The generation of infrasonic ambient noise in the ocean by nonlinear interactions of ocean surface waves

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Expressions for the spectra of infrasonic ocean noise and microseisms induced by nonlinear wave interaction are derived theoretically for an ocean environment modeled as a water layer overlying a solid half-space. A series of spectral transfer functions relating the source pressure field induced by the wave action to the underwater acoustic noise field and that of the microseisms this generates in the seabed are defined and calculated. The effect of bottom reflections on the transfer function is examined and an estimate of the contribution of the Rayleigh wave component of the microseism signal received onshore is also made.

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INTRODUCTION

In an earlier article,[1] the wave origin of infrasonic ocean noise in the frequency range of 0.1–1 Hz was confirmed by the close relation between ocean-wave spectra and those of the noise-pressure field derived from the induced microseisms. It was also shown that the acoustic-noise field and the microseism activity in the frequency range 0.1–5 Hz, both arise from nonlinear interactions of ocean surface waves. A large number of publications,[2–9] which are reviewed in Ref. 1, have considered the theoretical aspects of the generation of microseisms and ocean noise by such processes but their interrelation is not readily appreciated.

The purpose of this article is to present a somewhat more general analysis of the phenomena and draw attention to a number of aspects that do not appear to have been widely recognized previously. In particular, our analysis builds on earlier work by Hasselmann[5] and demonstrates that an acoustic field can arise not only from nonlinear interactions between ocean-wave components of nearly the same frequency and opposite directions of travel, but also between any two components satisfying certain, but less restrictive, conditions. We also examine the implications of some simplifying assumptions used in Ref. 1, whereby the microseism response was attributed solely to the vertical ground displacement, and the associated acoustic field in the water was derived from this by means of a simple impedance relation assuming the bottom to be a fluid half-space. The difficulties inherent in these assumptions in respect to shear-wave excitation, propagation outside the active ocean region, and signal enhancement through bottom reflectivity, were acknowledged in Ref. 1 (in Sec. V A) and reemphasised recently.[2,3]

To assist the smooth development of this discussion, a condensed derivation of the source pressure field induced by the nonlinear interaction of two ocean-wave trains is given in the first section of the article. The discussion in Sec. II is concerned with the derivation of the resulting acoustic-noise pressure field within a finite ocean volume, and that of the seabed motion this induces. In this analysis, the real ocean is modeled as a water layer overlying an elastic half-space, with the random source pressure field acting on the mean surface of the water layer. The frequency spectra of the source pressure field, the underwater noise field, and the vertical displacement of the seabed, for this simple geoacoustic model, are developed in Secs. III and IV. Finally, Sec. V is devoted to establishing an estimate of the contribution of Rayleigh wave energy induced by the source pressure field to the seismic signal received at a distance, and evaluating the effect of any fetch dependence of the wave field.

In a companion article[4] we review the experimental evidence presented in Ref. 1 in the light of this theoretical analysis. In later articles we will examine the directivity of the pressure field produced by nonlinear interactions, establish the relative levels of the homogeneous and inhomogeneous components, and consider the more general case of a multilayered seabed.

I. NONLINEAR INTERACTION BETWEEN PLANE SURFACE WAVE TRAINS AND THE RESULTING SOURCE PRESSURE FIELD

As was mentioned above, the nonlinear interaction of surface waves has been widely proposed as the main mechanism responsible for the generation of the double-frequency infrasonic pressure field and associated microseism activity at frequencies below 5 Hz. Several theoretical treatments[5–9] have been developed since Longuet-Higgin's original contribution to the subject.[10,11] Since these are based on fairly conventional perturbation approaches, only a simplified summary of them is given here before the present analysis and discussion is developed.

By denoting the velocity potential ϕ as (here and later, we use $\mathbf{x}$ to denote a vector and x its modulus)

$$\mathbf{v} = \nabla\phi , \tag{1}$$

the basic equations governing irrotational motion in a fluid (water) and the associated boundary conditions can be written as follows[12,13]:

momentum equation,

$$\frac{p}{\rho_1} + \frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + gz = 0; \tag{2}$$

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equation of continuity,

$$\frac{d\rho_1}{dt}+\rho_1\nabla^2\phi=0\,; \tag{3}$$

equation of elasticity,

$$\frac{dp}{d\rho_1}=\alpha_1^2\,; \tag{4}$$

free surface condition,

$$-\gamma G(\zeta)+\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\mathbf{v}^2\right)_\zeta+g\zeta=0\,; \tag{5}$$

and surface kinematic condition,

$$\left(\frac{d}{dt}(\zeta-z)\right)_\zeta=0\,, \tag{6}$$

where p is the pressure, $\mathbf{v}$ is the particle velocity vector, ρ_1 is the water density, α_1 is the sound speed in water, g is the gravitational acceleration, $\zeta=\zeta(x,y)$ is the displacement of the ocean surface from its equilibrium plane $z=0$, γ is the kinematic surface tension which is the ratio of the surface tension to water density, and $G(\zeta)=\nabla^2_=\zeta/[1+(\nabla_=\zeta)^2]^{3/2}$ is the principal radii of the surface curvature[12] [note $\nabla\equiv(\partial/\partial x,\partial/\partial y)$]. Even though it is responsible for the generation of waves, the turbulent air pressure acting on the ocean surface is neglected when discussing the interaction of established surface waves.

By applying the total derivative, $d/dt=\partial/\partial t+\mathbf{v}\cdot\nabla$, to Eqs. (2) and (5) and noting from Eqs. (3) and (4) that near the surface,

$$\frac{d}{dt}\left(\frac{p}{\rho_1}\right)=\frac{1}{\rho_1}\frac{d\rho_1}{dt}\left(\alpha_1^2-\frac{p}{\rho_1}\right)\approx\alpha_1^2\frac{1}{\rho_1}\frac{d\rho_1}{dt}=-\alpha_1^2\nabla^2\phi\,,$$

we can rewrite the momentum equation, and the boundary condition at $z=\zeta$ as

$$\alpha_1^2\nabla^2\phi=\frac{\partial^2\phi}{\partial t^2}+\frac{\partial}{\partial t}(\nabla\phi)^2+\frac{1}{2}\nabla\phi\cdot\nabla[(\nabla\phi)^2]+g\frac{\partial\phi}{\partial z} \tag{7}$$

and

$$\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial}{\partial t}(\nabla\phi)^2+\frac{1}{2}\nabla\phi\cdot\nabla[(\nabla\phi)^2]+g\frac{\partial\phi}{\partial z}\right)_{z=\zeta}-\gamma\frac{d}{dt}G(\zeta)=0. \tag{8}$$

As ζ is not known *a priori*, it is inconvenient to express the boundary condition at $z=\zeta$. This difficulty is overcome by expanding Eq. (8) about the plane $z=0$ as a power series of ζ and introducing the perturbation series (Ref. 6)

$$\phi=s\phi^{(1)}+s^2\phi^{(2)}+\cdots\ \text{and}\ \zeta=s\zeta^{(1)}+s^2\zeta^{(2)}+\cdots\,. \tag{9}$$

With s an adequately small quantity, this leads to the following series for the different orders of s:

$$(s^1):\quad \alpha_1^2\nabla^2\phi^{(1)}-\frac{\partial^2\phi^{(1)}}{\partial t^2}-g\frac{\partial\phi^{(1)}}{\partial z}=0,$$

$$\frac{\partial^2\phi^{(1)}}{\partial t^2}+g\frac{\partial\phi^{(1)}}{\partial z}-\gamma\nabla^2_=\frac{\partial\phi^{(1)}}{\partial z}=0\quad\text{at } z=0, \tag{11}$$

and

$$(s^2):\quad \alpha_1^2\nabla^2\phi^{(2)}-\frac{\partial^2\phi^{(2)}}{\partial t^2}-g\frac{\partial\phi^{(2)}}{\partial z}=\frac{\partial}{\partial t}(\nabla\phi^{(1)})^2\,, \tag{12}$$

$$\frac{\partial^2\phi^{(2)}}{\partial t^2}+g\frac{\partial\phi^{(2)}}{\partial z}-\gamma\nabla^2_=\frac{\partial\phi^{(2)}}{\partial z}=-\frac{\partial}{\partial t}(\nabla\phi^{(1)})^2-\zeta^{(1)}\frac{\partial^3\phi^{(1)}}{\partial z\,\partial t^2}-\zeta^{(1)}g\frac{\partial^2\phi^{(1)}}{\partial z^2}\quad\text{at } z=0\,, \tag{13}$$

in which use has been made of the notation $\nabla_=\equiv(\partial/\partial x,\partial/\partial y)$ and the approximation

$$\frac{dG(\zeta)}{dt}\approx\frac{d}{dt}\nabla^2_=\zeta=\nabla^2_=\frac{dz}{dt}=\nabla^2_=\frac{\partial\phi}{\partial z}.$$

The first-order solution $\phi^{(1)}$ can easily be found as

$$\phi^{(1)}=Q\exp[i(\mathbf{q}\cdot\mathbf{r}-\sigma t)+qz]\,, \tag{14}$$

which decays exponentially with depth, and further using Eq. (6) we obtain for $\zeta^{(1)}$ the first-order solution

$$\zeta^{(1)}(\mathbf{r},t)=(iQq/\sigma)\exp[i(\mathbf{q}\cdot\mathbf{r}-\sigma t)]\,, \tag{15}$$

where the amplitude Q is a constant, $\mathbf{q}$ and $\mathbf{r}$ are the horizontal wavenumber and position vectors, σ is the angular frequency, with

$$\sigma^2=gq(1+\gamma q^2/g)\,. \tag{16}$$

When $\gamma q\ll g/q$ (for an air–water interface at 20 °C, $\gamma=74$ cm^3 s^{-2}, see Ref. 14, p. 176), Eq. (16) reduces to the gravity-wave dispersion relation

$$\sigma^2=gq\ \text{or}\ \sigma=\sqrt{gq}\,. \tag{17}$$

The second-order field $\phi^{(2)}$ can be obtained by introducing the first-order field $\phi^{(1)}$ into the right-hand side of Eqs. (12) and (13) and by solving the resulting equations. These two equations can be separated, for example, as Brekhovskikh[6] has done, into the two parts ϕ_a^2 and $\phi_b^{(2)}$, which satisfy, respectively, an inhomogeneous equation with an homogeneous boundary condition and an homogeneous equation with an inhomogeneous boundary condition.

It can be shown without difficulty that both $\phi_a^{(2)}$ and $\phi_b^{(2)}$, obtained by simply substituting the first-order solutions $\phi^{(1)}$ and $\zeta^{(1)}$ into the second-order equations will be of the form $\exp[i2(\mathbf{q}\cdot\mathbf{r}-\sigma t)+2qz]$, which indicates that the attenuation rate in the direction of the negative z axis is twice that of the first-order field. Neither of these solutions is therefore of importance in the generation of microseisms and the underwater noise field in the deep ocean.

The nonlinear interaction between two trains of surface waves can, however, produce pressure waves that do not decay with depth. In fact, by substituting a first-order field $\phi^{(1)}$ consisting of any two trains of plane surface waves such that

$$\phi^{(1)}=Q_1(\mathbf{q}_1)\exp[i(\mathbf{q}_1\cdot\mathbf{r}-\sigma_1 t)+q_1 z]+Q_2(\mathbf{q}_2)\exp[i(\mathbf{q}_2\cdot\mathbf{r}-\sigma_2 t)+q_2 z]=\phi_1^{(1)}+\phi_2^{(1)}\,,$$

and

$$\zeta^{(1)}=\zeta_1^{(1)}+\zeta_2^{(2)}$$

into the right-hand side of the inhomogeneous boundary condition satisfied by $\phi_b^{(2)}$, we can finally write after some manipulation

$$\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z} - \gamma\nabla_=^2 \frac{\partial}{\partial z}\right)\phi_b^{(2)}$$
$$\approx i2Q_1(\mathbf{q}_1)Q_2(\mathbf{q}_2)[(q_1q_2 - \mathbf{q}_1\cdot\mathbf{q}_2)(\sigma_1+\sigma_2)$$
$$+\gamma\tfrac{1}{2}q_1q_2(q_1^3/\sigma_2 + q_2^3/\sigma_1)]$$
$$\times \exp\{i[(\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{r} - (\sigma_1+\sigma_2)t]\}. \quad (18)$$

A possible solution of $\phi_b^{(2)}$ is then

$$\phi_b^{(2)} = BQ_1Q_2 \times \exp\{i[(\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{r} - (\sigma_1+\sigma_2)t + \mu_0 z]\}. \quad (19)$$

By substituting Eq. (19) into Eq. (18), we get equations for μ_0 and B (Ref. 6):

$$-\alpha_1^2(|\mathbf{q}_1+\mathbf{q}_2|^2 + \mu_0^2) - ig\mu_0 + (\sigma_1+\sigma_2)^2 = 0 \quad (20)$$

and

$$B = \frac{2[(q_1q_2 - q_1\cdot q_2)(\sigma_1+\sigma_2) + \gamma q_1q_2(\sigma_1q_1^3 + \sigma_2q_2^3)/(2\sigma_1\sigma_2)]}{\mu_0(g+\gamma|\mathbf{q}_1+\mathbf{q}_2|^2) + i(\sigma_1+\sigma_2)^2}. \quad (21)$$

In Eq. (20), we choose the root μ_0 as

$$\mu_0 = -(1/2\alpha_1^2)\{ig + \sqrt{4\alpha_1^2[(\sigma_1+\sigma_2)^2 - \alpha_1^2|\mathbf{q}_1+\mathbf{q}_2|^2] - g^2}\} \quad (22)$$

to ensure that the wave-induced pressure field $\phi_b^{(2)}$ tends to zero when z goes to negative infinity. Moreover, as $\phi_b^{(2)}$ represents a downward traveling pressure wave, it follows that

$$4\alpha_1^2[(\sigma_1+\sigma_2)^2 - \alpha_1^2|\mathbf{q}_1+\mathbf{q}_2|^2] - g^2 > 0. \quad (23)$$

By denoting $\mathbf{q}_1\cdot\mathbf{q}_2 = q_1q_2\cos\theta$, $a_0 = g/(\alpha_1\sigma_1)$, $\eta_1 = 1+\gamma q_1^2/g$, $\eta_2 = 1+\gamma q_2^2/g$, and $\chi = \sigma_2/\sigma_1$, and solving the inequality (23) we can establish the range of crossing angles over which two plane surface waves interacting with each other will produce such a traveling acoustic wave. It follows from Eqs. (23) and (16) that

$$-1 < \cos\theta < -\frac{1}{2\chi^2}\frac{\eta_2}{\eta_1}\left[\left(1+\frac{\eta_1^2}{\eta_2^2}\chi^4\right) - a_0^2\eta_1^2(1+\chi)^2 + \frac{1}{4}a_0^4\eta_1^2\right]. \quad (24)$$

Since a_0 is the ratio of surface gravity wave velocity and the ocean sound velocity, and is generally small for the frequencies of interest, we can neglect the last term in Eq. (24). For the same reason it follows from Eq. (22) that $\phi_b^{(2)}$ will still be an inhomogeneous wave decaying exponentially in the negative z direction except when the horizontal wave vector $|\mathbf{q}_1+\mathbf{q}_2|\to 0$ (when $\theta\approx 180°$), and $\sigma_1\approx\sigma_2$.

It is of interest to establish the active range of the angle θ and the frequency ratio σ_2/σ_1 for which this condition would apply. For the sake of simplicity, we restrict discussion to gravity waves, assume $\eta_1\approx\eta_2 = 1$ and express the right-hand side of Eq. (24) as a function of frequency whereupon

$$v(\sigma_1,\sigma_2) = v(a_0,\chi) = (-1/2\chi^2)[(1+\chi^4) - a_0^2(1+\chi)^2]. \quad (25)$$

Denoting $\chi = 1-\Delta$, and neglecting terms involving $\Delta^s(s>2)$, we find after some manipulation that v has a maximum value

$$v_{\max} \approx -1 + 2a_0^2 \text{ at } \chi = 1 - a_0^2/2.$$

This value corresponds to an angular limit,

$$\theta_m = \cos^{-1}(-1+2a_0^2) \text{ or } \theta'_m = 180° - \theta_m \approx 2a_0.$$

Further, two roots of Eq. (25) can be found for the case $v=-1$, viz., $\chi_r = 1\pm a_0$. Figure 1 shows the relation between v and χ, and the shadow region represents the domain of the active values of the crossing angle θ and frequency raaio σ_2/σ_1.

The vector diagram corresponding to Fig. 1 is shown in Fig. 2. We see that, for a given wavenumber vector $\mathbf{q}_1$, the interacting wave vector $\mathbf{q}_2$ will lie in the cone $OBCD$ with its head in the shadow region D_g, which can be considered to approximate a circular disk. The resultant wave vector $\mathbf{q}_1+\mathbf{q}_2$, will lie in a disk, D_r, centered at point O and having the same shape as D_g.

From the above, it is seen that the nonlinear interaction of two trains of plane surface waves can induce a second–order potential $\phi_b^{(2)}$, corresponding to a homogeneous sound wave, if the wavenumber vectors of these two waves satisfy the condition (24). On the other hand, if the modulus $|\mathbf{q}_1+\mathbf{q}_2|$ is greater than the sound wavenumber ω/α_1, the induced field $\phi_b^{(2)}$ will be an inhomogeneous one decaying with depth, but at a rate less than that of the first-order field. The effect of $\phi_b^{(2)}$ can be established in terms of a source pressure field $P_0(\mathbf{r},t)$, acting on the mean surface $z=0$, which takes the form

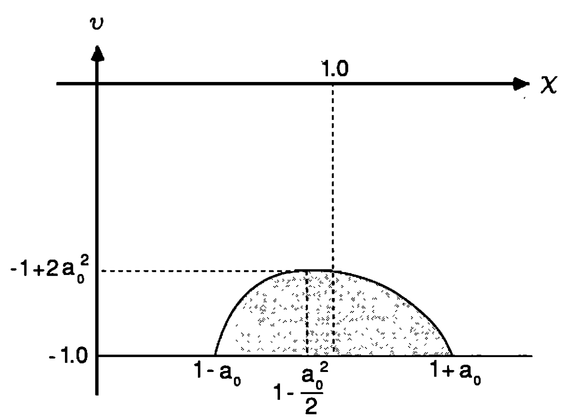

FIG. 1. Domain of the viable values of the crossing angle $\theta(v=\cos\theta)$ and frequency ratio $\sigma_2/\sigma_1(\chi=\sigma_2/\sigma_1)$.

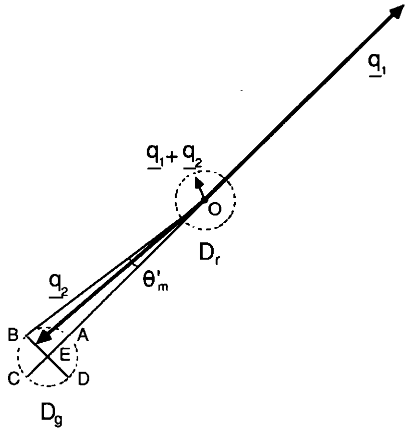

FIG. 2. Vector diagram of interacting waves.

$$P_0(\mathbf{r},t) = -\rho_1\left(\frac{\partial \phi_b^{(2)}}{\partial t}\right)_{z=0}$$
$$= iB\rho_1 Q_1 Q_2(\sigma_1+\sigma_2)$$
$$\times \exp\{i[\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{r}-(\sigma_1+\sigma_2)t]\}, \quad (26)$$

with B defined by Eq. (21). The analysis in subsequent sections is based on this concept of a source pressure field.

II. RESPONSE OF A LIQUID LAYER OVERLYING A SOLID HALF-SPACE TO A PLANE-WAVE PRESSURE FIELD

Before proceeding to discuss the wind–wave-induced component of underwater noise arising from wave interactions, in a practical situation, it will be helpful to derive the response of a typical ocean environment to a plane pressure wave acting on the surface. For this purpose, the ocean is modeled as a water layer of depth H, overlying a solid half-space. The geometry is shown in Fig. 3 where ρ_1 and ρ_2 are the densities, α_1 and α_2 are the compressional-wave velocities of the water and seabed, respectively, and β_2 is the shear-wave speed in the seabed.

Let us introduce displacement potentials[5,10] Φ_ν and $\boldsymbol{\Psi}_\nu$ [in Eqs. (27) and (28), $\boldsymbol{\Psi}_2$ denotes a vector of which the modulus is Ψ_2] ($\nu = 1,2$) such that the particle-displacement vector $\mathbf{S}_\nu$ takes the form

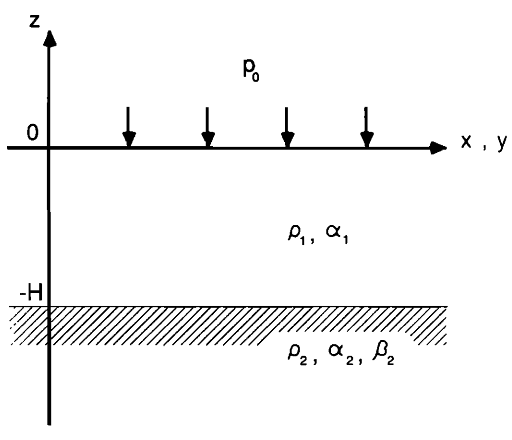

FIG. 3. Geometry of the geoacoustic model.

$$\mathbf{S}_\nu(u_\nu, v_\nu \omega_\nu) = \nabla\Phi_\nu + \nabla\times\boldsymbol{\Psi}_\nu \quad (27)$$

where u_ν, u_ν, ω_ν are, respectively, the x, y, and z components of $\mathbf{S}^\nu$. Since water cannot sustain a shear stress, it follows from the continuity of stress that no horizontally polarized wave component (SH wave) will exist in the solid half-space. Accordingly, we can write[5]

$$\boldsymbol{\Psi}_2 = (\mathbf{k}/k)\times\mathbf{n}\Psi_2, \quad (28)$$

where $\mathbf{k}$ is the horizontal wavenumber vector and $\mathbf{n}$ the unit vector along the z axis. Here, Φ_ν and Ψ_2 (the amplitude of vector $\boldsymbol{\Psi}_2$) satisfy the wave equations,

$$\nabla^2\Phi_\nu + (\omega^2/\alpha_\nu^2)\Phi_\nu = 0 \quad (29)$$

and

$$\nabla^2\Psi_2 + (\omega^2/\beta_2^2)\Psi_2 = 0, \quad (30)$$

where $\omega = 2\pi f$ is the angular frequency. Substituting the assumed solutions

$$\left.\begin{aligned}\Phi_1 &= B_1\exp(-ik_1'z) + C_1\exp(ik_1'z)\\ \Phi_2 &= B_2\exp(-ik_2'z)\\ \Psi_2 &= D_2\exp(-ik_2''z)\end{aligned}\right\}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)], \quad (31)$$

into the boundary conditions at $z=0$ and $z=-H$, we obtain

at $z = 0$

$$P_{033} = P_{133}:\ \lambda_1\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial \omega_1}{\partial z}\right) = P_0(\mathbf{k},\omega,\mathbf{r},t);$$

at $z = -H$

$$P_{133} = P_{233}:$$
$$\lambda_2\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial \omega_2}{\partial z}\right) + 2\mu_2\frac{\partial \omega_2}{\partial z}$$
$$= \lambda_1\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial \omega_1}{\partial z}\right);$$
$$P_{223} = P_{232} = 0:\ \frac{\partial v_2}{\partial z}+\frac{\partial \omega_2}{\partial y} = 0; \quad (32)$$
$$P_{231} = P_{213} = 0:\ \frac{\partial \omega_2}{\partial x}+\frac{\partial u_2}{\partial z} = 0;$$
$$\omega_1 = \omega_2;$$

where

$$\mathbf{k} = (k_x, k_y), \quad \mathbf{r} = (x,y),$$
$$k_\nu' = \sqrt{(\omega/\alpha_\nu)^2 - k^2}, \quad k_2'' = \sqrt{(\omega/\beta_2)^2 - k^2},$$
$$\alpha_1^2 = \frac{\lambda_1}{\rho_1}, \quad \alpha_2^2 = \frac{(2\mu_2+\lambda_2)}{\rho_2}, \quad \beta_2^2 = \frac{\mu_2}{\rho_2},$$

and λ_ν, μ_2 are the elastic constants of the media, P_0 is the source pressure field acting on the water surface, and $P_{\nu lm}$ is the pressure component acting at the interfaces of the two media.

The substitution of Eqs. (31) into (32) leads, after solving the system of algebraic equations for coefficients B_1, B_2, C_1, and D_2, to an expression for the vertical displacement of the seabed; viz.,

$$W_{-H}(\mathbf{k},\omega,\mathbf{r},t)=\frac{\partial\Phi_1}{\partial z}$$
$$=-\frac{1}{\rho_1\omega^2}\frac{\sqrt{\omega^2/\alpha_1^2-k^2}}{J(k^2,\omega^2)}P_0$$
$$\times\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\,, \qquad (33)$$

where

$$J(k^2,\omega^2)=\sin(k_1'H)$$
$$+i\frac{k_1'\beta_2^4\rho_2}{k_2'\omega^4\rho_1}[(k^2-k_2''^2)^2$$
$$+4k^2k_2'k_2'']\cos(k_1'H)\,. \qquad (34a)$$

Equations (33) and (34a) can also be derived directly by using the bottom reflection coefficient R_b so that

$$J(k^2,\omega^2)=\sin(k_1'H)$$
$$+i[(1+R_b)/(1-R_b)]\cos(k_1'H)\,. \qquad (34b)$$

In the same way, we can establish an expression for the pressure field in the water layer,

$$P(\mathbf{k},\omega;\mathbf{r},z,t)=[J(k^2,\omega^2,z)/J(k^2,\omega^2)]P_0(\mathbf{k},\omega;\mathbf{r},t)\,, \qquad (35)$$

where

$$J(k^2,\omega^2,z)=\sin[k_1'(H+z)]$$
$$+i\frac{k_1'}{k_2'}\frac{\beta_2^4}{\omega^4}\frac{\rho_2}{\rho_1}[(k^2-k_2''^2)^2$$
$$+4k^2k_2'k_2'']\cos[k_1'(H+z)] \qquad (36a)$$

and

$$J(k^2,\omega^2)=J(k^2,\omega^2,z)_{z=0}\,,$$

or, more generally,

$$\frac{J(k^2,\omega^2,z)}{J(k^2,\omega^2)}=\frac{\exp[-ik_1'(H+z)]+R_b\exp[ik_1'(H+z)]}{\exp(-ik_1'H)+R_b\exp(ik_1'H)}\,. \qquad (36b)$$

III. THE UNDERWATER INFRASONIC NOISE FIELD INDUCED BY NONLINEAR INTERACTIONS IN SURFACE WIND WAVES

Our discussion relating to the underwater pressure field produced by two trains of plane surface waves can be extended readily to the case of a real sea in which wave components are distributed over a certain range of frequencies and directions. As is usually done, we suppose that a real surface can be described as a time-stationary and space homogeneous stochastic process, so that the first-order velocity potential $\phi^{(1)}$ and the surface displacement $\zeta^{(1)}$ (later in this article we will be mainly interested in surface gravity waves) can be expressed in terms of Winer's generalized harmonic analysis[5,12,15] as

$$\phi^{(1)}(\mathbf{r},z,t)=\int\exp[i(\mathbf{q}\cdot\mathbf{r}-\sigma t)+qz]dQ(\mathbf{q}) \qquad (37)$$

and

$$\zeta^{(1)}(\mathbf{r},t)=\int\exp[i(\mathbf{q}\cdot\mathbf{r}-\sigma t)]dZ(\mathbf{q})\,, \qquad (38)$$

where, according to the boundary condition Eq. (18),

$$dZ(\mathbf{q})\equiv i(q/\sigma)dQ(\mathbf{q}),\quad \sigma=\sqrt{gq}\,. \qquad (39)$$

The two-dimensional, variance-density spectrum of the surface displacement is then

$$f_\zeta(\mathbf{q})=\frac{\langle|dZ(\mathbf{q})|^2\rangle}{d\mathbf{q}}=\frac{\langle|dZ|^2\rangle}{q\,dq\,d\theta}\,, \qquad (40)$$

with $\langle\ \rangle$ denoting the ensemble average.

In Sec. I, we show that two interacting trains of plane surface waves satisfying certain conditions will induce a source pressure field acting on the mean surface of the sea given by

$$P_0(\mathbf{r},t)=iB\rho_1Q_1(\mathbf{q}_1)Q_2(\mathbf{q}_2)(\sigma_1+\sigma_2)$$
$$\times\exp\{i[(\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{r}-(\sigma_1+\sigma_2)t]\}\,. \qquad (41)$$

If consideration is restricted to gravity waves only, then from Eq. (21),

$$B=2(q_1q_2-\mathbf{q}_1\cdot\mathbf{q}_2)(\sigma_1+\sigma_2)/[\mu_0g+i(\sigma_1+\sigma_2)^2]\,, \qquad (42)$$

or, alternatively, using the definition of μ_0 given by Eq. (22),

$$B=[2(q_1q_2-q_1\cdot\mathbf{q}_2)(\sigma_1+\sigma_2)/\sqrt{(\sigma_1+\sigma_2)^4-g^2(\mathbf{q}_1+\mathbf{q}_2)^2}]\exp(i\epsilon)\,, \qquad (43)$$

where ϵ is a phase factor.

For the case of a real sea, the source pressure field induced by any two of the stochastic wave components with nearly opposite propagation directions and nearly equal frequencies, can then be expressed as a Stieltjes integral, i.e.,

$$P_0(\mathbf{r},t)=\int\int M(\mathbf{q}_1+\mathbf{q}_2)\exp\{i[(\mathbf{q}_1,\mathbf{q}_2)\cdot\mathbf{r}-(\sigma_1+\sigma_2)t]\}$$
$$\times dZ_1(\mathbf{q}_1)\,dZ_2(\mathbf{q}_2)\,, \qquad (44)$$

where

$$M(\mathbf{q}_1,\mathbf{q}_2)=[\rho_1(q_1q_2-\mathbf{q}_1\cdot\mathbf{q}_2)(\sigma_1+\sigma_2)^2\sigma_1\sigma_2/q_1q_2\sqrt{(\sigma_1+\sigma_2)^4-g^2|q_1+\mathbf{q}_2|^2}]$$
$$\times\exp(i\epsilon')\,, \qquad (45)$$

with $\epsilon'=\epsilon-\frac{1}{2}\pi$.

Further, if the pressure field is a zero-mean process, the wavenumber frequency spectrum of the field is then established as the Fourier transform of the covariance function of the source pressure field $R(\tilde{\rho},\tau)$:

$$f_p(\mathbf{k},\omega)=\int\int R(\tilde{\boldsymbol{\rho}},\tau)\exp[-i(\mathbf{k}\cdot\tilde{\boldsymbol{\rho}}-\omega\tau)]d\tilde{\boldsymbol{\rho}}\,d\tau$$
$$=\int|M(\mathbf{q}_1,\mathbf{k}-\mathbf{q}_1)|^2$$
$$\times\delta[\sqrt{g}(\sqrt{|\mathbf{q}_1|}+\sqrt{|\mathbf{k}-\mathbf{q}_1|})-\omega]f_\zeta(q_1)$$
$$\times f_\zeta(\mathbf{k}-\mathbf{q}_1)d\mathbf{q}_1, \qquad (46a)$$

in which use has been made of the orthogonality

$$\langle dZ(\mathbf{q})dZ^*(\mathbf{q}')\rangle = \begin{cases} 0, & \text{for } \mathbf{q}\neq\mathbf{q}', \\ \langle|dZ(\mathbf{q})|^2\rangle, & \text{for } \mathbf{q}=\mathbf{q}'. \end{cases}$$

By substituting $\mathbf{q}_1+\mathbf{q}_2=\mathbf{k}$, $\sigma_1+\sigma_2=\omega$, $\sigma_2/\sigma_1=\chi$, and $f_s(\mathbf{q}_v) = g^2F_a(\sigma_v)H(\theta_v/(2\sigma_v^3)$ into Eq. (46a) we obtain, after some manipulations,

$$f_p(\mathbf{k},\omega) = \frac{1}{2}\omega\rho_1^2g^2\int_0^{2\pi}\frac{[1-\cos(\theta_1-\theta_2)]^2\chi^2}{(1+\chi)^2(1-\frac{1}{4}\hat{m}^2)[\chi^2-(\frac{1}{2}\hat{m}\hat{t}+1)\chi+(1-\frac{1}{2}\hat{m}\hat{t})]} \times F_a\left(\frac{\omega}{1+\chi}\right)F_a\left(\frac{\chi\omega}{1+\chi}\right)H(\theta_1)\,H(\theta_2)\,d\theta_1, \tag{46b}$$

where $\hat{m}=2kg/\omega^2$, $\hat{t}=\cos(\theta_1-\theta_k)$, and θ_1, θ_2, and θ_k are, respectively, the angles of the wave vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{k}$. Also, $H(\theta)$ and $F_a(\sigma)$ are defined by Eq. (49). For the case of homogeneous waves, i.e., when $k\ll\omega/\alpha_1$, we can use the "double frequency" and "opposite-direction" approximation in the calculation of Eq. (46b), and put

$$\mathbf{q}_1+\mathbf{q}_2\approx 0 \text{ and } \sigma_1+\sigma_2\approx 2\sigma_1.$$

Noting further that,

$$\int A(\sigma_1)\delta(2\sigma_1-\omega)d\sigma_1 = \frac{1}{2}A\left(\frac{1}{2}\omega\right)$$

and that

$$|M|^2\approx 4\rho_1^2\sigma_1^4,\quad d\mathbf{q}_1 = q_1\,dq_1\,d\theta = (2\sigma_1^3/g^2)d\theta\,d\sigma_1,$$

we obtain from Eq. (46b) that

$$f_p(\mathbf{k},\omega) = \frac{\rho_1^2\omega^7}{32g^2}\int_{-\pi}^{\pi} f_\zeta(\mathbf{q}_1)f_\zeta(-\mathbf{q}_1)d\theta. \tag{47}$$

Equation (47) can be expressed alternatively as [see Ref. 5, Eq. (2.15)]

$$f_p(\mathbf{k},\omega) = \tfrac{1}{2}\rho_1^2g^2\omega F_a^2(\tfrac{1}{2}\omega)I(\omega), \tag{48}$$

by use of the following definitions and relations:

$$\begin{aligned} f_\zeta(\mathbf{q}_1) &\equiv g^2F_a(\sigma_1)H(\theta)/2\sigma_1^3, \\ f_\zeta(-\mathbf{q}_1) &\equiv g^2F_a(\sigma_1)H(\theta+\pi)/2\sigma_1^3, \end{aligned} \tag{49}$$

and

$$I(\omega) = \int_{-\pi}^{\pi} H(\theta)\,H(\theta+\pi)\,d\theta,$$

where $H(\theta)$ is the normalized spreading function of the wave spectral energy, and is defined as

$$H(\theta) = \frac{1}{H_0}G(\theta) \text{ and } H_0 = \int_{-\pi}^{\pi} G(\theta)\,d\theta.$$

Integration of the wavenumber spectrum $f_p(\mathbf{k},\omega)$ with respect to the wavenumber vector $\mathbf{k}$ establishes the frequency spectrum of the source pressure field $F_p(\omega)$ as

$$F_p(\omega) = \int_0^{2\pi}\int_0^{k_{\max}} f_p(\mathbf{k},\omega)k\,dk\,d\theta_k.$$

It can be shown[16,17] that for the second-order interaction discussed here $k_{\max}$ is ω^2/g and that $\omega^2/(2g)$ can be used as a reasonable numerical limit. Defining $\eta_{ir}(\omega,z)$ as the ratio of the inhomogeneous [integrating $\omega/\alpha_1\to\omega^2/(2g)$] and the homogeneous (integrating $0\to\omega/\alpha^1$) components, we can write

$$\begin{aligned} F_p(\omega) &= [1+\eta_{ir}(\omega,0)]\int_0^{2\pi}\int_0^{\omega/\alpha_1} f_p(\mathbf{k},\omega)k\,dk\,d\theta_k \\ &= [1+\eta_{ir}(\omega,0)]\frac{\pi\rho_1^2g^2\omega^3}{2\alpha_1^2}F_a^2\left(\frac{1}{2}\omega\right)I(\omega). \end{aligned} \tag{50}$$

The spectrum of the noise-pressure field arising from the wave interactions can then be expressed as

$$F_N(\omega,z) = [1+\eta_{ir}(\omega,z)] \times(\pi\rho_1^2g^2\omega^3/\alpha_1^2)F_a^2(\tfrac{1}{2}\omega)I(\omega)I_N, \tag{51}$$

where

$$\eta_{ir}(\omega,z) = \frac{\int_0^{2\pi}\int_{\omega/\alpha_1}^{\omega^2/(2g)}|J(k^2,\omega^2,z)/J(k^2,\omega^2)|^2f_p(\mathbf{k},\omega)k\,dk\,d\theta_k}{\int_0^{2\pi}\int_0^{\omega/\alpha_1}|J(k^2,\omega^2,z)/J(k^2,\omega^2)|^2f_p(\mathbf{k},\omega)k\,dk\,d\theta_k} \tag{52}$$

and

$$I_N(\omega) = \int_0^1\left|\frac{J(k^2,\omega^2,z)}{J(k^2,\omega^2)}\right|^2\kappa\,d\kappa,\quad \text{in which } \kappa = k\alpha_1/\omega. \tag{53}$$

Equation (50) with $\eta_{ir}(\omega,0)=0$ can also be derived from Hasselmann's Eq. (2.15),[5] Brekhovskikh's Eq. (53),[6] and Hughes' Eq. (33)[7] if due allowance is made for the use of a single- or double-sided spectrum. Equation (51) is the exact solution of the medium response for the model assumed and incorporates the effects of seabed reflection and enhancement.[7,9] As expected, Eq. (51) reduces to Eq. (50) when $z=0$.

It has been shown[16,17] that $\eta_{ir}(\omega,z)$ decreases rapidly with increasing depth and frequency. For a shallow depth, say at 100 m, the quantity 10 log η_{ir} can be as high as 25–30 dB at 0.2 Hz, dropping to less than 5 dB at 1.0 Hz. At a depth of 1000 m these values reduce to 0 and − 40 dB, respectively. The implications of this depth dependence in respect of the New Zealand experiment (and others) has been examined. It has been established that the wave-induced pressure field described in Refs. 1 and 2 is not seriously in error even though consideration was restricted to the homogeneous component only. However, a detailed discussion of this question is deferred to our companion article.[4]

By a similar procedure, we can also establish the frequency spectrum of the vertical displacement of the seabed. This leads finally to the expression

$$F_M(\omega) = (\pi g^2\omega/\alpha_1^4)[1+\eta_{ir}(\omega,-H)] \times F_a^2(\omega/2)I(\omega)I_M(\omega)\,, \quad (54)$$

where

$$I_M(\omega) = \int_0^1 \frac{(1-\kappa^2)}{|J(\kappa^2,\omega^2)|^2}\,\kappa\, d\kappa\,. \quad (55)$$

IV. THE SPECTRAL TRANSFER FUNCTIONS AND THEIR APPROXIMATIONS

A. The transfer functions

We can now define a series of spectral transfer functions for this two-layered model. For simplicity, the transfer functions are expressed in terms of the homogeneous component only. To obtain the full transfer function, the factor $[1+\eta_{ir}(\omega,z)]$ should be included.

(1) $T_{PN}(\omega)$, the transfer function relating the spectrum of the source pressure field to that of underwater noise field:

$$T_{PN}(\omega) \equiv F_N(\omega,z)/F_P(\omega)\,. \quad (56)$$

By referring to Eqs. (50) and (51) we obtain

$$T_{PN}(\omega) = 2\int_0^1 \left|\frac{J(\kappa^2,\omega^2,z)}{J(\kappa^2,\omega^2)}\right|^2 \kappa\, d\kappa\,. \quad (57)$$

(2) $T_{PM}(\omega)$, the transfer function relating the spectrum of the source pressure field at the surface to that of the microseism response, the seabed displacement spectrum:

$$T_{PM}(\omega) \equiv F_M(\omega)/F_P(\omega)\,. \quad (58)$$

Recalling Eqs. (50) and (54) it follows that

$$T_{PM}(\omega) = \frac{2}{(\rho_1\omega\alpha_1)^2}\int_0^1 \frac{(1-\kappa^2)}{|J(\kappa^2,\omega^2)|^2}\,\kappa\, d\kappa\,. \quad (59)$$

(3) $T_{MN}(\omega)$, the transfer function relating the seabed displacement spectrum to that of the underwater noise field:

$$T_{MN}(\omega) = F_N(\omega,z)/F_M(\omega)\,. \quad (60)$$

Since $F_N(\omega,z) = F_P(\omega)T_{PN}(\omega,z)$ and $F_M(\omega) = F_P(\omega) \times T_{PM}(\omega)$, we can write

$$T_{MN}(\omega) = \frac{T_{PN}(\omega,z)}{T_{PM}(\omega)} = (\rho_1\omega\alpha_1)^2 \frac{\int_0^1 |J(\kappa^2,\omega^2,z)/J(\kappa^2,\omega^2)|^2\kappa\, d\kappa}{\int_0^1 (1-\kappa^2)/|J(\kappa^2,\omega^2)|^2\kappa\, d\kappa}\,. \quad (61)$$

1. Steep angle and fluid-bottom approximations

In the situation when $\beta_2 = 0$, Eq. (36) becomes

$$J(\kappa^2,\omega^2,z) = \sin\left(\frac{\omega}{\alpha_1}\sqrt{1-\kappa^2}(z+H)\right) + i\,\frac{m\sqrt{1-\kappa^2}}{\sqrt{n^2-\kappa^2}}\cos\left(\frac{\omega}{\alpha_1}\sqrt{1-\kappa^2}\,(z+H)\right), \quad (62)$$

where H is the water depth, $n = \alpha_1/\alpha_2$, and the other symbols are as previously defined.

Further, from Fig. 4, the variable κ, defined as the ratio $k(\omega/\alpha_1)$, is seen to be just the sine of the incident angle of the plane pressure wave, so that $\kappa = \sin\Theta$. If $\kappa < n$, we can therefore write

$$|J(\kappa^2,\omega^2,z)|^2 = \left(\frac{1+R_b}{1-R_b}\right)^2 \times\left[1 - \frac{4R_b}{(1+R_b)^2}\sin^2\left(\frac{\omega}{\alpha_1}(H+z)\cos\Theta\right)\right], \quad (63)$$

where

$$R_b = \frac{m\cos\Theta - \sqrt{n^2-\sin^2\Theta}}{m\cos\Theta + \sqrt{n^2-\sin^2\Theta}}$$

is the plane-wave reflection coefficient of the seabed.

It is of interest to the results obtained by Kibblewhite and Ewans[1,2,4] to now consider the transfer function relating the microseism wavenumber frequency spectrum to that of the noise-pressure field, i.e.,

$$T_{MN}(\mathbf{k},\omega) = f_N(\mathbf{k},\omega)/f_M(\mathbf{k},\omega)\,. \quad (64)$$

From Eq. (51), we have

$$f_N(\omega,z) = \frac{1}{2}\rho_1^2 g^2\omega F_a^2\left(\frac{1}{2}\,\omega\right)I(\omega)\left|\frac{J(k^2,\omega^2,z)}{J(k^2,\omega^2)}\right|^2,$$

which when combined with the equivalent expression for $f_M(\mathbf{k},\omega)$ and Eq. (63) gives

$$T_{MN}(\mathbf{k},\omega) = \frac{(\rho_1\alpha_1\omega)^2}{\cos\Theta}\left(\frac{1+R_b}{1-R_b}\right)^2 \times\left[1 - \frac{4R_b}{(1+R_b)^2}\sin^2\left(\frac{\omega}{\alpha_1}(H+z)\cos\Theta\right)\right]. \quad (65)$$

In the case where $\Theta \to 0$, and the whole water–seabed interface moves in phase, it follows that

$$T_{MN}(\mathbf{k},\omega) = (\rho_1\alpha_1\omega)^2\left(\frac{1+R_b}{1-R_b}\right)^2 \times\left[1 - \frac{4R_b}{(1-R_b)^2}\sin^2\left(\frac{\omega}{\alpha_1}\,(H+z)\right)\right], \quad (66)$$

and that the corresponding inverse transfer function relating the pressure field to the microseism field will be

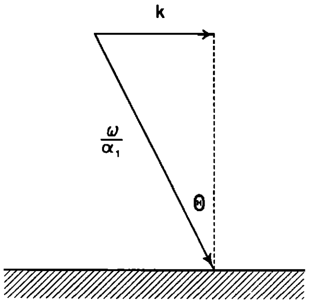

FIG. 4. Geometry of the incidence of the pressure field on the seabed.

$$T_{NM}(\mathbf{k},\omega) = \frac{1}{(\rho_1\alpha_1\omega)^2}\left(\frac{1-R_b}{1+R_b}\right)^2 \times\left[1-\frac{4R_b}{(1+R_b)^2}\sin^2\!\left(\frac{\omega}{\alpha_1}(H+z)\right)\right]^{-1}. \quad (67)$$

Equations (66) and (67) reduce to the transfer functions proposed by Urick[9] and used by Kibblewhite and Ewans[1] when, as a first step approximation, R_b is taken as 0.

It is also of interest to establish an approximate expression for the transfer function $T_{PM}(\omega)$, in the case where $\beta_2 \to 0$. By using the approximation,

$$(\omega/\alpha_1)H\sqrt{1-\kappa^2} \ll 1,$$

it can be shown[18,16] that, in this case, the approximate transfer function becomes

$$T_{PM}(\omega) \approx a/(\rho_1\alpha_1\omega)^2, \quad (68)$$

with

$$a = (1/2m^2)(1-2n^2+2n^4). \quad (69)$$

Plots of $T_{PM}(\omega)$ for a selection of geoacoustic models typical of the New Zealand region are given in Fig. 7 of our companion article.[4] The curves were calculated numerically using the full expression, while the straight lines correspond to Eqs. (68) and (69). Within the limitations of the simple two-layered model, and accepting the restriction to the homogeneous component only, it appears that over the frequency range of present interest ($0.1 < f < 1.0$ Hz) the approximate transfer function as used by Kibblewhite and Ewans[1] adequately reflects the frequency dependence involved, but that, as expected, the absolute levels are influenced by the geoacoustic properties of the environment. A detailed discussion of these questions and their significance in respect of the source field arising from wave–wave interactions, is given in our companion article.[4]

V. THE SHEAR-WAVE CONTRIBUTION

In previous sections, we have discussed the spectrum of the pressure/microseism field as it would be observed on the seafloor inside the active region. A more important case in practice is that considered in Ref. 1 when the sensor is a (vertical) seismometer installed outside the active region. The purpose of this section is thus to establish a preliminary estimate of the microseism field at a distance from the source region, where the shear-wave components are expected to make important contributions. This analysis is again based on the simple two-layered model. It is recognized, however, that a more realistic statement must await the multilayered analysis that is in progress.

It is difficult to obtain an exact assessment of this contribution in the experiment described in Ref. 1, since the vibrating interface is partly covered by water and partly in contact with air; moreover, while the observation point is not inside the storm region, it is not very far from it. Some simplifications have to be made to obtain even a rough estimate and we first assume that the active ocean region can be modeled as a viscoelastic half-space in contact with the air at the plane boundary $z = -H$, on which is acting a source pressure field of finite size. The justification for using this Rayleigh wave model[5] is based on the fact that the depth of the water layer within the active region is sufficiently small compared with the wavelengths involved that its effect on the pressure field can be neglected. Furthermore, based on Fig. 1 of the earlier article,[1] we describe the active region as a rectangular area and set the negative x and positive y axes to coincide with the coastline as shown in Fig. 5. Finally, in spite of the close proximity of the active region and the measurement site, we shall still use the farfield asymptotic expression of the Hankel function in the subsequent calculations, noting that this is sufficiently accurate when the propagation distance is larger than 1 wavelength. The problem now becomes one of solving the homogeneous wave equations governing the displacement potentials, with inhomogeneous boundary conditions, the calculation being based on the diffraction theory approximation.[19]

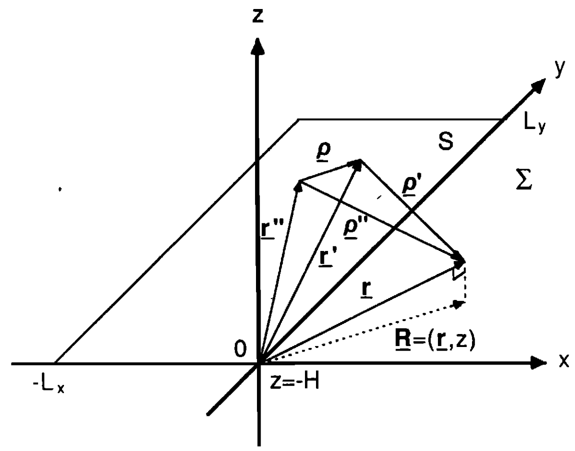

FIG. 5. The geometry for the active storm region and the onshore observation point.

From Sec. II, we can establish the governing equations and approximate boundary conditions for the displacement potential at point $\mathbf{R}$ within the solid half-space ($\mathbf{r}$ lies on the interface). Using a cylindrical coordinate system[20] centered at point $\mathbf{r}'$, letting $|\mathbf{r}-\mathbf{r}'| = \rho'$ and noting that Σ is that part of the plane $z = -H$ excluding S (see Fig. 5), we can substitute the suggested solutions

$$\tilde{\Phi}_2(\mathbf{R},\mathbf{r}') = \int_0^\infty A\exp[\eta_a(z+H)]J_0(\xi\rho')\xi\, d\xi,$$
$$\tilde{\Psi}_2(\mathbf{R},\mathbf{r}') = \int_0^\infty B\exp[\eta_b(z+H)]J_0(\xi\rho')\xi\, d\xi, \quad (70)$$

into the boundary conditions. After some manipulation[18,16] we obtain solutions of Φ_2 and Ψ_2 as

$$\Phi_2(\mathbf{R}) = \frac{1}{2\pi\mu_2}\int P_{zz}(\mathbf{r}')d\mathbf{r}'\int_0^\infty \frac{[2\xi^2-(\Omega/\beta_2)^2]\exp[\eta_a(z+H)]}{D(\xi)}J_0(\xi\rho')\xi\, d\xi \quad (71)$$

and

$$\Psi_2(\mathbf{R}) = -\frac{1}{\pi\mu_2}\int_s P_{zz}(\mathbf{r}')d\mathbf{r}'\int_0^\infty \frac{\eta_a}{D(\xi)} \times \exp[\eta_b(z+H)]J_0(\xi\rho')\xi\,d\xi\,, \quad (72)$$

where

$$D(\zeta) = [2\xi^2 - (\Omega/\beta_2)^2]^2 - 4\xi^2\eta_a\eta_b\,,$$

and we have used the assumption $\alpha_2^2 = 3\beta_2^2$. In both Eqs. (71) and (72) the integrands of the inner integrations are, apart from a multiplying term ξ, even functions of ξ, so by use of the formula

$$J_0(\xi\rho') = \tfrac{1}{2}[H_0^{(1)}(\xi\rho') + H_0^{(2)}(\xi\rho')]\,,$$

they can be converted to path integrals containing the whole real axis.

As $D(\xi)$ has a zero point $\xi = \xi_R = \Omega/(0.9194\beta_2)$ corresponding to the wavenumber of the free Rayleigh wave.[20] by neglecting the contribution from the cutlines, the path integration can be carried out in terms of the residue theorem.[20] Since

$$D'(\xi_R) = 8\xi_R[2\xi_R^2 - (\Omega/\beta_2)^2 - 4.93\eta_{aR}\eta_{bR}]\,,$$

the vertical displacement of the seabed vibration $W_{-H}(\mathbf{r},t)$ can now be expressed in terms of the potentials $\Phi_2(\mathbf{R})$ and $\Psi^2(\mathbf{R})$ as

$$W_{-H}(\mathbf{r},t) = \exp(-i\Omega t)\left(\frac{\partial\Phi_2}{\partial z} - \frac{\partial^2\Psi_2}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi_2}{\partial r}\right)_{z=-H}$$
$$\approx \frac{\Omega^2\eta_a R}{8\sqrt{2\pi\xi_R}\,\mu_2\beta_2^2(2\xi_R^2 - (\Omega/\beta_2)^2 - 4.93\eta_{aR}\eta_{bR})}\int_s P_{zz}(\mathbf{r}')\frac{\exp[i(\xi_R\rho' - \Omega t + \pi/4)]}{\sqrt{\rho'}}d\mathbf{r}'$$
$$\approx \frac{0.183}{\mu_2}\sqrt{\frac{\xi_R}{(2\pi)}}\int_R P_{zz}(\mathbf{r}')\frac{1}{\sqrt{\rho'}}\exp\left[i\left(\xi_R\rho' - \Omega t + \frac{\pi}{4}\right)\right]d\mathbf{r}'\,. \quad (73)$$

We have up to now been considering a simple harmonic source. For the case of wave–wave interactions, the term $\sqrt{\xi_R}P_{zz}(\mathbf{r}')\exp(i[\xi_R\rho' - \Omega t])$ in Eq. (73) should be replaced by the stochastic integral:

$$P_R(\mathbf{r}',\mathbf{r},t) = \int\int\sqrt{\xi_R}M(\mathbf{q}_1',\mathbf{q}_2')\exp\{i[(\mathbf{q}_1'+\mathbf{q}_2')\cdot\mathbf{r}' + \boldsymbol{\xi}_R\cdot\tilde{\boldsymbol{\rho}}' - (\sigma_1'+\sigma_2')t]\}dZ(\mathbf{q}_1'\cdot\mathbf{r}')\,dZ(\mathbf{q}_2'\cdot\mathbf{r}')\,,$$

where $\boldsymbol{\xi}_R = \xi_R\tilde{\boldsymbol{\rho}}'/|\tilde{\boldsymbol{\rho}}'|$, if we assume that the wave field has local stationarity, so that position dependent spectra exist. The autocorrelation function of the vertical displacement (which is assumed to be a zero-mean value process),

$$R_W(\mathbf{r},\tau) = \langle W_{-H}(\mathbf{r},t)W^*_{-H}(\mathbf{r},t-\tau)\rangle\,,$$

is then

$$R_W(\mathbf{r},\tau) = \frac{0.0335}{\mu_2^2 2\pi}\int_s\int_s\frac{1}{\sqrt{\rho'\rho''}}\mathrm{R_P}(\tilde{\rho},\tau,\mathbf{r})d\mathbf{r}'\,d\mathbf{r}''\,, \quad (74)$$

where

$$R_P(\tilde{\boldsymbol{\rho}},\tau,\mathbf{r}) = \langle P_R(\mathbf{r}',t,\mathbf{r})P_R^*(\mathbf{r}'-\tilde{\boldsymbol{\rho}},t-\tau,\mathbf{r})\rangle\,,$$

with $\tilde{\boldsymbol{\rho}} = \mathbf{r}' - \mathbf{r}''$.

The frequency spectrum of the microseisms, as the Fourier transform of the autocorrelation function, takes the form (for the detail see Ref. 18):

$$F_M(\omega,\mathbf{r}) = \frac{0.0335}{2\pi\mu_2^2}\int_s\frac{d\mathbf{r}'}{\rho'}\int\int\xi_R 4\rho_1^2\sigma_1^4 f_\zeta(\mathbf{q}_1,\mathbf{r}') \times f_\zeta(-\mathbf{q}_1,\mathbf{r}')f_\zeta(-\mathbf{q}_1,\mathbf{r}')\delta(\omega-2\sigma_1) \times(2\sigma_1^3/g^2)\,d\theta\,d\sigma_1$$
$$= \frac{0.0091}{\pi m^2\beta_2^5}g^2\omega^2 I(\omega)\int\frac{F_a^2(\frac{1}{2}\omega,\mathbf{r}')}{\rho'}d\mathbf{r}'\,, \quad (75)$$

in which we have used the relation $\mu_2 = \rho_2\beta_2^2$, with the definitions $m = \rho_2/\rho_1$, and

$$I(\omega) = \int_0^{2\pi}H(\theta)H(\theta+\pi)d\theta$$

and

$$f_\zeta(\mathbf{q}_1,\mathbf{r}') = \frac{g^2}{2\sigma_1^3}F_a\left(\frac{1}{2}\omega,\mathbf{r}'\right)H(\theta)\,.$$

By referring to Fig. 5 and assuming that the wave spectrum depends only on the coordinate y, the integral involved in Eq. (75) can be written as

$$\int_s\frac{F_a^2(\frac{1}{2}\omega,\mathbf{r}')}{\rho'}d\mathbf{r}' = \int_0^{L_y}F_a^2\left(\frac{1}{2}\omega,y'\right) \times\int_{-L_x}^0\frac{dx'}{\sqrt{(x-x')^2+(y-y')^2}}dy' \equiv L_y\,\overline{F_a^2(\tfrac{1}{2}\omega)}, \quad (76)$$

where

$$\overline{F_a^2\left(\frac{1}{2}\omega\right)} = \frac{1}{L_y}\int_0^{L_y}I(y')F_a^2\left(\frac{1}{2}\omega,y'\right)dy'$$

is defined as the square of the wave spectrum averaged over the fetch L_y, and we have

$$I(y') = \ln[(x+L_x) + \sqrt{(y-y')^2+(x+L_x)^2}] - \ln[x+\sqrt{(y-y')^2+x^2}]\,.$$

The spectral function of the microseism field then becomes

$$F_M(\omega,\mathbf{r}) = (0.0091/\pi m^2\beta_2^5)g^2\omega^2 I(\omega)L_y\,\overline{F_a^2(\tfrac{1}{2}\omega)}\,. \quad (77)$$

By recalling Eq. (50), the spectral transfer function of the Rayleigh wave can be established as

$$T_{PMR}(\omega,\mathbf{r}) = F_M(\omega,\mathbf{r})/F_P(\omega)$$
$$= 0.018\alpha_1^2 L_y \eta(s,\omega)/\pi^2 m^2 \rho_1^2 \beta_2^5 \omega , \quad (78)$$

where $\eta(s,\omega)$ is the ratio of the fetch average of the square of the wave spectra to the square of the local wave spectrum, or simply the ratio of the wave spectrum square; i.e.,

$$\eta(s,\omega) = \overline{F_a^2(\tfrac{1}{2}\omega)}/F_a^2(\tfrac{1}{2}\omega,y) . \quad (79)$$

The transfer function for the Rayleigh wave,

$$T_{PMR} \sim \alpha_1^2 \rho_1^{-2} \beta_2^{-5} \omega^{-1} ,$$

thus differs from that defined by Hasselmann (Ref. 5, p. 185).

$$\widetilde{T}^{r}_{sv} \sim \rho_1^{-2} \beta_2^{-5} \omega ,$$

by an extra factor $\alpha_1^2 \omega^{-2}$. This is because, here, we have used as the denominator the frequency spectrum $F_p(\omega) \sim \rho_1^2 g^2 \omega^3$ [see Eq. (50)] instead of the wavenumber frequency spectrum used by Hasselmann, $F_p(k,\omega) \sim \rho_1^2 g^2 \omega$.

Figure 6 shows the ratio $\eta(s,\omega)$ as a function of frequency for theparticular case where the size of the storm region is taken as $L_x L_y = 100\,000 \times 400\,000$ m, with $x = 1000$ m, and the fetch y used in calculating the local spectrum is taken to be the y coordinate of the observation point of the wave field in the New Zealand experiment ($y = 194\,000$ m) under southeasterly conditions.[1] The curves seem to suggest that, in this case, the ratio for wind speeds from 5–30 m/s and for frequencies higher than 0.2 Hz tends to a constant value, $\eta(s,\omega) \approx 1.26$ ($10 \log_{10} \eta \approx 1$ dB). This result is not surprising in view of the proximity of the recording site to the active ocean region. It also explains why the use of the standard version of the Maui wave spectrum, in the analyses reported by Kibblewhite and Ewans,[1] was a reasonable simplification.

Comparison with the approximate spectral transfer function for the local compressional wave given in Eqs. (68) and (69) allows us to establish the ratio of the two transfer functions:

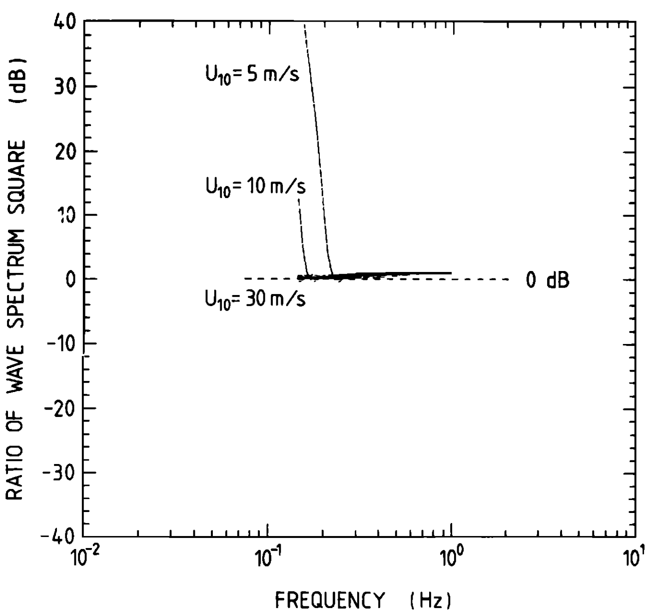

FIG. 6. The ratio of the "wave spectrum square" for wind speeds from 5–30 m/s.

$$\eta_T(\omega) \equiv \frac{T_{PMR}}{T_{PM}} = \frac{T_{PMR}}{a} \rho_1^2 \alpha_1^2 \omega^2$$
$$= \frac{0.036\alpha_1^4}{\pi^2(1 - 2n^2 + 2n^4)} \frac{L_y \omega}{\beta_2^5} \eta(s,\omega) , \quad (80)$$

in which we have used (69) for the factor a.

Equation (80) indicates that the ratio of the two transfer functions depends on the size of the storm region, the elastic properties of the medium, and the frequency. To establish some feel for the value of $\eta_T(\omega)$ we consider two geoacoustical models. In both cases, we take $L_y = 400\,000$ m and $\eta(s,\omega) \approx 1$. For case 1, we assume values of $n = 0.19$ and $\beta_2 = 4560$ m/s (Ref. 20, Sec. 4.4) as an extreme situation, and, for case 2, we take $n = 0.55$ ($\alpha_2 = 2{,}727$ m/s), $\beta_2 = \alpha_2/\sqrt{3} = 1{,}575$ m/s, which are representative values for the area involved in Ref. 1. Equation (80) then establishes the range of $\eta_T(\omega)$ for the two cases over the frequency range $0.1 < f < 1$Hz, as $2.5 \times 10^{-3} < \eta_T(\omega) < 2.5 \times 10^{-2}$ and $0.8 < \eta_T(\omega) < 8.0$, respectively.

In case 1, the Rayleigh component of the induced ground motion is apparently some two orders of magnitude lower than the compressional wave. This case probably represents the extreme if the true basement in the Maui area[1] is the significant interface but the situation in the New Zealand experiment proves to be more realistically represented by case 2. Certainly horizontal ground motions could be detected.[22]

A detailed examination of the influence of the interface wave component on the seabed displacement based on the simple geophysical model assumed, is deferred to our companion article.[4] It will be shown there that no significant errors appear to have followed from the approximations used in Ref. 1 to estimate the acoustic noise field from the compressional component of the seabed displacements. However, a frequency-dependent contribution from a shear-wave component, at the level indicated for case 2, together with an adjustment for multiple sea interactions, does appear to resolve the minor ambiguities described in the earlier article.

VI. SUMMARY

In an earlier article[1] evidence was presented to establish nonlinear interactions between ocean-surface waves as the dominant source of the infrasonic noise field in the ocean and of wave induced microseisms. The experimental data confirmed the essential characteristics predicted by the various theoretical treatments then available, but the analaysis was frustrated to some degree by uncertainties in the various transfer functions involved. The resulting uncertainties so introduced in the comparison of the experimental and theoretical spectral levels were discussed in Sec. V A of that article.

Some progress in resolving these residual difficulties was reported subsequently[23] but a full understanding had to await a more comprehensive theoretical treatment of the phenomena involved. This artilce attempts to provide a step towards that situation. The ocean is modeled as a water layer of finite depth overlying an elastic half-space. The first step in the analysis is to establish the source pressure field at the ocean surface, created by nonlinear interactions of ocean waves. The solution of the wave equation using appropriate boundary conditions leads from this to expressions for both the underwater acoustic noise field and the vertical displacement of the seabed (the microseism field). The spectra of these wave fields are then derived by assuming that the motion of the ocean surface is a space homogeneous and time stationary process. Expressions identical to those of porevious treatments are established, but the analysis is extended to develop a set of spectral transfer functions relating the source pressure field, the underwater noise field, and the microseism field in a variety of geophysical situations. For the particular case of a water layer overlying a solid half-space, it is shown that when compressional wave components only are involved the transfer function relating the microseism spectrum to that of the source pressure field, $T_{MP}(\omega)$ lies very close to $\rho_1^2\alpha_1^2\omega^2/a$, where a is a parameter dependent on the geoacoustical properties of the medium. This formula can be regarded as an extension of the transfer function $\rho_1^2\alpha_1^2\omega^2$ proposed by Urick,[9] which, as a first-order approximation, neglects the effect of reflection from the seabed.

An expansion of the theoretical analysis then examines the consequences of shear-wave excitation in this simple geoacoustic model. This establishes that the transfer function linking the observed microseism field to its underwater noise equivalent must be modified when both compressional and shear-wave components are present in the ground displacement.

In a companion article,[4] we review experimental measurements and theoretical predictions in the light of our current understanding of the processes involved. This analysis includes a discussion of the influence of the transfer functions, the role of the spreading function describing the angular distribution of wave energy, and the influence of interactions within single and multiple seas. The overall agreement is shown to be good and previous anomalies appear now to be better understood.

However, the apparent agreement between the absolute values of the theoretical and experimental pressure spectra in the present case must still be regarded as somewhat fortuitous, as this analysis is restricted to a simple two-layer geoacoustic model. Subsequent work to be reported is extending the analysis to a multilayered model, and also examines in detail the influence of the inhomogeneous component of the source field on the seismoacoustic response of the environment. Until this work is complete, some uncertainty must remain.

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