation of the effect of surface shear, as was noted in the results for linear shear currents in section 3.

The prediction of stopping currents leads to the exact formula

$$S_E = (\kappa \coth \kappa - 1)^{1/2} \qquad (107)$$

and the approximations

$$S_1 = \kappa^{1/2}\left(1 - \frac{(1 - e^{-2\kappa})}{2\kappa}\right) \qquad (108)$$

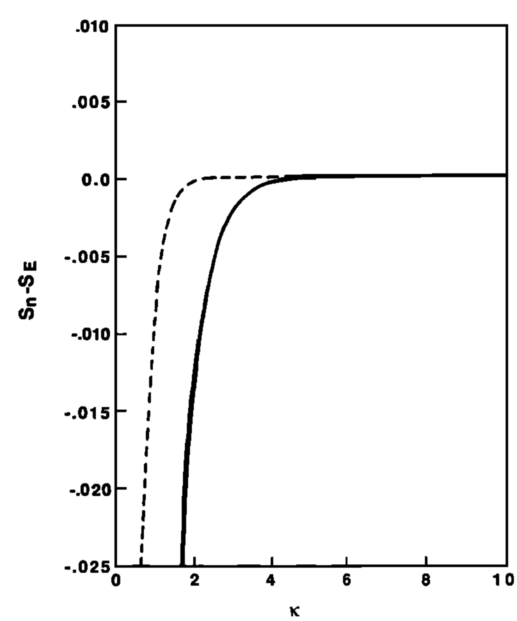

Fig. 9. Absolute error $S_n - S_E$ for predicted stopping current, for opposing depth-limited current in deep water. The solid line shows the $O(\varepsilon^2)$ approximation (n=2); the dashed line is the $O(\varepsilon)$ approximation (n=1).

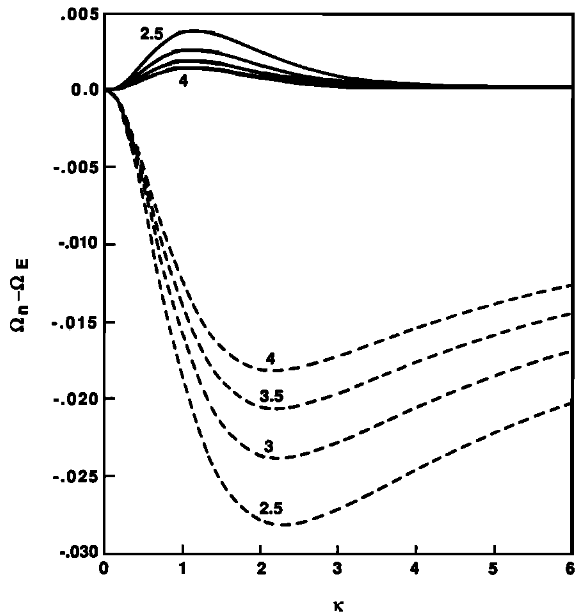

Fig. 10. As in Figure 8 for depth-limited, opposing linear shear current in deep water.

$$S_2 = \frac{S_1}{2}\{1 + [1 - \frac{2}{S_1^2}(\frac{1}{4\kappa}(1 - e^{-4\kappa}) - e^{-2\kappa})]^{1/2}\} \tag{109}$$

The expression under the square root in (109) again has a zero at $\kappa_{CR} = 0.78633$, which corresponds to a stopping current $S_E = 0.44506$. The second-order approximation gives no prediction of S below this value of κ_{CR}. Plots of absolute error $S_E - S_n$ are included in Figure 11. For this case, the error in predicted stopping current obtained from the second-order approximation is reduced essentially to zero for $\kappa > 5$, which corresponds to $S = 2.0001$ from (107). The approach of the first-order prediction to the exact solution is deferred to much higher values of κ and is not as qualitatively satisfactory, possibly again because of inadequate representation of the effects of surface current shear.

5.2 Exponential and Linear Shear Profiles

Wave-current interaction assumes a role of great importance in the theory of generation of wind waves. Waves generated by wind action interact with a wind-driven, sheared current profile with a thickness of the order of the wavelength. Several recent studies have shown that the dispersion properties of the initial wavelets are not strongly dependent on the form of the current chosen as long as the current profile reproduces the value of the current and shear existing at the surface. (See Gastel et al. [1985] for a recent contribution.)

In this section, we compare the dispersion relation for an exponential current profile

$$U(z) = U_s e^{z/d} \qquad z < 0 \tag{110}$$

to the dispersion relation for a depth-limited profile having the same velocity and surface shear, namely,

$$U(z) = U_s\left(1 + \frac{z}{d}\right) \qquad -d < z < 0$$
$$U(z) = 0 \qquad z < -d \tag{111}$$

These profiles have total mass flux rates differing by a factor of 2 but have very similar structure close to the surface, where the linear profile neglects terms of $O(z/d)^2$ in (110) where d is the e-folding length scale for decay of the exponential profile. To the second order of approximation in the present theory, the dispersion relations for the two profiles are given by

$$\Omega_{exp} = S\kappa^{1/2} + \frac{2\kappa^2}{(2\kappa+1)} + \frac{\kappa^{3/2}}{S}\left\{\frac{\kappa}{2(\kappa+1)} - \frac{2\kappa^2}{(2\kappa+1)^2}\right\} \tag{112}$$

$$\Omega_{\ell in} = S\kappa^{1/2} + \kappa - \frac{1}{2}(1 - e^{-2\kappa}) + \frac{\kappa^{1/2}}{2S}\left\{\frac{1}{4\kappa}(1 - e^{-4\kappa}) - e^{-2\kappa}\right\} \tag{113}$$

where U_s is defined positive for a following current (k>0) and where the notation of the previous section is retained. Figure 12 shows a plot of Ω versus κ for a range of S values. There is close agreement between the two approximate dispersion relationships. This result suggests that a velocity potential solution based on a depth-limited linear shear profile could be used to some advantage in the study of initial wave growth, since three-dimensional effects could be handled more simply than is possible when the analysis is based on a stream function.

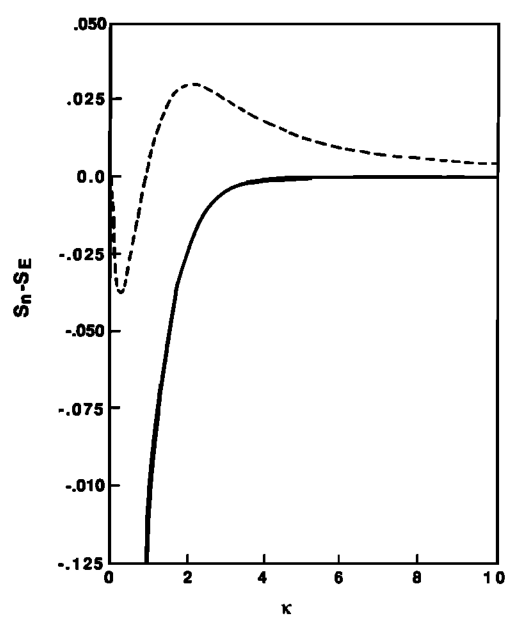

Fig. 11. As in Figure 9 for depth-limited opposing linear shear current in deep water.

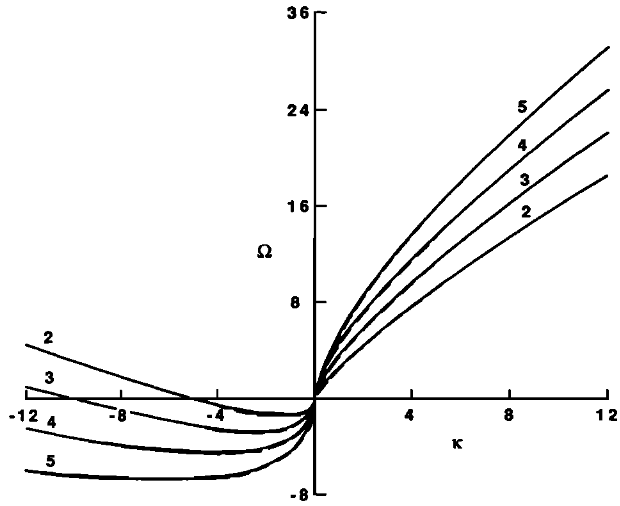

Fig. 12. Comparison of dispersion relations for exponential profile (equation (110), solid lines) and linear profile (equation (111), dashed lines) having equal speed and shear at z=0.

We remark that agreement between the first-order approximations for the two profiles considered is also close but that there is a general overall deviation between the dispersion curves for the first and second approximations, reflecting the reduced accuracy of the first-order approximation. In reference to the discussion of the limited range of validity of the second-order approximation, we consider the basic no-wave state described in Figure 1 of Gastel et al. For this case, d is approximately 5 mm with $U_s = 0.08$ m s^{-1}, yielding a value S = 2.77, which is well up into the range of validity of the present approximations. Capillarity is neglected here and would significantly alter the expressions (112)-(113) at the length scale for this particular example.

6. Comments on Action Flux Conservation

One of the chief applications for approximate dispersion relations for wave-current interaction is in the construction of models for waves in slowly varying domains. Such an application deserves a detailed analysis in its own respect and will be the subject of further work, which in any case is necessitated by the findings below; here, we can provide some initial results, using the results of the O(ε) problem in the context of irrotational wave theory. In particular, Skop [1987] suggests that the velocity $\tilde{U}$ obtained in section 2.2 may be used as the basis for the wave-current interaction in propagation models, but he provides no further analysis or support. Here, we proceed using such an assumption and then analyze the results for the special case of a linear shear current, using the results of Jonsson et al. [1978] as the basis for analytic comparisons.

We consider a linear wave riding on a flow of uniform-over-depth velocity $\tilde{U}$, given by

$$\phi = \mathrm{Re}\left\{-\frac{iga}{2\tilde{\sigma}}\frac{\cosh k(h+z)}{\cosh kh}e^{i\psi}\right\} + \int \tilde{\underset{\sim}{U}}\cdot d\underset{\sim}{x} \qquad (114)$$

$$\eta = \mathrm{Re}\{ae^{i\psi}\} - \frac{|\tilde{U}|^2}{2g} \qquad (115)$$

where

$$\underset{\sim}{k} = \nabla_h\psi \qquad \omega = -\psi_t \qquad (116)$$

subject to the dispersion relation

$$\omega = \tilde{\sigma} + \underset{\sim}{k}\cdot\tilde{\underset{\sim}{U}} \qquad \tilde{\sigma} = (gk\tanh kh)^{1/2} \qquad (117)$$

which is a simple extension of (21) to two dimensions. Following Kirby [1984], (114) may be used as a trial function in a variational principle due to Luke [1967], leading to a wave equation

$$\frac{D^2\tilde{\phi}}{Dt^2} + (\nabla_h\cdot\tilde{\underset{\sim}{U}})\frac{D\tilde{\phi}}{Dt} - \nabla_h\cdot(cc_g\nabla_h\tilde{\phi}) + (\tilde{\sigma}^2 - k^2cc_g)\tilde{\phi} = 0 \qquad (118)$$

where

$$\phi = \tilde{\phi}\cosh k(h+z)/\cosh kh \qquad (119)$$

and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \tilde{\underset{\sim}{U}}\cdot\nabla_h \qquad c = \frac{\tilde{\sigma}}{k} \qquad c_g = \frac{\partial\tilde{\sigma}}{\partial k} \qquad (120)$$

We allow $\tilde{\underset{\sim}{U}}$ and h to have slow spatial derivatives and a (the amplitude) to vary slowly in space and time. Taking a and ψ to be real functions allows (118) to be reduced to an eikonal equation for ψ and a transport equation given by

$$\left(\frac{\tilde{E}}{\tilde{\sigma}}\right)_t + \nabla_h\cdot\left(\frac{\tilde{E}}{\tilde{\sigma}}(\underset{\sim}{c}_g + \tilde{\underset{\sim}{U}})\right) = 0 \qquad (121)$$

where $\tilde{E}$ is given by the simple expression

$$\tilde{E} = \frac{1}{2}\rho ga^2 \qquad (122)$$

and where $\underset{\sim}{c}_g = c_g\underset{\sim}{k}/k$. The quantity $\tilde{E}/\tilde{\sigma}$ is an estimate of the wave action density, and (121) expresses the conservation of flux of wave action. The question to be addressed is whether

$$\tilde{F} = \frac{\tilde{E}}{\tilde{\sigma}}(\underset{\sim}{c}_g + \tilde{\underset{\sim}{U}}) = \frac{\tilde{E}}{\tilde{\sigma}}\tilde{\underset{\sim}{c}}_{g_a} \qquad (123)$$

is a proper estimate of action flux to the level of approximation considered here.

Jonsson et al. [1978] give an exact expression for action flux on a linear shear current in one direction, which we write here as

$$F = \frac{E}{\bar{\sigma}}c_{g_a} \qquad (124)$$

where

$$\bar{\sigma} = \omega - k\bar{U} \qquad (125)$$

and

$$c_{g_a} = c_{g_{rs}} + U_s = \frac{\partial}{\partial k}(kc_{rs}) + U_s \tag{126}$$

where c_{rs} is the phase speed relative to the surface current, given by (40) as

$$c_{rs} = \left(\frac{g \tanh kh}{k}\right)^{1/2}\left(1 + \frac{\omega_0^2 \tanh kh}{4gk}\right)^{1/2} - \frac{\omega_0 \tanh kh}{2k} \tag{127}$$

Considering terms only to $O(\varepsilon)$, it is apparent that

$$c_{rs} + U_s = c_0 + \tilde{U} + O(\varepsilon^2) \tag{128}$$

and hence $\tilde{c}_{g_a}$, given by

$$\tilde{c}_{g_a} = \frac{\partial}{\partial k}\{kc_0 + k\tilde{U}\} = c_{g_a} + O(\varepsilon^2) \tag{129}$$

is the correct advection velocity to $O(\varepsilon)$. However, note that

$$\tilde{c}_{g_a} = \frac{\partial\tilde{\sigma}}{\partial k} + \tilde{U} + k\frac{\partial\tilde{U}}{\partial k} \tag{130}$$

is not equivalent to the simple estimate obtained from irrotational theory, where $\tilde{U}$ is entered simply as a local estimate of depth-uniform velocity, and hence is not apparently a function of k. It is necessary to take this dependence into account explicitly in arriving at the correct expression for the group velocity in the absolute reference frame. Details of a comparison of the expressions for c_{g_a} and $\tilde{c}_{g_a}$ are given in the appendix.

Turning to the expression for wave action, we may write the exact expression for E for a linear shear current (from Jonsson et al.) as

$$E = \frac{1}{2}\rho g a^2 \left(1 - \frac{\omega_0 c_{rs}}{2g}\right)\frac{\bar{\sigma}}{\sigma_s} \tag{131}$$

where

$$\sigma_s = \omega - kU_s \tag{132}$$

is the frequency relative to the surface current. To $O(\varepsilon)$, we may then write E as

$$E = \tilde{E}\left(1 - \frac{\omega_0 c_0}{2g}\right)\frac{\bar{\sigma}}{\sigma_s} + O\left(\frac{\omega_0 h}{(gh)^{1/2}}\right)^2 \tag{133}$$

The expression for wave action density is $E/\bar{\sigma}$, which then gives

$$\frac{E}{\bar{\sigma}} = \frac{\tilde{E}}{\sigma_s}\left(1 - \frac{\omega_0 c_0}{2g}\right) + O\left(\frac{\omega_0 h}{(gh)^{1/2}}\right)^2 \tag{134}$$

Examining σ_s, we have

$$\begin{aligned}\sigma_s &= \omega - kU_s = \omega - k\tilde{U} - k(U_s - \tilde{U})\\ &= \tilde{\sigma} - \frac{\omega_0 \tanh kh}{2}\\ &= \tilde{\sigma} - \frac{\omega_0 k c_0^2}{2g}\end{aligned} \tag{135}$$

Factoring out $\tilde{\sigma} = kc_0$ then gives

$$\sigma_s = \tilde{\sigma}\left(1 - \frac{\omega_0 c_0}{2g}\right) \tag{136}$$

Substituting (135) in (134) finally gives

$$\frac{E}{\bar{\sigma}} = \frac{\tilde{E}}{\tilde{\sigma}} + O\left(\frac{\omega_0 h}{(gh)^{1/2}}\right)^2 \tag{137}$$

and we see that $\tilde{E}/\tilde{\sigma}$ is a proper estimate of action density to the required order. The final expression for wave action flux is then given by

$$F = \frac{\tilde{E}}{\tilde{\sigma}}\left\{\frac{c_0}{2}(1+G) + \tilde{U} + \frac{\omega_0 c_0^2}{2g}(1-G)\right\} + O\left(\frac{\omega_0 h}{(gh)^{1/2}}\right)^2 \tag{138}$$

where we have used (A1) and where G is defined in (A2).

It is apparent that the derivation of a wave propagation model based on irrotational theory and using $\tilde{U}$ as the local uniform-over-depth velocity does not produce a consistent model at the order of the expansion considered. The construction of proper wave equations or evolution equations depends on further investigation of the full rotational problem in the context of a slowly varying, one- or two-dimensional (in plan) domain. Direct use of $\tilde{U}$ as a depth-averaged velocity in existing evolution equation models based on irrotational theory will incur an error of $O(\varepsilon)$ in action flux conservation, thus rendering the models invalid over accumulated distances of $O(\varepsilon^{-1})$. However, the expression (138) (or alternate forms of (129) for nonconstant shear) may be used in eikonal-transport models for refraction calculations, with consistency maintained up to $O(\varepsilon^2)$.

7. Conclusions

This study has provided approximate dispersion relations to $O(\varepsilon^2)$ for waves propagating on weak currents $U(z) = O(\varepsilon c)$. In contrast to approximate results for deep water, where $O(\varepsilon)$ approximations are quite sufficient [Stewart and Joy, 1974; Skop, 1987], the results here indicate that approximations to $O(\varepsilon^2)$ are required for any degree of accuracy to be obtained in finite water depth, except for very weak current conditions or for cases where vorticity is confined near the bed and waves are relatively short. The $O(\varepsilon^2)$ results provide the next correction to the results of Skop [1987] and provide significant improvements for cases where vorticity is distributed more or less evenly over the depth. Additional analysis indicates that an expansion procedure for arbitrarily strong currents with weak vorticity yields equivalent results to the weak current case; this conjecture is proven here only for the case of a linear shear current.

A consideration of the formulae for action flux resulting from the $O(\epsilon)$ approximation and the exact solution for a linear shear current indicates that the use of the $O(\epsilon)$ average velocity $\tilde{U}$ as an estimate of depth-averaged velocity in existing wave models incurs an error of $O(\epsilon)$ in the action flux, rendering existing models invalid for length scales of $O(\epsilon^{-1})$. Correction of this problem awaits further research on rotational waves in slowly varying domains.

Appendix: Action Flux Velocity for Linear Shear Current

Based on the results in (127)-(130), we consider the equivalence of the advection velocity c_{g_a} between the $O(\epsilon)$ solution and the expansion of Jonsson et al.'s [1978] exact solution to that order. (Here, we take the viewpoint of the large current, small vorticity expansion so that $\epsilon = O(\omega_0 h/(gh)^{1/2}) \ll 1$.) Evaluating $\tilde{c}_{g_a}$ from (130) gives

$$\tilde{c}_{g_a} = \frac{c_0}{2}(1 + G) + U_s - \frac{\omega_0 \tanh kh}{2k} G$$

$$= \frac{c_0}{2}(1 + G) + \tilde{U} + \frac{\omega_0 \tanh kh}{2k}(1 - G) \qquad \text{(A1)}$$

where

$$G = 2kh/\sinh 2kh \qquad \text{(A2)}$$

From Jonsson et al., we have

$$c_{g_a} = \frac{\partial}{\partial k}(kc_{rs}) + U_s$$

$$= \frac{c_{rs}}{2}\frac{[(1 + G) - (\omega_0 c_{rs} G/g)]}{[1 - (\omega_0 c_{rs}/2g)]} + U_s \qquad \text{(A3)}$$

To the required order, we have

$$c_{rs} = c_0 + \tilde{U} - U_s = c_0 - \frac{\omega_0 c_0^2}{2g} \qquad \text{(A4)}$$

Using (A4) in (A3) and retaining terms only to first order in ω_0 gives back (A1), indicating the desired result

$$c_{g_a} = \tilde{c}_{g_a} + O\left(\frac{\omega_0 h}{(gh)^{1/2}}\right)^2 \qquad \text{(A5)}$$

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T.-M. Chen, Department of Civil Engineering and Operations Research, Princeton University, Princeton, NJ 08540.

J. T. Kirby, Ocean Engineering Group, Department of Civil Engineering, University of Delaware, Newark, DE 19716.

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