The theoretical description of wave–wave interactions as a noise source in the ocean

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The historical development of the theory governing the generation of ULF ocean noise by random surface fluctuations is reviewed. Emphasis is directed at the intrinsic relation between the two main theories currently in vogue, the perturbation procedure and that using an integral procedure based on Lighthill's equation. The relative contributions to the underwater pressure field from the direct radiation of turbulent air motion and the interaction of surface waves is examined. It is shown that even though the direct radiation can approach or even exceed the contribution from wave interactions in the early stages of sea development, overall it is much smaller than the latter. The use of the perturbation expansion in the theoretical analysis of wave interactions at low frequencies is justified. An estimate of the contributions from wave interactions of different order shows that the spectral levels decrease rapidly with increasing order, an increase in the order from m to $m+1$ ($m \geqslant 2$), resulting in a decrease in the spectral peak level by about 25 dB. At the same time the peak frequency shifts from $m\sigma_p$ to $(m+1)\sigma_p$, where σ_p is the peak frequency of the ocean wave spectrum. With the basic importance of the second-order wave–wave interactions in the generation of the noise field confirmed, the formulas currently used to describe this interaction are reexamined and extended to include other wave effects including the contributions of the inhomogeneous component. This leads finally to a more comprehensive expression for the wave-generated ULF noise spectrum.

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INTRODUCTION

The acoustic noise field in the ocean at low frequencies has been the subject of a large number of investigations over the past 50 years. One of the most significant features established has been a dependence on wind and sea state over most of the spectrum.[1-3] Many complementary theoretical studies within this period have explored the nature of the physical mechanisms responsible for this wind and sea-state dependence of the observed noise field, and it is now well recognized that at low frequencies turbulent air motion contributes to the underwater noise field through two different processes. In the first the fluctuations in the air flow are considered to couple directly into the ocean through the air–sea interface, while in the second energy is transferred indirectly through the action of ocean waves. The first process is basically a linear one, but the second involves nonlinear interactions within the wind-induced wave field.

Because of the complexity of the processes involved it is not surprising that the various theoretical treatments placing different emphasis on these two basic mechanisms can lead to different and even contradictory conclusions. At this stage in the development of the subject, it is thus of interest to examine existing analyses in respect of the ultra-low-frequency (ULF) spectrum below 1 Hz and extend these where possible to account for new experimental data relating to the upper layer of the ocean. This paper attempts to do this.

We begin in Sec. I with a review of past work and then proceed to examine a suggestion relating to the relative significance of air turbulence and second-order wave–wave interactions to the generation of the ULF noise field. It is shown that in well-developed sea states the noise field is basically dominated by nonlinear wave interactions, and that only in the early stages of the sea development is the other source important. The application of the perturbation expansion at the frequencies of interest is also justified.

In Sec. II we begin by examining the effects of the higher-order nonlinear wave interactions. This analysis shows that the spectral peak level of the acoustic field decreases by about 25 dB with each increase in order, and justifies the usual neglect of the high-order interactions in practical calculations. Section II then continues with the derivation of a more comprehensive expression for the noise spectrum resulting from the second-order wave interactions. It incorporates for the first time the contributions from all wave interactions, and examines the nature of the inhomogeneous component of the wave-induced noise in the upper levels of the ocean. A short summary is given in Sec. III.

A comparison of the theoretical noise spectra so derived with published data, and a further theoretical extension to incorporate the shallow water case, will be presented in subsequent papers.

I. THEORETICAL TREATMENT OF ULF NOISE GENERATION

A. Review of earlier work

The generation of underwater noise by turbulent air motion above the sea surface is a very complicated physical

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process. Energy within the air flow cannot only pass directly into the water but can also be transmitted indirectly through the mechanism of surface wave–wave interactions. Furthermore the generation of surface waves is itself a complex fetch and duration-dependent process[4] requiring for its description equations and boundary conditions which are nonlinear. It is not surprising therefore that the treatments used to establish the wind dependence of the surface generated noise field at low frequencies have been diverse. The central step in all theoretical treatments, however, is the linearization of the nonlinear fluid mechanical equation and the associated boundary conditions on the perturbed sea surface. Two main procedures have been developed. The first is the perturbation expansion in which the basic equation and boundary conditions are expanded as a power series and then regrouped to form a set of linear equations and boundary conditions according to the order of smallness of the quantities involved.[5–7] The second is that introduced originally by Lighthill[8] in which the hydrodynamic equation is organized as a linear acoustic wave equation, all the nonlinear terms being confined to the right-hand side of the equation as an inhomogeneous (source) term.

In the first analysis the solutions to the linearized equations of different order are usually found by determining the amplitude and phase of each plane-wave solution for a specified frequency and wave-number vector. In the second procedure the solution is usually constructed in the form of a Helmholtz integration involving the corresponding Green's functions and source distributions. The relation between these two approaches is thus analogous to that between the spectral (frequency or wave-number vector) response and impulse (time or space) response of a linear system. They can be adopted to serve different purposes. For the noise field arising from an infinite distribution of (overhead) sources, the perturbation procedure with plane wave solutions is the more convenient to use, though both procedures lead to the same result. When dealing with distant and finite sources, on the other hand, the Lighthill–Helmholtz equation has definite advantages. In addition to these two, other procedures have been developed for particular purposes. These too will be discussed briefly in the review that follows.

1. The perturbation approach

Historically the study of the wave-induced pressure field began when the contributions of Miche[5] and Longuet-Higgins[6] led to an understanding of the origin of microseismic ground motion. In his paper Longuet-Higgins began with the basic fluid dynamic equation and boundary conditions,

$$\alpha_1^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} - g \frac{\partial \phi}{\partial z} - \frac{\partial u^2}{\partial t} - \mathbf{u} \nabla \left(\frac{1}{2} \mathbf{u}^2 \right) = 0,$$

$$\frac{\partial \phi}{\partial z} = 0 \text{ for } z = h \text{ and } p = 0 \text{ and } \frac{Dp}{Dt} = 0 \text{ at } z = \zeta,$$

and their first- and second-order perturbation approximations,

$$\alpha_1^2 \nabla^2 \phi^{(1)} - \frac{\partial^2 \phi^{(1)}}{\partial t^2} - g \frac{\partial \phi^{(1)}}{\partial z} = 0,$$

$$\frac{\partial \phi^{(1)}}{\partial z} = 0 \text{ at } z = h, \text{ and } \nabla^2 \phi^{(1)} = 0 \text{ at } z = 0,$$

and

$$\alpha_1^2 \nabla^2 \phi^{(2)} - \frac{\partial^2 \phi^{(2)}}{\partial t^2} - g \frac{\partial \phi^{(2)}}{\partial z} = \frac{\partial}{\partial t} (\mathbf{u}^{(1)})^2,$$

$$\frac{\partial \phi^{(2)}}{\partial z} = 0 \text{ at } z = h,$$

$$\text{and } \nabla^2 \phi^{(2)} = -\zeta_1 \frac{\partial}{\partial z} \nabla^2 \phi^{(1)} \text{ at } z = 0.$$

In the above (and later) α_1 is the sound velocity of sea water and g the gravitational acceleration, $z = \zeta$ is the sea-surface displacement and $z = h$ the depth of the sea floor, p, $\mathbf{u}$, and ϕ are, respectively, the pressure, the velocity, and potential of the wave-induced particle motion, while ϕ_j and ζ_j are the potential and surface displacement of order j. He then calculated the nondecaying pressure field contained in the standing gravity wave field as a second-order effect. He also considered the seismic response of an elastic bottom that supports propagating Stoneley waves, and without naming it used what is essentially a Green's function technique to calculate the far-field displacement of the seabed, placing the sources of the acoustical field at the base of the "gravity layer," half a gravity wavelength down from the surface.[7] This enabled him to calculate the main component of the far field. Further Longuet-Higgins introduced a random directional wave field and its spectrum in calculating the order of magnitude of the seismoacoustic response of the wave activity. He also calculated the effects of an elastic sea bottom on the pressure field and introduced the well-known form of wave directivity that is critical to all later calculations in the subject. His analyses and Hasselmann's later work (see below) have since been adopted as a standard procedure in the calculation of wave-induced pressure and microseismic fields.

In the early 1960's, Hasselmann[9] treated the problem statistically. Based on spectral transfer functions and the local energy balance equation, he derived an expression for the spectrum of the wave-induced pressure field and discussed the effects of seismic wave propagation. Although he too restricted consideration to the homogeneous component he noted that any interacting wave components, which generate a pressure wave with a phase velocity greater than the sound velocity in water, will contribute to the noise field. He also discussed the direct radiation from the turbulent motion in the air (using Lighthill's equation), as well as the first-order (primary frequency, PF) wave-induced pressure field in the ocean.

In a later analysis Brekhovskikh[10] also investigated the ULF noise field in the ocean. He adopted the same perturbation procedure and obtained results consistent with those of Longuet-Higgins and Hasselmann. Brekhovskikh also established an explicit expression for the condition under which two interacting surface wave trains will produce a nondecaying or homogeneous acoustic pressure field.

Further evidence of the growing interest of the acoustic community in low-frequency noise predictions was provided when Hughes[11] re-examined nonlinear wave interactions as a generation mechanism of the ocean noise field below 10 Hz. The basic equations and the perturbation procedure were again used (although the pressure was chosen as the main variable) and the calculated noise spectra were compared with measured data. It is notable that, in his predictions, Hughes used well established and explicit forms of the developed ocean-wave spectra. He obtained reasonably good agreement with available noise data below 10 Hz, which suggested that it was sufficient to use the second-order perturbation approximation to predict the ULF noise field in real situations. This result is consistent with that of Marsh,[1] who had found good agreement between the Knudsen's noise spectrum and that calculated using Longuet-Higgins' expressions and a classical form of the wave spectrum.

There have been several more recent studies. As part of a long-term investigation of the wind-wave spectrum and microseismic noise measured on land nearby, Kibblewhite and Ewans,[2,12] established a set of ocean-noise pressure spectra in the frequency band 0.1–2.0 Hz for wind speeds varying from 7.5 to 35 m/s. Comparable spectral levels and characteristics were soon reported in other published hydrophone and seismic data,[13–15] all being in reasonable agreement with the theoretical predictions based on the perturbation procedures.[16–18] Adair[13] and Webb and Cox[15] have also made theoretical assessments of the ocean noise and microseismic fields, based on the second-order perturbation, with similar success.

On the basis of these studies over four decades, the perturbation procedure, involving the second-order approximation, has been shown able to provide a reasonable explanation for the ocean noise observed below 10 Hz. However an intrinsic uncertainty remained, relating to the significance of the contribution of the higher-order terms. An analysis that addresses this question and leads to a more comprehensive description of the acoustic effects of nonlinear wave interactions is presented in the next section. However before this is presented we pause to examine the generation of ocean noise by the direct radiation of turbulent air motion, to consider other treatments differing slightly from the classical perturbation procedure, and to look at the application of the Lighthill procedure in the investigation of the ocean-noise field.

2. Air turbulence and other treatments

In his 1963 analysis, Hasselmann[9] included a treatment of the direct energy transfer from atmospheric turbulence to the underwater-noise field. He reduced the problem to the evaluation of the source-pressure spectrum at the free surface of a layered elastic space. The pressure field was assumed to be induced by nonlinear interaction in the turbulent boundary layer above the surface, where the air motion is described by the Lighthill equation:

$$\frac{1}{\alpha_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2}{\partial x_i\,\partial x_j}(\rho u_i u_j)$$

with α_0 and ρ the sound velocity and density of the air, respectively. From a comparison of his theoretical estimation with the observed microseism spectra generated by storms, Hasselmann concluded that the direct contribution of atmospheric turbulence is generally negligible. In his calculation the interaction between the air turbulence and the wave surface was neglected.

This additional interaction was treated later by Isakovich and Kur'yanov,[19] who used a slightly different approach calling on experience gathered from surface roughness studies. Based upon the fact that when an air flow passes over a solid rough surface, the normalized spatial correlation function of the pressure fluctuation has a universal form and the radius of correlation is determined by the ratio of the flow velocity to the frequencies involved, they expressed the time-space spectrum of the wind-pressure field in terms of the parameters of the ocean surface-wave spectrum and then calculated the noise spectrum in the air and water. Their calculation (later improved slightly by Wilson[20]) also showed that the sound power radiated from the air turbulence to the water is very small, being nearly five orders of magnitude less than that radiated into the air. However, as Adair[21] comments, in their calculation Isakovich and Kur'yanov employed the same source function to describe both the generation of wind waves and the underwater noise field. We agree with him that this is inappropriate since the growth of a developing sea is controlled largely by nonlinear processes.

In another analysis during this period, Harper and Simpkins[22] calculated the wave-induced noise field by dividing the water body into an inner region (scale dimension of a surface wavelength below the surface) and an outer region, and matching the acoustic field in the outer region with the perturbation solutions of the inner region. In another analysis, Kuo[23] modeled the sea surface as patches of capillary waves riding on the forward faces of the longer carrier waves and expressed the high-frequency noise spectrum as the convolution of the spectrum of the carrier waves and the high-frequency random radiation. In this way he avoided the difficulty encountered at higher frequencies of restricting the perturbation procedure to the second-order only. In another development Goncharov,[24] by converting the Lighthill equation to an Helmholtz integration, calculated the nonlinear interaction between surface waves and the turbulence in the adjacent surface layer in the water. All these developments have greatly enriched our understanding of the physical processes involved.

3. The Lighthill formulation

We now return to the 1950's. In 1952 Lighthill published his famous paper,[8] which has since been widely recognized as the foundation of sound generation and propagation by fluid motion. In this theory the basic fluid mechanical equation is expressed as an acoustic wave equation with an external stress term as the source:

$$\frac{\partial^2 \rho}{\partial t^2} - \alpha_0^2 \nabla^2 \rho = \frac{\partial^2}{\partial x_i\, x_j} T_{ij},$$

where

$$T_{ij} = \rho u_i u_j + p_{ij} - \alpha_0^2 \rho \delta_{ij}$$

and α_0 is again the sound velocity in air. This approach laid the foundation of sound generation by turbulent motion. Later, in 1955, Curle[25] extended the application of Lighthill's equation to include the case where interfaces are present, by expressing it in the form of the Helmholtz integration. In a subsequent development involving the introduction of the generalized function theory, Ffowcs-Williams and Hawkings[26] further extended the Lighthill–Curle analysis to include arbitrary convective motion and moving boundary surfaces. This development has proved to be very effective in handling situations encountered in the case of the real ocean-noise field.

Although Lighthill's equation has been used widely for nearly 30 years to estimate the ocean-noise field associated with atmospheric turbulence, its application to the noise generated by surface wave action was considered only recently. In 1981, Lloyd,[27] using Ribner's modified version of the Lighthill theory in which the overpressure was chosen as the field variable and $-(1/\alpha_0^2)(\partial^2 p_0/\partial t^2)$ as the source term, calculated the noise field induced by wave–wave interactions and obtained a far-field acoustic spectrum of the same form as that established by Hughes.[11] More recently, in one of a series of papers dealing with wind and wave-induced noise, Guo,[28] in an analysis based on the Lighthill–Ffowcs-Williams theory, compared the ocean noise generated by atmospheric turbulence with that produced by weak nonlinear interactions of surface waves. About the same time Cato[29] reported similar investigations using the Lighthill–Curle approach. In both these treatments, the noise field is finally expressed in terms of a distribution of quadrupole, dipole and monopole sources, corresponding respectively to the direct radiation of air flow turbulence, the second-order wave interactions and first-order pressure fluctuations. The Green's functions Guo used are more general in that they include the effects of gravity and thus resonant wave generation. Cato's expression however will be more convenient in treating nonsteady-state cases.

Guo also discussed the relative importance of the underwater noise field arising from the second-order interactions of the surface-wave field produced by the air flow (for the case of low wind speeds) and that generated by the direct radiation from the turbulence in the air motion into the water. The small value of the ratio so obtained, and the additional consideration that at higher wind speeds the surface waves will depart progressively from the assumed conditions of single-value and small slope required by the "weak-interaction" process, led him to conclude that such nonlinear interactions in the ocean-wave field are probably not significant sources of underwater sound. This conclusion seems to be in conflict with many of the other theoretical and experimental investigations,[3] and it will be appropriate to examine it before proceeding.

B. Wave–wave interactions vs air turbulence

To resolve this inconsistency we first establish the value of Guo's ratio in a real situation where the sea is developed and can be well described by one of the widely recognized forms of the ocean-wave spectrum. We follow Eq. (5.5) of Ref. 28, which represents the sound power, W_s, arising from an area of the sea surface in unit frequency band, as:

$$W_s = \rho_w w^6/12\pi c_w^3 (2\pi)^3 \times \int_\infty \widehat{\Pi}(\mathbf{q},\sigma)\widehat{\Pi}(-\mathbf{q},\omega-\sigma) q\, dq\, d\theta\, d\sigma, \qquad (1)$$

where

$$\overline{\zeta^2(\mathbf{x},t)} = \frac{1}{(2\pi)^3}\int \widehat{\Pi}(\mathbf{q},\sigma) d\mathbf{q}\, d\sigma.$$

By referring to Hasselmann's expressions [see Eqs. (2.6) and (2.7) of Ref. 9]

$$\zeta(\mathbf{x},t) = \int\int e^{i\mathbf{q}\mathbf{r}}[dZ_+(\mathbf{q})e^{-i\sigma t} + dZ_-(\mathbf{q})e^{i\sigma t}]$$

and

$$f_\zeta(\mathbf{q}) = 2\langle |dZ_+|^2\rangle/d\mathbf{q}, \quad \sigma = \sqrt{gq}$$

it is easy to see that

$$\widehat{\Pi}(\mathbf{q},\sigma) = (2\pi)^3 f_\zeta(\mathbf{q})\delta(\sigma-\sqrt{gq}),$$

where $f_\zeta(\mathbf{q})$ is the surface-wave spectrum satisfying

$$f_\zeta(\mathbf{q}) = (g^2/2\sigma^3)F_a(\sigma)H(\theta).$$

Here $F_a(\sigma)$ is the frequency spectrum of the surface wave field and

$$H(\theta) \equiv G(\theta) \Big/ \int_{-\pi}^{\pi} G(\theta)d\theta$$

is the normalized distribution function of the wave energy. By substituting

$$\widehat{\Pi}(\mathbf{q},\sigma) = (4\pi^3 g^2/\sigma^3)F_a(\sigma)H(\theta)\delta(\sigma-\sqrt{gq})$$

into Eq. (1) we obtain after straightforward manipulations

$$W_s = (4\pi^2\rho_w w^3 g^2/3c_w^3)F_a^2(\omega/2)\hat{I}(\omega), \qquad (2)$$

where

$$\hat{I}(\omega) \equiv \int_{-\pi}^{\pi} H(\theta)H(\theta+\pi)d\theta = \pi^{-1/2}2^{-2s-1}[\Gamma(s+1)/\Gamma(s+0.5)]$$

and $\Gamma(x)$ is the gamma function,[30,31] describes the effect of the angular distribution of the ocean wave energy on the noise field and follows the form $\cos^{2s}(\theta/2)$. By using Guo's Eq. (5.4) for the second-order field arising from air turbulence, $W_t(\omega)$ [but noting that in Guo's Eqs. (5.4) and (5.8), ω^4 and ω^{15} should properly be ω^3 and ω^{13}, respectively], and Eq. (2) above for $W_s(\omega)$, we obtain for the case of $\omega > g/U$ (U is the wind speed) the ratio corresponding to that Guo did using his Eqs. (5.4) and (5.8), namely,

$$\eta(\omega) = \frac{1}{8\pi^3}\left(\frac{\rho_a}{\rho_w}\right)^2 \frac{l^2\Delta^2 u^4}{c_a^2 g^2 F_a^2(\omega/2)\hat{I}(\omega)}.$$

Substituting the PM spectrum,[32] $F_a(\sigma) = \alpha g^2\sigma^{-5} \times \exp[-\beta(g/U\sigma)^4]$, it follows that

$$\eta(\omega) = \frac{1}{8\pi^3}\left(\frac{\rho_a}{\rho_w}\right)^2 \frac{l^2\Delta^2 u^4}{c_a^2 g^6 \alpha^2 \hat{I}(\omega)} \times \left(\frac{\omega}{2}\right)^{10} \exp\left[2\beta\left(\frac{2g}{U\omega}\right)^4\right].$$

Further, taking $(\rho_a/\rho_w)^2 = 10^{-6}$, $c_a^2 = 10^5\ \text{m}^2/\text{s}^2$, $\alpha = 8.1\times 10^{-3}$, $\beta = 0.74$, $l \simeq \Delta = g/\omega^2$, $u \simeq U\sqrt{0.003}$

$= 0.055U$,[28] the value of $\exp[2\beta(2g/U\omega)^4]$ at the peak frequency, and a value of 0.05 as an estimate of $\hat{I}(\omega)$, the ratio of the two pressures around the spectral peak will be

$$(p_t/p_s)^{(1)} < 10^{-6}\omega U^2.$$

This is obviously very small for the frequencies and wind speeds of present interest (actually it can be as small as $10^{-9}\omega U^2$, here we have taken $\omega/2$ in the expression of l and exaggerated the term U) and contrasts markedly with the ratio Guo calculated using his model [Eq. (5.11) of Ref. 28],

$$(p_t/p_s)^{(2)} \simeq 10^{10} L^{-1} U^{-2} \omega^{-5},$$

where L is the fetch. For instance, if we invoke the usual units and take $\omega = 1$, $U = 10$, and $L = 10^5$, then $(p_t/p_s)^{(2)} \simeq 10^3$, while $(p_t/p_s)^{(1)}$ is less than 10^{-4}.

This analysis suggests that, even if the direct radiation from air turbulence can cause ULF noise levels in the water as high as those induced by the second-order wave interactions at the early stages in the sea development ($U \ll 10$ m/s) as Guo claims, in the important case of a developed sea the contribution from air turbulence becomes negligible in comparison. The findings of many investigations are therefore substantiated and the nonlinear wave–wave interaction process is reaffirmed as the more important noise generation mechanism at very low frequencies when the sea is developed.[12-17] Cato's[29] recently reported lake measurements are the latest to show a good agreement between prediction (based on Lighthill's procedure in this case) and measured wave and noise data.

The questions left then are (i) whether in real situations involving developed sea states it is proper to use the perturbation expansion and (ii) if the expansion is justified whether it is sufficient to take into account terms up to the second order only.

Considering the first question, if we are prepared to accept some relaxation in mathematical rigor, we can justify the perturbation expansion of the surface boundary conditions by recognizing that even in real situations the average slope of the sea surface is still reasonably small. For instance, when fetch and duration are both unlimited, energetic seas when fully developed can be well described by the Pierson–Moskowitz (PM) spectrum at frequencies below 0.5 Hz. This spectrum predicts the average slope, defined as the ratio of the significant wave height, $H_{1/3}$, and the wavelength at the spectral peak frequency, L_m,[32,33] to be

$$\text{slope} = H_{1/3}/L_m = (2.14 \times 7.54) \times 10^{-2}/2\pi \simeq 0.026.$$

Before the second question can be answered it is necessary to discuss some general properties of nonlinear wave interaction. These are examined in the next section.

II. SECOND-ORDER WAVE–WAVE INTERACTIONS AS A SOURCE OF ULF NOISE

In the previous section we saw that, in considering the ULF noise field, it was mathematically reasonable to expand the boundary conditions imposed on the random surface of the developed sea as a power series of the sea surface displacement. Since this series has hitherto only been treated up to the second order in the literature, it is appropriate to seek a justification for this restriction. In this section it will be shown that the contribution to the acoustic field from higher-order interactions is usually negligible and can be neglected. We will then proceed to the development of a generalized expression describing the acoustical effects of nonlinear interactions between ocean-gravity waves, which was presented without detailed discussion in an earlier paper.[16]

A. The relative importance of higher-order interactions

It is easy to see that the solution of the perturbation equation and boundary conditions for the mth (integer) order will contain the term of the form

$$\prod_{j=1}^{m} \exp[i(\mathbf{q}_j\mathbf{r} - \sigma_j t)] = \exp\left[i\sum_{j=1}^{m} \mathbf{q}_j\mathbf{r} - i\left(\sum_{j=1}^{m} \sigma_j\right)t\right],$$

where $\mathbf{q}_j$ and σ_j are the wave vectors and angular frequencies of the surface wave trains satisfying, in the case of gravity waves,the dispersion relation,

$$\sigma_j^2 = g|\mathbf{q}_j|.$$

Because of this dispersion relation, it can be shown that two gravity waves cannot produce another gravity wave through nonlinear interaction. In fact, if this were not so it would follow that $g|\mathbf{q}_1 + \mathbf{q}_2| = (\sigma_1 + \sigma_2)^2$ so that, $(\sigma_1^4 + \sigma_2^4 - 2\sigma_1^2\sigma_2^2 \cos\theta) = (\sigma_1 + \sigma_2)^4$, which leads to the impossible expression, $\cos\theta = -(3 + 2\sigma_1/\sigma_2 + 2\sigma_2/\sigma_1)$. Instead, the result of such an interaction is a kind of inhomogeneous wave, characterized by a speed higher than that of the gravity waves from which it arises and a subsurface pressure field which decays more slowly with depth than occurs in the case of ordinary gravity waves. Only when the amplitude of the resultant wave vector is sufficiently small that the phase velocity $(\sigma_1 + \sigma_2)/|\mathbf{q}_1 + \mathbf{q}_2|$ becomes greater than the sound velocity in water, does the resultant wave become an acoustic wave. The same argument applies when more than two gravity waves interact with each other.

Accordingly, since the sound velocity in water is much greater than the value g/σ at the frequencies of interest, in the following analysis of the relative contributions of interactions of different order we can approximate the resultant wave vector as $\Sigma_j \mathbf{q}_j = 0$. Further, since the wave spectrum for a given wind speed displays a single peak, the maximum of the acoustic field arising from the interaction of the order m will appear when all these interacting waves are of the same frequency $\sigma_j = \sigma_p$ (the frequency of the spectral peak). The peak values of the potentials of different order thus take the form

$$\phi^{(m)}(\sigma_p) \sim F_a^{m/2}(\sigma_p) e^{-im\sigma_p(t - z/\alpha_1)} \quad (m \geqslant 2),$$

where α_1 is the sound velocity in water. Since the frequency spectrum of the surface displacement, $F_a(\sigma_p)$, has the dimension L^2T, the proper form of the spectrum of the mth order pressure field will be

$$F_p^{(m)}(\sigma_p)|^2 = B_m(g,\sigma_p,\alpha_1)[(\sigma_p^5/g^2)F_a(\sigma_p)]^m.$$

If the Pierson–Moskowitz spectral parameters are as before,

$\alpha = 8.10\times 10^{-3}$ and $\beta = 0.74$,[32] the ratio of the peak values of two successive orders is then

$$\eta = (\sigma_p^5/g^2)F_a(\sigma_p) = \alpha e^{-\beta} = 3.86\times 10^{-3}.$$

This suggests that the peak value decreases by about 25 dB with each unity increase in the order. At the same time the peak of the noise spectrum shifts to $m\sigma_p$, where σ_p is the frequency of the wave spectral peak. Furthermore, since a wave spectrum usually has a sharp front before the peak, the highest order which can make a significant contribution to the noise at a given frequency σ is defined by the integer $M = (\sigma/\sigma_p)$, where σ_p depends on wind speed. Therefore, for an angular frequency $\sigma < m\sigma_p = mg/U_{10}$, the mth order interaction can be neglected (U_{10} is the wind speed at 10 m above the sea surface). For example, at frequencies lower than 0.5 Hz the third-order interaction will contribute little to the noise field.

The above discussion shows clearly that, for frequencies of present interest, nonlinear wave interactions of order higher than 2 will make little contribution to the noise pressure field. For the sake of simplicity, they can thus be omitted from any calculation without introducing significant error. In the next section, calculations based on the second-order interaction only will be extended to include the contributions from both homogeneous and inhomogeneous waves. The interacting waves will no longer be restricted to diametrically opposite directions and identical frequencies. For convenience we will call this analysis the "generalized standing-wave solution" to distinguish it from the traditional analysis mentioned earlier.

B. The generalized standing-wave solution

Having justified the application of the perturbation procedure to the analysis of the ULF noise field associated with developed sea states, and the truncation of the expansion series to the second-order, we now reexamine the calculation of this field. Since we are mainly interested in the wave generated noise the effects of turbulent motion in the water will be neglected. As in all developments,[6,10,11,16] we start with the second-order equation and associated boundary conditions,

$$\alpha_1^2\nabla^2\phi^{(2)} - \frac{\partial^2\phi^{(2)}}{\partial t^2} - g\frac{\partial\phi^{(2)}}{\partial z} = 0,$$

$$\frac{\partial^2\phi^{(2)}}{\partial t^2} + \frac{\partial\phi^{(2)}}{\partial z} = -\frac{\partial}{\partial t}(\nabla\phi^{(1)})^2 - \zeta^{(1)}\frac{\partial^3\phi^{(1)}}{\partial z\,\partial t^2} - \zeta^{(1)}g\frac{\partial^2\phi^{(1)}}{\partial z^2}, \quad z = 0. \tag{3}$$

Substituting the first-order solution

$$\phi^{(1)} = Q\exp[i(\mathbf{q}\cdot\mathbf{r} - \sigma t) + qz] \tag{4}$$

into the right-hand side of Eq. (3) we can find the second-order solution of the form

$$\exp\{i[\mathbf{q}_1 + \mathbf{q}_2)\cdot\mathbf{r} - (\sigma_1 + \sigma_2)t + \mu_0 z]\} \tag{5}$$

and the resulting pressure field

$$P_0(\mathbf{r},t) = i\rho_1(\sigma_1 + \sigma_2)BQ_1Q_2\exp\{i[\mathbf{q}_1 + \mathbf{q}_2)\cdot\mathbf{r} - (\sigma_1 + \sigma_2)t + \mu_0 z]\}, \tag{6}$$

where

$$\mu_0 = -(1/2\alpha_1^2)\times\{ig + \sqrt{4\alpha_1^2[(\sigma_1 + \sigma_2)^2 - \alpha_1^2|\mathbf{q}_1 + \mathbf{q}_2|^2] - g^2}\} \tag{7}$$

and

$$B = \frac{2(q_1q_2 - \mathbf{q}_1\mathbf{q}_2)(\sigma_1 + \sigma_2)}{\sqrt{(\sigma_1 + \sigma_2)^4 - g^2|\mathbf{q}_1 + \mathbf{q}_2|^2}}e^{i\epsilon}. \tag{8}$$

In the above, ρ_1, α_1, and g are, respectively, the ocean density, sound speed, and gravitational acceleration, and ϵ is a phase factor.

Equation (6) simply says that the second-order pressure wave will have a horizontal wave number $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$ and angular frequency $\omega = \sigma_1 + \sigma_2$, and that when μ_0 has a nonzero real part $\Re\mu_0$ and satisfies the relation

$$(\Re\mu_0)^2 + |\mathbf{q}_1 + \mathbf{q}_2|^2 = (\sigma_1 + \sigma_2)^2/\alpha_1^2$$

the result is an ordinary acoustic wave, while when $\Re\mu_0 = 0$ the pressure wave becomes inhomogeneous and decays in the z direction.

A simple description of this process has been presented in Refs. 16 and 18 but discussion was restricted to the case where $\Re\mu_0 \neq 0$. We now remove this restriction. For convenience we develop the discussion by seeking to determine the wave vectors for two interacting surface-wave trains when the acoustic frequency ω and the horizontal wave vector $\mathbf{k}$ are defined.

Since any pair of interacting vectors must satisfy the relations

$$\sigma_j^2/|\mathbf{q}_j| = g, \quad j = 1,2, \tag{9}$$

$$\sigma_1 + \sigma_2 = \omega, \tag{10}$$

$$\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{k}, \tag{11}$$

they can be represented geometrically in a three-dimensional $\omega - \mathbf{k}$ space, as shown in Figs. 1–5. To appreciate these figures it is helpful to first recognize the significance of the cone $OBCDEB$ centered on the ω axis — see Fig. 1. Any point within the surface of this cone defines an ordinary plane acoustic wave satisfying the dispersion relation

$$|\mathbf{k}| = \epsilon\omega/\alpha_1, \quad 0\leqslant\epsilon\leqslant 1. \tag{12}$$

The cone is a representation of the relation $k = \omega/\alpha_1$ when $\epsilon = 1$.

The two horn-like surfaces whose apexes O and O' are also centered on the ω axis, represent, respectively, the two dispersion surfaces describing the relations $\sigma_1 = \sqrt{gq_1}$ and $\sigma_2 = \sqrt{gq_2}$ for the case $\mathbf{k} = 0$, when the additional constraint $\sigma_1 + \sigma_2 = \omega$ is also applied. (If $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{k} \neq 0$ then O and O' do not both lie on the ω axis, and in Fig. 2 the more general case of $k \neq 0$ is considered.) The intersection of these two rotational surfaces is indicated by the spatial curve L, the projection of which in the $\mathbf{k}$ plane is the curve L'. Corresponding to each point on L', say G' in the lower part of Fig. 1, there are two diametrically opposed vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ which add to give $\mathbf{k} = 0$.

In the more general case where $\mathbf{k} \neq 0$, the induced acoustic wave is propagating in a direction other than normal to

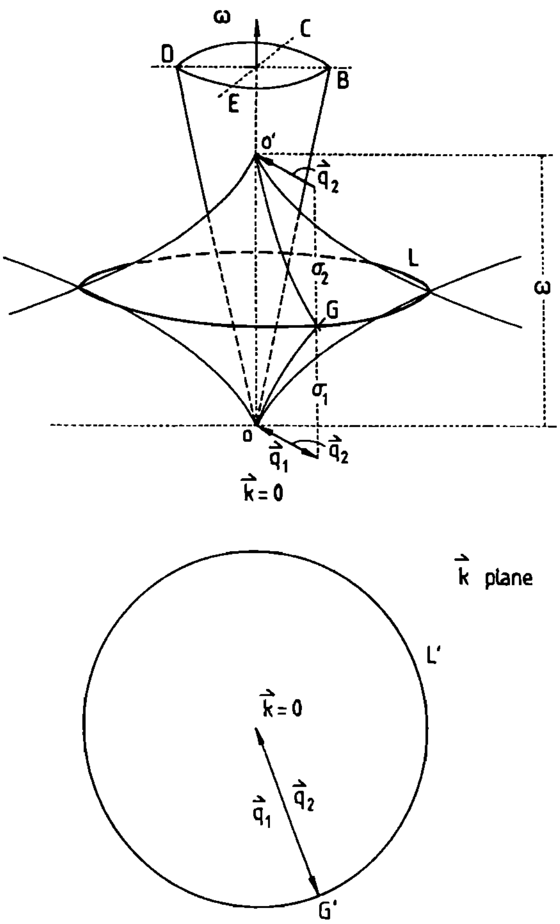

FIG. 1. The representation of the nonlinear interaction of two surface gravity waves inducing a plane acoustic wave with horizontal wave-number vector **k** for **k** = 0.

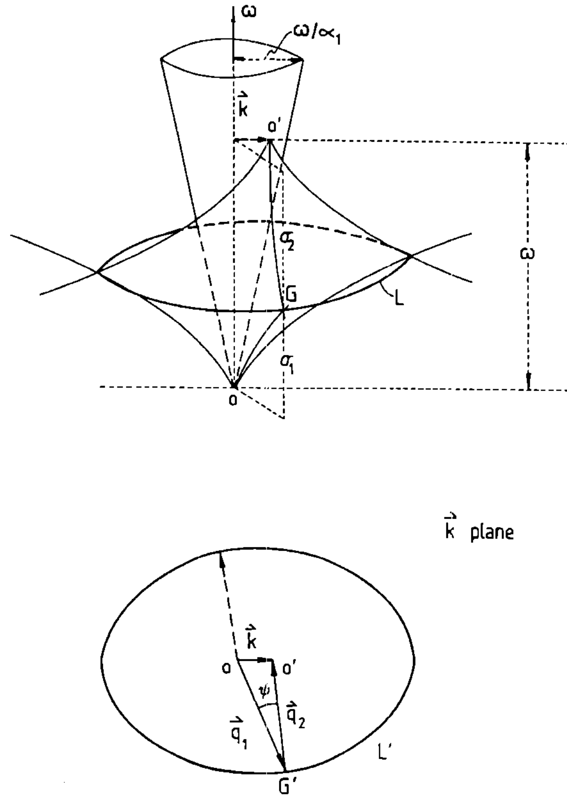

FIG. 2. The representation of the nonlinear interaction of two surface gravity waves inducing a plane acoustic wave with horizontal wave-number vector **k** for $0 \leqslant |\mathbf{k}| \leqslant \omega/\alpha_1$.

the surface of the sea. We first consider the case of a homogeneous wave (Fig. 2), where the end point of **k** lies inside the surface of the cone defined by Eq. (12). In this situation the apex of the upper horn is displaced as shown and the projection L', of the intersecting curve L in the **k** plane, is slightly different from a circle. The two interacting vectors $\mathbf{q}_1$, $\mathbf{q}_2$, associated with point G', now differ slightly in magnitude and direction.

When $|\mathbf{k}|$ equals the acoustic wave number ω/α_1, the induced pressure field becomes a traveling plane wave propagating horizontally. For $|\mathbf{k}| > \omega/\alpha_1$ the acoustic response degenerates to an inhomogeneous wave decaying with increasing depth (distance from the surface). The intersection curve L and its projection L' gradually change from the dumbbell-like shape of Fig. 3 to the situation in Fig. 4 where two closed curves centered in O and O' exist. In both situations it is clear that for each point G' on L', up to three pairs of interacting vectors $\mathbf{q}_1, \mathbf{q}_2$ can lead to the resultant **k**. This is indicated in exaggerated form in Fig. 4. For a given **k** and a defined direction for $\mathbf{q}_1$ three possible pairs of vectors exist—(i) $\mathbf{q}_1 = \mathbf{OG}'$, $\mathbf{q}_2 = \mathbf{G'O'}$; (ii) $\mathbf{q}_1 = \mathbf{OF}$, $\mathbf{q}_2 = \mathbf{FO}'$; and (iii) $\mathbf{q}_1 = \mathbf{OH}$, $\mathbf{q}_2 = \mathbf{HO}'$. Finally when **k** reaches a value $k = \omega^2/g$, the only points remaining common to the two

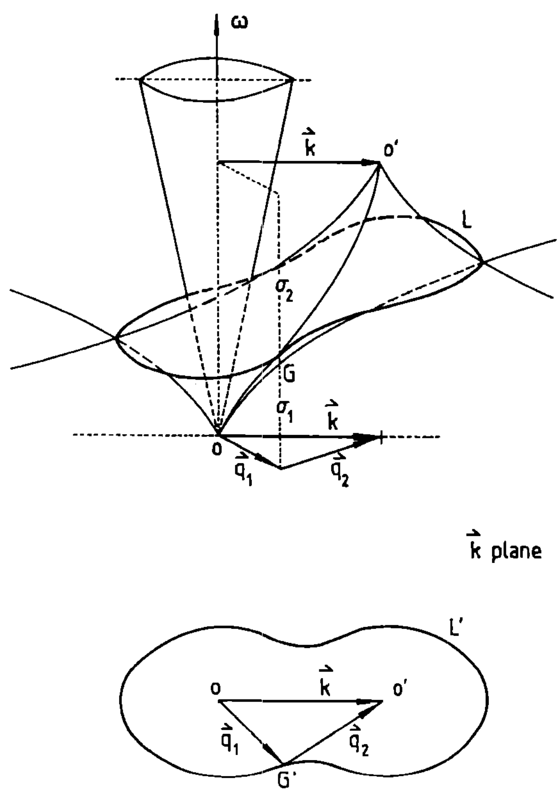

FIG. 3. The representation of the nonlinear interaction of two surface gravity waves inducing a plane acoustic wave with horizontal wave-number vector **k** for $\mathbf{k} \geqslant \omega/\alpha_1$.

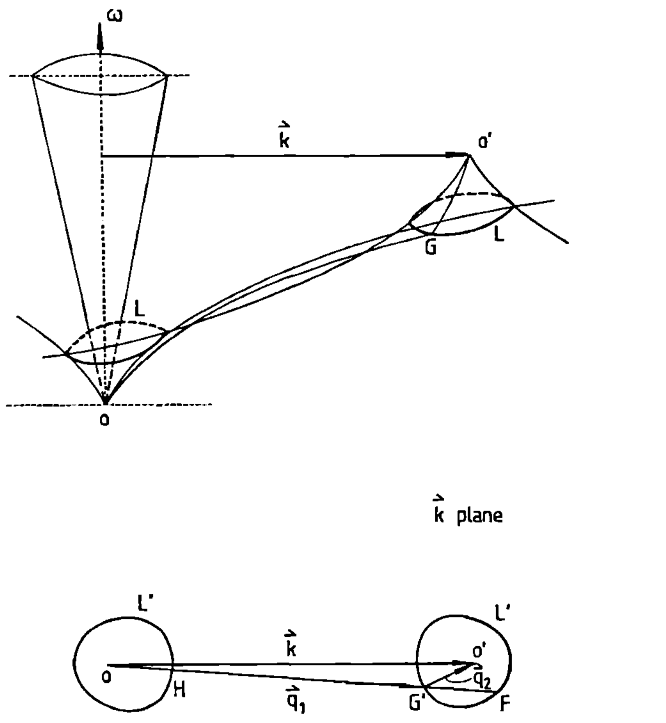

FIG. 4. The representation of the nonlinear interaction of two surface gravity waves inducing a plane acoustic wave with horizontal wave-number vector **k** for $\mathbf{k} \gg \omega/\alpha_1$.

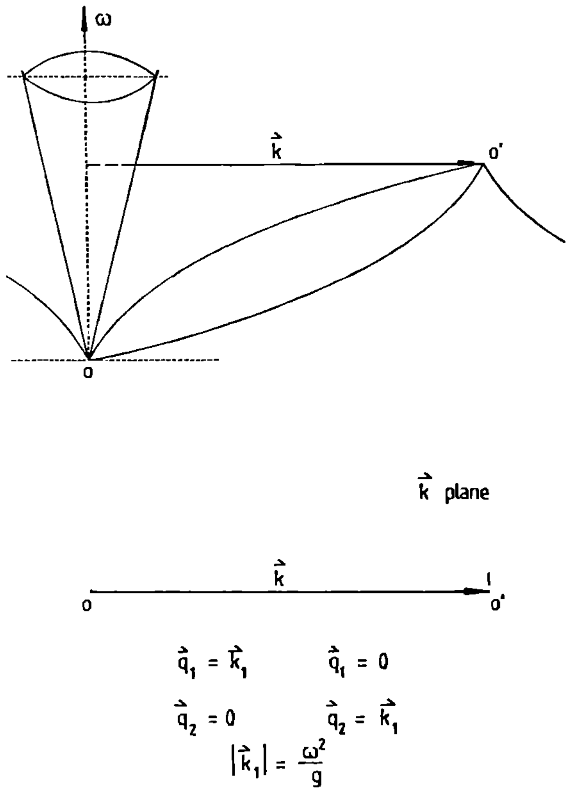

FIG. 5. The representation of the nonlinear interaction of two surface gravity waves inducing a plane acoustic wave with horizontal wave-number vector **k** for $\mathbf{k} = \omega^2/g$.

curved surfaces are O and O'. In this critical case, represented in Fig. 5, we have $\sigma_1 = 0$, $\sigma_2 = \omega$, or $\sigma_1 = \omega$, $\sigma_2 = 0$, whereupon $\mathbf{q}_1 = 0$ and $\mathbf{q}_2 = \omega^2/g$ or $\mathbf{q}_1 = \omega^2/g$ and $\mathbf{q}_2 = 0$. Beyond this point the two dispersion curves separate completely from each other and the wave–wave interaction ceases.

The geometrical description outlined above shows that the horizontal wave number of the induced pressure field cannot become infinite but is limited to the value ω^2/g uniquely determined by the frequency. This is a property of the noise source induced by the wave interactions. This feature is markedly different from conventional point source distributions.

C. Calculation of the total spectrum

Having established an appreciation of the physical processes involved we now turn to the quantitative calculation of the spectrum of the pressure field induced by the second-order wave interactions. Starting from Eq. (6) and repeating the procedures related to correlation and Fourier transformation, we can write the spectrum of the induced pressure field in the frequency–horizontal wave-number domain, $f_{p0}(\mathbf{k},\omega)$, as an integration of $d\mathbf{q}_1$ (see Refs. 16 and 18):

$$f_{p0}(\mathbf{k},\omega) = \int_\Sigma |M(\mathbf{q}_1,\mathbf{k}-\mathbf{q}_1)|^2 \delta(\sqrt{gq_1} + \sqrt{g|\mathbf{k}-\mathbf{q}_1|} - \omega) \times f_\xi(\mathbf{q}_1) f_\xi(\mathbf{k}-\mathbf{q}_1) d\mathbf{q}_1, \qquad (13)$$

where

$$|M(\mathbf{q}_1,\mathbf{k}-\mathbf{q}_1)|^2 = \frac{\rho_1^2\sigma_1^2(\omega-\sigma_1)^2(1-\cos\theta_{12})^2}{1-g^2k^2/\omega^4},$$

$$f_\xi(\mathbf{q}) = \frac{g^2F_a(\sigma)H(\theta-\theta_\omega)}{2\sigma^3},$$

$\delta(x)$ is the δ function, θ_{12} is the angle between $\mathbf{q}_1$ and $\mathbf{q}_2$, θ_ω is the bearing of the prevailing wind, and Σ is the defined domain of the integrand in the $\mathbf{q}_1$ plane. By denoting $\chi = \sigma_2/\sigma_1$, $\hat{m} = 2kg/\omega^2$, $\hat{t} = \cos\theta_{1k}$, and $d\mathbf{q}_1 = q_1\, dq_1\, d\theta_1$, we obtain from the Appendix that

$$f_{p0}(\mathbf{k},\omega) = \frac{1}{2}\omega\rho_1^2 g^2 \int_0^{2\pi} \frac{[1-\cos(\theta_{12})]^2\chi^2}{(1+\chi)^2(1-\frac{1}{4}\hat{m}^2)[\chi^2-(\frac{1}{2}\hat{m}\hat{t}+1)\chi+(1-\frac{1}{2}\hat{m}\hat{t})]} \times F_a[\omega/(1+\chi)]F_a[\chi\omega/(1+\chi)]H(\theta_1-\theta_\omega)H(\theta_2-\theta_\omega)d\theta_1, \qquad (14)$$

where

$$\theta_2 = \theta_1 - (\pi - \psi)\,\mathrm{sgn}[\sin(\theta_1 - \theta_k)],$$
$$\psi = \cos^{-1}\{[1 - \tfrac{1}{2}\hat{m}\hat{t}(1+\chi)^2]/\chi^2\}$$

and χ is the root of the cubic equation

$$(\chi^2+1)(\chi-1) - \tfrac{1}{4}\hat{m}^2(1+\chi)^3 + \hat{m}\hat{t}(1+\chi) = 0. \tag{15}$$

Equation (14) appears as Eq. (46b) in Ref. 16 but its full derivation was not presented there.

From a handbook of algebra we can find the solution of Eq. (15). It is of interest to note that the cubic equation reflects the fact that, when k is sufficiently large, three pairs of interacting waves can exist for a given **k** and ω — see Fig. 4. It is also well known that Eq. (15) can have only one real root under certain conditions. This in turn corresponds to the case when **k** is sufficiently small and only one pair of interacting waves contributes to the induced field defined by ω and **k**. From algebraic manipulation we establish the condition $\delta > 1$ (δ is defined below) for which Eq. (15) will have a single root

$$\chi = \xi - b/3a,$$

where

$$\xi = (-\tfrac{1}{2}q + \sqrt{\delta})^{1/3} + (-\tfrac{1}{2}q - \sqrt{\delta})^{1/3},$$
$$\delta = (\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3, \quad p = (3a\rho^2 - 2b\rho + c)/a,$$
$$q = (b\rho^2 - a\rho^3 - c\rho + d)/a,$$
$$\rho = b/(3a), \quad a = (1 - \tfrac{1}{4}\hat{m}^2), \quad b = -(\tfrac{3}{4}\hat{m}^2 + 1),$$
$$c = (1 - \tfrac{3}{4}\hat{m}^2 + \hat{m}\hat{t}), \quad d = (\hat{m}\hat{t} - \tfrac{1}{4}\hat{m}^2 - 1).$$

In the last section we established that the maximum value of $|\mathbf{k}|$ is ω^2/g, so that $0 \leqslant \hat{m} \leqslant 2$. It can be shown that when $\hat{m}$ is less than a certain value $\hat{m}_c$ (≈ 0.98), for all $\hat{m}$ and $\hat{t}$ the cubic equation will be positive and then χ has only one real root. The region in which δ is negative and χ can have three different real roots is very narrow around $\hat{t} = 1$ (i.e., $\theta_1 = \theta_k$). This can be well explained by Fig. 4 where we see that only in a narrow sector (i.e., $\theta_1 - \theta_k \approx 0$) can there be three pairs of interacting wave trains for a given $\mathbf{q}_1$; and that this sector becomes narrower with increasing $\hat{m}$. On the other hand when k is less than the value for which the curves L and L' retain the single closed form, there will only be a single pair of interacting wave trains. Moreover, since the value of $f_{p0}(\mathbf{k},\omega)$ is proportional to $(1 - \cos\theta_{12})^2$ [see Eq. (14)], it will become smaller as $\hat{m}$ increases and the angle θ_{12} decreases. Therefore we can take $k_{\max} = \omega^2/(2g)$ (and thus $\hat{m} = 1$) as the upper limit of k, without introducing significant error. Then by direct integration of the wavenumber–frequency spectrum $f_{p0}(\mathbf{k},\omega)$ over the **k** plane we can obtain the frequency spectrum of the total noise pressure field

$$\begin{aligned} F_p(\omega,z) &= F_{pr}(\omega,z) + F_{pi}(\omega,z) \\ &= \int_0^{\omega/\alpha_1}\int_0^{2\pi} f_{p0}(\mathbf{k},\omega)k\,d\theta_k\,dk + \int_{\omega/\alpha_1}^{\omega^2/2g}\int_0^{2\pi} f_{p0}(\mathbf{k},\omega) \\ &\quad \times \exp[-2\sqrt{k^2 - (\omega/\alpha_1)^2}|z|]k\,d\theta_k\,dk \quad z \leqslant 0. \end{aligned} \tag{16}$$

If we set $\chi = 1$, $k = 0$, $\theta_{12} = \pi$, and $\theta_2 = \theta_1 + \pi$ in Eq. (14), Eq. (16) degenerates to the conventional result,

$$F_{p0}(\omega) = (\pi\rho_1^2 g^2 \omega^3 / 2\alpha_1^2) F_a^2(\tfrac{1}{2}\omega) I(\omega), \tag{17}$$

where

$$I(\omega) \equiv \int_{-\pi}^{\pi} H(\theta)H(\theta+\pi)\,d\theta$$

as derived in the literature by different authors.[6,9–11,16]

The spectra numerically calculated from Eq. (16), using the JONSWAP form of the wave spectrum[34] but with parameters appropriate to the New Zealand sea conditions,[2]

$$F_a(\sigma) = \alpha g^2 (2\pi)^{-4} \sigma^{-5} \exp[-\tfrac{5}{4}(\sigma/\sigma_m)^{-4}] \times \gamma^{\exp[-(\sigma/\sigma_m - 1)^2/(2c^2)]},$$

where

$$\alpha = 0.07(U_{10}^2/gX)^{0.27},$$
$$\sigma_m = 2.59(g^{0.72}/U_{10}^{0.44}X^{0.28}),$$
$$\gamma = 2.9,$$
$$c = 0.10, \quad \text{for } \sigma \leqslant \sigma_m \tag{18}$$
$$= 0.13, \quad \text{for } \sigma > \sigma_m \tag{19}$$

are shown in Fig. 6(a) for the selected observation depths of $z = -10$, -50, -100, -500, and -1000 m and a wind speed of 30 m/s. Similar spectra based on the Pierson–Moskowitz formalism of the wave spectra are given in Fig. 6(b).

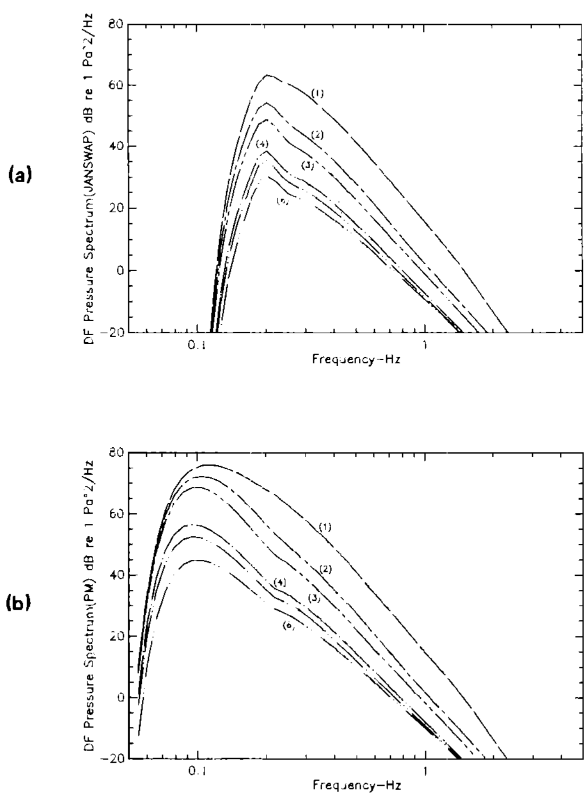

FIG. 6. Spectra of the wave-induced pressure field for a wind speed 30 m s^{-1} at the observation depths of (1) $z = -10$ m, (2) $z = -50$ m, (3) $z = -100$ m, (4) $z = -500$ m, (5) $z = -1000$ m, (6) $z =$ infinity. (a) For the JONSWAP spectrum, (b) For the PM spectrum.

In these figures curve (6) represents the spectrum of the homogeneous component of the noise field. This component, which is independent of depth, corresponds closely to the spectrum we presented earlier in Ref. 17. (A normalization factor, 2π, of the power spectrum has been adopted here.) However it is also apparent from Fig. 6 that, while the inhomogeneous component decreases with depth and frequency, it augments the induced pressure field substantially in the upper levels of the ocean. Accordingly, as has been pointed out by Schmidt and Kuperman,[36] in certain circumstances this component will be significant and must be taken into account in considering the total noise field. It appears however that at great observation depth in deep water, the peak of the wave-induced ULF field can still reach levels of about 30–40 dB *re*:1 Pa^2/Hz. The points they raise clearly require examination. A more-detailed discussion of the overall noise field, its various components, and their relation to the spectra already presented in Refs. 2, 16, and 17 and elsewhere[37] will be given in other papers in preparation.[38,39]

III. SUMMARY

An examination of the historical development of the subject has shown that while a general agreement exists between the observed ULF ocean- noise, pressure-field, and theoretical predictions based on the second-order wave–wave interactions, questions still remained about the validity of applying the perturbation procedure to real ocean situations and the justification of restricting the expansion series to the second-order terms only. These uncertainties had recently compounded a debate about the dominant mechanism of ULF ocean-noise generation.[35]

A review of the various theoretical treatments presented to date has been made to demonstrate the particular virtues of the two main analysis procedures, the classical perturbation expansion and the integral equation based on Lighthill's equation.

Numerical calculations have shown that in the deep ocean the average slope of the developed wave surface at the peak frequency of the Pierson–Moskowitz spectrum is sufficiently small to justify the main assumptions adopted in the perturbation analysis at the frequencies of present interest.

The relative importance of the pressure field resulting from the direct radiation of air turbulent motion in the atmosphere and that generated through the interaction of surface waves, has also been examined. The analysis shows that even though the direct radiation can be equal to and even more important than that produced by the interaction of ocean waves at the very early stage of sea growth, it is much smaller than the latter when the sea is even moderately developed. The dominance of the wave–wave interaction process as the source of ULF noise in realistic sea states is thus confirmed. Further an estimate of the peak values of the noise spectra generated by wave–wave interactions of different order m has shown that the noise spectrum level generated by these interactions will be $(m-1)\times 25$ dB less than that of the second-order interaction, and occur at a frequency $m\sigma_p$, where σ_p is the peak frequency of the surface wave spectrum. It follows that the truncation of the perturbation series to the second order is acceptable for most practical purposes.

With the second-order interaction confirmed as the most dominant mechanism of ULF noise generation, the currently used formula for the noise spectrum has been extended to include an exact expression for both the homogeneous and inhomogeneous components of the noise field resulting from second-order interactions. The analysis includes a geometrical description of the wave–wave interaction process. The more comprehensive expression established seems to provide for a more complete understanding of the processes involved.

ACKNOWLEDGMENT

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APPENDIX: DERIVATION OF $dq_1/d\omega$

By denoting $\chi=\sigma_2/\sigma_1$, $\hat{m}=2kg/\omega^2$, $\hat{t}=\cos\theta_{1k}$, and carrying out the integration of $\delta(\sigma_1+\sigma_2-\omega)$ by setting $dq_1=(dq_1/d\omega)d\omega$ we get

$$f_{p0}(\mathbf{k},\omega) = \frac{1}{4}\rho_1^2 g^4 \int\int_\Sigma \frac{[1-\cos(\theta_{12})]^2}{(1-\frac{1}{4}\hat{m}^2)} F_a(\sigma_1)F_a(\omega-\sigma_1) \tag{A1}$$

$$\times H(\theta_1-\theta_w)H(\theta_2-\theta_w)q_1\left(\frac{dq_1}{d\omega}\right)d\theta_1. \tag{A2}$$

Further, from the trigonometric relations between $\mathbf{k}$, $\mathbf{q}_1$, and $\mathbf{q}_2$ (see Fig. 2)

$$q_2^2=q_1^2+k^2-2kq_1\cos\theta_{1k},$$

$$\sigma_1=\omega/(1+\chi) \text{ and } q_1=\omega^2/g(1+\chi)^2,$$

where $\theta_{1k}=\theta_1-\theta_k$ is the angular difference, we can denote $\hat{t}=\cos\theta_{1k}$ and write:

$$\chi^4=1+\frac{k^2}{q_1^2}-\frac{2k\hat{t}}{q_1} = 1+\frac{k^2g^2(1+\chi)^4}{\omega^4}-\frac{2kg(1+\chi)^2\hat{t}}{\omega^2},$$

$$(\chi^4-1)=\tfrac{1}{4}\hat{m}^2(1+\chi)^4-\hat{m}\hat{t}(1+\chi)^2, \tag{A3}$$

$$(\chi^2+1)(\chi-1)-\tfrac{1}{4}\hat{m}^2(1+\chi)^3+\hat{m}\hat{t}(1+\chi)=0.$$

A careful examination of the geometrical relations between $\mathbf{k}$, $\mathbf{q}_1$, and $\mathbf{q}_2$ in Fig. 2 shows that θ_2 can be expressed in terms of θ_1 through the following relations:

$$\theta_2=\theta_1-\pi+\psi, \quad \text{for } \theta_k<\theta_1<\theta_k+\pi,$$

$$\theta_2=\theta_1+\pi-\psi, \quad \text{for } \theta_k-\pi<\theta_1<\theta_k,$$

or simply,

$$\theta_2=\theta_1-(\pi-\psi)\,\mathrm{sgn}[\sin(\theta_1-\theta_k)],$$

where $\mathrm{sgn}(\chi)=\chi/|\chi|$. The acute angle ψ can be obtained by using the trigonometric relations $q_1^2+q_2^2-2q_1q_2\cos\psi=k^2$ and $q_1^2+k^2-2q_1k\cos(\theta_1-\theta_k)=q_2^2$, whereupon

$$\cos\psi=[1-\hat{m}\hat{t}(1+\chi)^2/2]/\chi^2.$$

Finally to get the explicit form of the term $(dq_1/d\omega)$ we start from the equation:

$$\chi^4 = \left(1 - \frac{2k}{q_1}\hat{t} + \frac{k^2}{q_1^2}\right),$$

so that

$$4\chi^3 \frac{d\chi}{dq_1} = \frac{2k\hat{t}}{q_1^2} - \frac{2k^2}{q_1^3} = \frac{2k}{q_1^2}\left(\hat{t} - \frac{k}{q_1}\right),$$

$$\frac{d\chi}{dq_1} = \frac{k}{2\chi^3 q_1^2}\left(\hat{t} - \frac{k}{q_1}\right).$$

The required derivative $dq_1/d\omega$ can be obtained as follows:

$$\frac{dq_1}{d\omega} = \frac{2\omega}{g}(1+\chi)^{-2} - 2\frac{\omega^2}{g}(1+\chi)^{-3}\frac{d\chi}{dq_1}\frac{dq_1}{d\omega}$$

or

$$\frac{dq_1}{d\omega} = \frac{2\omega/g(1+\chi)^2}{1 + [2\omega^2/g(1+\chi)^3]\cdot(d\chi/dq_1)} = \frac{2\omega}{g(1+\chi)^2 + [2\omega^2/(1+\chi)]\cdot(k/2\chi^3 q_1^2)(\hat{t} - k/q_1)}.$$

By using the relations

$$\frac{k}{q_1} = \frac{\hat{m}}{2}(1+\chi)^2 \text{ and } q_1 = \frac{\omega^2}{g(1+\chi)^2}$$

or

$$\omega^2/q_1 = g(1+\chi)^2$$

we establish

$$\frac{dq_1}{d\omega} = \frac{2\omega/g}{(1+\chi)^2 + (1+\chi)\chi^{-3}(k/q_1)(\hat{t} - k/q_1)}$$
$$= \frac{2\omega\chi^3}{g(1+\chi)^2[\chi^2 - (\frac{1}{2}\hat{m}\hat{t} + 1)\chi + (1 - \frac{1}{2}\hat{m}\hat{t})]}$$

in which use has been made of the equalities (22). Substitution of the expression for $dq_1/d\omega$ into (21) leads straightforwardly to Eq. (14).

[1]W. Marsh, "Origin of the Knudsen spectra," J. Acoust. Soc. Am. **35**, 409–410 (1963).

[2]A. C. Kibblewhite and K. C. Ewans, "Wave–wave interactions, microseisms and infrasonic ambient noise in the ocean," J. Acoust. Soc. Am. **78**, 981–994 (1985).

[3]A. C. Kibblewhite, "Ocean noise spectrum below 10 Hz — mechanisms and measurements," in *Natural Mechanisms of Surface-generated Ambient Noise in the Ocean*, edited by B. Kerman (Plenum, New York, 1988).

[4]O. M. Phillips, *The Dynamics of the Upper Ocean* (Cambridge U. P., Cambridge, 1977), 2nd ed.

[5]M. Miche, "Mouvements ondulatoires de la mer en profondeur constante ou decroissante," Ann. Ponts Chaussees **114**, 25–78 (1944).

[6]M. S. Longuet-Higgins, "A theory of the origin of microseisms," Philos. Trans. R. Soc. London Ser. A **243**, 1–35 (1950).

[7]M. S. Longuet-Higgins, "Can sea waves cause microseisms? Symposium on Microseisms, September 1952, NAS-NRC Pub. No. 306, pp. 74–93 (1953).

[8]M. J. Lighthill, "On sound generated aerodynamically I: General theory," Proc. R. Soc. London Ser. A **211**, 564–587 (1952).

[9]K. Hasselmann, "A statistical analysis of the generation of microseisms," Rev. Geophys. **1**, 177–210 (1963).

[10]L. M. Brekhovskikh, "Underwater sound waves generated by surface waves in the ocean," Izv. Atmos. Ocean Phys. **2**, 582–587 (1966).

[11]B. Hughes, "Estimates of underwater sound (and infra sound) produced by non-linearly interacting ocean waves," J. Acoust. Soc. Am. **60**, 1032–1039 (1976).

[12]K. C. Ewans, "Ocean waves, microseisms and their interrelations," Doctoral dissertation, University of Auckland, New Zealand (1984).

[13]R. G. Adair, "Low frequency noise observations in the deep ocean," J. Acoust. Soc. Am. **80**, 633–645 (1986).

[14]S. C. Webb, "Coherent pressure fluctuations observed at two sites on the seafloor," Geophys. Res. Lett. **13**, 141–144 (1986).

[15]S. C. Webb and C. S. Cox, "Observations and modelling of seafloor microseisms," J. Geophys. Res. **91**, 7343–7358 (1986).

[16]A. C. Kibblewhite and C. Y. Wu, "The generation of microseisms and infrasonic ambient noise by nonlinear interactions of ocean surface waves," J. Acoust. Soc. Am. **85**, 1935–1945 (1989).

[17]A. C. Kibblewhite and C. Y. Wu, "A reexamination of the role of wave-wave interactions in ocean noise and microseims," J. Acoust. Soc. Am. **85**, 1946–1957 (1989).

[18]C. Y. Wu, "Wave-wave interactions and the infrasonic pressure field in the ocean," Doctoral dissertation, University of Auckland, New Zealand (1988).

[19]M. A. Isakovich and B. F. Kur'yanov, "Theory of low frequency noise in the ocean," Sov. Phys. Acoust. **16**, 49–58 (1970).

[20]J. H. Wilson, "Wind-generated noise modelling," J. Acoust. Soc. Am. **73**, 211–216 (1983).

[21]R. G. Adair, "Comments on the infrasonic noise theory of Isakovich and Kur'yanov and its modifiction by Wilson," J. Acoust. Soc. Am. **81**, 1192–1195 (1987).

[22]E. Y. Harper and P. G. Simpkins, "On the generation of sound in the ocean by surface waves," J. Sound Vib. **37**, 185–193 (1974).

[23]E. Y. T. Kuo, "Deep-sea noise due to surface motion," J. Acoust. Soc. Am. **43**, 1017–1024 (1968).

[24]V. V. Goncharov, "Sound generation in the ocean by the interaction of surface waves and turbulence," Izv. Atmos. Ocean. Phys. **6**, 1189–1196 (1970) (translated by F. Goodspeed).

[25]N. Curle, "The influence of solid boundaries upon aerodynamic sound," Proc. R. Soc. London Ser. A **231**, 505–514 (1955).

[26]J. E. Ffowcs-Williams and D. L. Hawkings, "Sound generation by turbulence and surfaces in arbitrary motion," Philos. Trans. R. Soc. London Sect. **246**, 312–342 (1969).

[27]S. P. Lloyd, "Underwater sound from surface waves according to the Lighthill–Ribner theory," J. Acoust. Soc. Am. **69**, 425–435 (1981).

[28]Y. P. Guo, "On sound generation by weakly non-linear interactions of surface gravity waves," J. Fluid Mech. **181**, 311–328 (1987).

[29]D. H. Cato, "Sound generation in the vicinity of the sea surface: Source mechanisms and the coupling to the received sound field," J. Acoust. Soc. Am. **89**, 1076–1095.

[30]M. S. Longuet-Higgins, D. E. Cartwright, and N. D. Smith (LCS), "Observations of the directional spectrum of sea waves using the motions of a floating buoy," in *Ocean Wave Spectra* (Prentice Hall, Englewood Cliffs, NJ, 1963), pp. 111–136.

[31]G. L. Tyler, C. C. Teague, R. H. Stewart, A. M. Peterson, W. H. Munk, and J. W. Joy, "Wave directional spectra from synthetic aperture observations of radio scatter," Deep Sea Res. **21**, 989–1016 (1974).

[32]W. J. Pierson, Jr., "A proposed spectral form for fully developed wind sea based on the similarity theory of S. A. Kataigorodskii," J. Geophys. Res. **69**, 5181–5189 (1964).

[33]W. J. Pierson, Jr., "The interpretation of wave spectrums in terms of the wind measured at a constant height," J. Geophys. Res. **69**, 5191–5203 (1964).

[34]K. Hasselmann, J. P. Barnett, E. Bouws, H. Carlson, D. E. Cartwright, K. Enke, J. A. Ewing, H. Glenapp, D. E. Hasselmann, P. Kruseman, A. Meerburg, P. Muller, D. J. Olbers, K. Richter, W. Sell, and H. Walden, "Measurements of wind-wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP)," Dent. Hydrogr. **2**, Suppl. A 8, No. 12 (1973).

[35]Panel discussion, in *Natural Mechanisms of Surface Generated Noise in the Ocean*, edited by B. R. Kerman (NATO ASI Series, Series C, Mathematical and Physical Science: 238) Proceedings of the NATO Advance Research Workshop held in Lerici, Italy, 15–19 June 1987.

[36]H. Schmidt and W. A. Kuperman, "Estimation of surface-noise level from low frequency seismo-acoustic ambient noise measurements," J. Acoust. Soc. Am. **84**, 2153–2162 (1988).
[37]C. S. Cox and D. C. Jacobs, "Cartesian diver observations of double frequency pressure fluctuations in the upper levels of the ocean," Geo. Res. Lett. **16**, 807–810 (1989).
[38]A. C. Kibblewhite and C. Y. Wu, "An analysis of the ULF acoustic noise field in the ocean", in *Proceedings of the Conference on Natural Physical Sources of Underwater Noise* (Cambridge U. P., Cambridge, 1990).
[39]A. C. Kibblewhite and C. Y. Wu, "The ULF pressure spectrum in shallow water environments" (in preparation).