SOME ASPECTS OF THE KINEMATICS OF SHORT WAVES OVER LONGER GRAVITY WAVES ON DEEP WATER

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ABSTRACT. Modulations in wave number and in amplitude of short waves propagating over the surface of longer gravity waves on deep water is known to be of main importance for various fundamental and applied problems. They can be studied either by integrating the ray equations coupled with the action conservation principle established by Bretherton and Garrett (1969) or by treating the stability of the long waves to superharmonic perturbations. The aim of this note is to check the validity of the former approach when the nonlinearity of the longer wave is taken into account together with the capillarity effect on the short waves. A system of first order differential equations is derived from approximate formulation and solved numerically along the characteristics. A numerical investigation of superharmonic normal mode perturbations of finite amplitude Stokes wave has been developed by using the QZ algorithm. The variations of wave number and amplitude of the short waves over the profile of the long wave, obtained by the two different methods are similar, showing that ray equations and action conservation principle can be applied for strongly nonlinear inhomogeneous moving media.

1. INTRODUCTION

The energy and the wavelength of short waves are known to be affected by longer waves. How their amplitude and their wave vector are distributed along the phase of longer waves is a crucial question in fundamental and applied hydrodynamic fields. In this study the short waves are considered as linear waves and the longer wave is assumed to be fully nonlinear. The back reaction of the small waves on the larger wave is ignored.

Henyey et al. (1988) have studied the dynamics of small waves riding on longer waves using a canonical formulation. They extended the calculation of Longuet-Higgins (1987) to include gravity-capillary waves and to allow three dimensional wave field. In this note the results given by the classical ray equations and the Bretherton and Garrett principle are compared to those derived from the stability computations.

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Firstly the method of calculation of the surface velocities and orbital accelerations in the longer wave is briefly presented. Then, the approximate formulation and the stability study are developed. The last section reports on the results for pure gravity waves and gravity-capillary waves.

2. FINITE AMPLITUDE GRAVITY WAVES

The long gravity surface wave considered herein is a Stokes wave on deep water. In a frame of reference moving with constant speed C, the unperturbed surface defined for a wavenumber K = 1 is given by the usual parametric representation :

$$x = -\frac{\varphi}{C} - \sum_{1}^{\infty} H_n \sin n\frac{\varphi}{C}$$
$$z = \frac{H_o}{2} + \sum_{1}^{\infty} H_n \cos n\frac{\varphi}{C} \qquad (2.1)$$

where x and z are respectively the horizontal and the vertical coordinates and φ is the potential velocity. The unknown coeffficients H_n and the phase velocity C are calculated by an iterative scheme developed by Longuet-Higgins (1985). Then the wave amplitude writes :

$$A = H_1 + H_3 + H_5 + \ldots$$

The orbital velocity and the orbital acceleration are given by :

$$U^2 = -2z \qquad (2.2)$$

$$a = -U^6 (x_\varphi + i z_\varphi)(x_{\varphi\varphi} + i z_{\varphi\varphi}) \qquad (2.3)$$

3. FORMULATION OF THE PROBLEM

3.1. Hamilton equations and action conservation

In this section we consider the dynamics of the small waves in a coordinate system **s** (s_1, s_2) tied to the longer wave. Wavelets superimposed upon and interacting with a much longer wave are considered as slowly varying wave trains of small amplitude propagating in an inhomogeneous moving medium as shown in figure 1. The vector wavenumber **k** (k_1, k_2) of the short waves is parallel to the long wave surface. The amplitude a of the small waves is defined in terms of normal distance from the long wave surface. We assume :

$$ak \ll 1 \qquad K \ll k$$

where ak is the wave steepness of the short waves and K the wavenumber of the finite amplitude long waves.

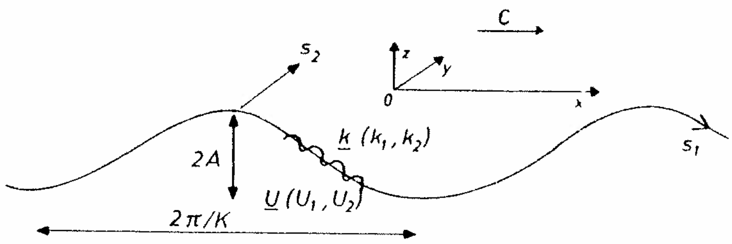

Figure 1. Definition sketch

The ray trajectories are given by :

$$\frac{dk_1}{dt} = -\frac{\partial W}{\partial s_1} \quad ; \quad \frac{ds_1}{dt} = \frac{\partial W}{\partial k_1} \qquad \text{(a)}$$

$$\frac{dk_2}{dt} = -\frac{\partial W}{\partial s_2} \quad ; \quad \frac{ds_2}{dt} = \frac{\partial W}{\partial k_2} \qquad \text{(b)} \qquad (3.1)$$

$$W(\mathbf{k}, \mathbf{s}, t) = \sigma + \mathbf{k}.\mathbf{U} \qquad (3.2)$$

$$\sigma^2 = g' \, (k_1{}^2 + k_2{}^2)^{\frac{1}{2}} + \frac{T}{\rho} \, (k_1{}^2 + k_2{}^2)^{3/2} \qquad (3.3)$$

$$\mathbf{g'} = \mathbf{g} - \mathbf{a} \qquad (3.4)$$

$$g' = |\mathbf{g'}| \qquad (3.5)$$

where s_1, s_2 and t are the space variables and the time variable, $W(\mathbf{k},\mathbf{s},t)$ is the local dispersion relation, σ is the intrinsic frequency, U is the local orbital velocity in the long wave as seen by an observer travelling with the long wave speed C, $\mathbf{g'}$ is the effective vector gravity, $\mathbf{a}$ is the orbital acceleration in the long wave and T is the surface tension coefficient. Since the pressure gradient of the long wave has no component tangential to the surface, $\mathbf{g'}$ is always normal to the free surface.

In the frame of reference chosen, the long gravity wave is steady and the free surface is a streamline. In consequence the velocity $\mathbf{U}$ is always tangent to the free surface. In the case of a Stokes wave the component U_2 is zero. The component U_1 and the acceleration g' are determined from equations (2.2), (2.3), (3.4) and (3.5). In order to integrate equations (3.1) we write U_1 and g' as functions of the variable x by using Fourier transforms. Then the derivatives dU_1/dx and dg'/dx can be easily obtained.

Let $z = \eta(x)$ be the equation of the long wave surface. Then we can write :

$$ds_1 = (1 + \eta^2_x)^{\frac{1}{2}}\, dx$$

$$ds_2 = dy$$

and use these expressions to define x and y as independent variables instead of s_1 and s_2. Equations (3.1) are integrated numericaly with x, y and t as variables. The initial conditions are taken at the mean surface level of the long wave. The solutions give the change in wavenumber along the rays.

In order to determine the amplitude modulation we use the Bretherton and Garrett (1969) formulation. These authors established the action conservation principle as :

$$\frac{\partial}{\partial t}\left(\frac{E}{\sigma}\right) + \nabla.\ [(\mathbf{U} + \mathbf{C}_g\ )\ \frac{E}{\sigma}\ ] = 0 \qquad (3.6)$$

where $$E = \frac{1}{2}\rho\, g'\, a^2 \left(1 + \frac{T(k_1{}^2 + k_2{}^2)}{\rho\, g'}\right)$$

is the wave energy density in a frame of reference moving with the local current velocity and $\mathbf{C}_g$ is the group velocity of the short waves defined as :

$$\mathbf{C}_g = \frac{\partial\sigma}{\partial\mathbf{k}}$$

In the present case the medium is time independant, so that the frequency is conserved along the trajectories and the equation (3.6) may be written as :

$$\nabla.\ [(\mathbf{U} + \mathbf{C}_g)\ \frac{E}{\sigma}\ ] = 0 \qquad (3.7)$$

3.2. Stability

A more extensive calculation relative to the modulation of short wave is derived using the numerical investigations of Kharif and Ramamonjiarisoa (1988). The motion of deep water waves on an inviscid irrotational incompressible fluid obeys a well-known set of a linear equation and nonlinear boundary conditions. In first place we ignore the effect of the capillarity.

In a frame of reference moving at some constant speed C, the basic equations are :

$$\nabla^2\varphi = 0 \; , \quad -\infty < z < \eta \; , \quad \lim_{z \to -\infty} \varphi = - Cx \tag{3.8}$$

$$\varphi_t + \eta + \frac{1}{2}(\varphi_x{}^2 + \varphi_y{}^2 + \varphi_z{}^2) = \frac{1}{2} C^2 \qquad \text{(a)}$$

$$\text{on } z = \eta(x,y,t) \tag{3.9}$$

$$\eta_t + \varphi_x \eta_x + \varphi_y \eta_y - \varphi_z = 0, \qquad \text{(b)}$$

where x, y are the horizontal coordinates, z is the vertical coordinate, t is the time, $\varphi(x,y,z,t)$ is the velocity potential, and $z = \eta(x,y,t)$ is the free surface. As usual, the gravitational acceleration and the wavelength are taken, respectively, to be unity and 2π without loss of generality.

The system above is known to admit two-dimensional steady solutions (Stokes waves with phase speed C). This study deals with the stability of these solutions to two and three dimensional perturbations. Let :

$$\eta = \bar{\eta} + \eta' \qquad \varphi = \bar{\varphi} + \varphi' \tag{3.10}$$

where $(\bar{\eta}, \bar{\varphi})$ and (η', φ') correspond, respectively, to the unperturbed and infinitesimal perturbative motions ($\eta' \ll \bar{\eta}$, $\varphi' \ll \bar{\varphi}$).

The first order pertubation equations can be written as :

$$\nabla^2 \varphi' = 0 \; , \qquad -\infty < z < \bar{\eta}$$

$$\varphi'_t = - \eta' - \bar{\varphi}_x \varphi'_x - \bar{\varphi}_z \varphi'_z - (\bar{\varphi}_x \bar{\varphi}_{xz} + \bar{\varphi}_z \bar{\varphi}_{zz})\, \eta' \qquad \text{(a)}$$

$$\text{on } z = \bar{\eta}(z) \tag{3.11}$$

$$\eta'_t = - \bar{\varphi}_x \eta'_x - \bar{\eta}_x \varphi'_x - (\bar{\varphi}_{xz} \bar{\eta}_x - \bar{\varphi}_{zz})\, \eta' + \varphi'_z \qquad \text{(b)}$$

with solutions of the following form :

$$\eta' = e^{-i\sigma t}\, e^{i(px + qy)} \sum_{-\infty}^{+\infty} a_j\, e^{ijx} \qquad (a)$$

$$\varphi' = e^{-i\sigma t}\, e^{i(px + qy)} \sum_{-\infty}^{+\infty} b_j\, e^{ijx}\, e^{\sqrt{(p+j)^2 + q^2}\, z} \qquad (b) \qquad (3.12)$$

An eigenvalue problem for σ with eigenvector $\mathbf{u} = [a_j, b_j]$ is derived from (3.11) :

$$(\mathbf{A} - i\sigma\, \mathbf{B})\, \mathbf{u} = 0 \qquad (3.13)$$

where A and B are complex matrices depending on the unperturbed wave steepness AK and the arbitrary real numbers p and q. The eigenvalue satisfies :

$$\left| \, i\sigma\, \mathbf{B} - \mathbf{A} \, \right| = 0$$

The unperturbed surface is given by the usual parametric representation (2.1).

The purpose is to study the modulations of perturbations with wavelengths smaller than that of the umperturbed wave, so we will only considered the case of superharmonic stability (p = 0). The deflection due to the perturbations may be written as :

$$\eta' = e^{-i\sigma t}\, e^{iqy}\, \alpha(x)\, e^{i\phi(x)}$$

where $\alpha(x)$ is the envelope and $\phi(x)$ the phase of the short waves.

Then, with simple transformations, it is easy to calculate the wavenumber and the amplitude of the short wave as defined in section 3.1, along the surface coordinates of the long wave.

If the effect of capillarity is taken into account only for the short waves, we have to add supplementary terms to the equation (3.11.a) :

$$\varphi'_t = -\eta' - \bar{\varphi}_x \varphi'_x - \bar{\varphi}_z \varphi'_z - (\bar{\varphi}_x \bar{\varphi}_{xz} + \bar{\varphi}_z \bar{\varphi}_{zz})\eta'$$
$$+ \kappa\, [(1 + \bar{\eta}_x{}^2)^{-\frac{1}{2}}\, \eta'_{yy} + (1 + \bar{\eta}_x{}^2)^{-3/2}\, \eta'_{xx} - 3(1 + \bar{\eta}_x{}^2)^{-5/2}\, \bar{\eta}_{xx}\bar{\eta}_x \eta'_x] = 0$$
$$, \quad z = \bar{\eta}(x)$$

where $\kappa = K^2 T/\rho g$ is the non dimensional surface tension coefficient. Then the procedure is similar to that presented previously.

Let us consider the special case in which the unperturbed wave has zero amplitude. The dispersion relation may be written as :

$$\sigma_n = -\, n \pm [n^2 + q^2]^{1/4}\, [1 + \kappa(n^2 + q^2)]^{\frac{1}{2}}$$

where n is the wave number of the perturbation in the x direction. When n and q become significantly large we cannot neglect the term $\kappa (n^2 + q^2)$, even if $\kappa << 1$.

4. RESULTS

Modulations in wave number and in amplitude of superharmonic modes travelling in the same sense (positive sense) or in opposite sense (negative sense) with respect to the long basic gravity wave, have been computed for various values of the steepness AK using two different methods. The results presented in this paper deal with the two dimensional mode n = 10 and the three dimensional mode (n,q) = (10,10).

Figures 2 and 3 display respectively the evolution of the normalized amplitude $a/\bar{a}$ ($\bar{a}$ is the amplitude of the short waves at the mean level of the long wave) and wave number k of short gravity waves (two dimensional mode n = 10) propagating in the positive sense, as a function of the horizontal coordinate x between the crest and the trough of the long wave.

With the linear dispersion relation (3.2) and the Bretherton and Garrett equation (3.7) the amplitude and wave number evolutions are found to be very close to the evolutions obtained from the stability calculations of the long wave to superharmonic perturbations.

Similarly, figures 4 and 5 show the evolution of the relative amplitude and wave number of gravity capillary waves (two dimensional mode n = 10) travelling in the positive sense. In this case small difference appears at the crest for AK = 0.40. Figures 6 and 7 display the curves corresponding to gravity capillary waves (two dimensional mode n = 10) travelling in the negative sense. In figures 8 and 9 are plotted the curves corresponding to the three dimensional mode (p,q) = (10,10).

As a main conclusion an equivalence exists between the ray equations based on the linear dispersion relation coupled with the action conservation principle and the eigenvalue problem derived from the exact equations.

Generally, instabilities arise when frequencies of two modes of opposite signatures collide (McKay and Saffman, 1986). According to the previous conclusion, it seems possible to predict the instability of wave of permanent form by using the linear dispersion relation with the signature added, for any value of the wave steepness AK.

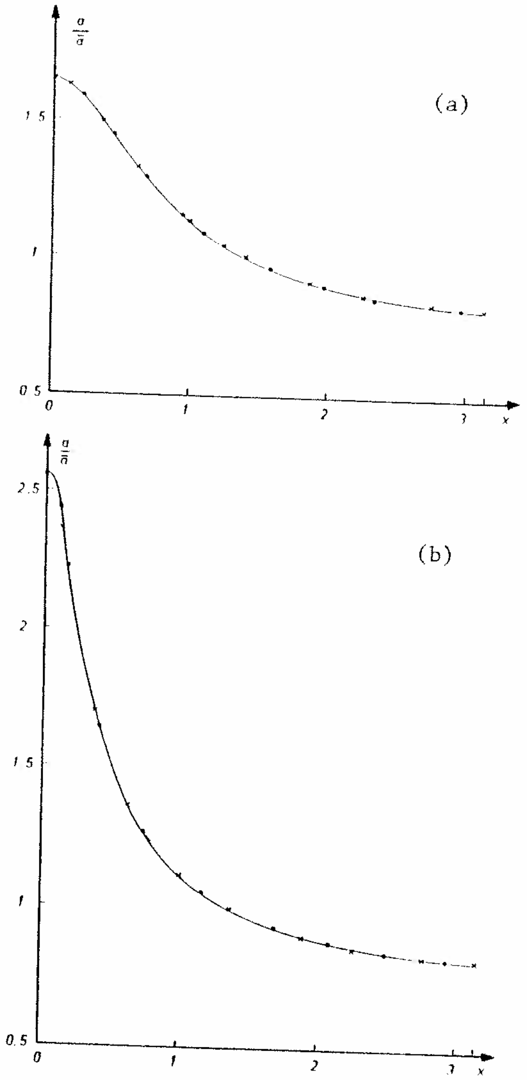

Figure 2. Evolution of the normalized amplitude $a/\bar{a}$ of two dimensional short gravity waves propagating in the <u>positive</u> sense as a function of the horizontal coordinate x between the crest and the trough of the long wave. K = 1 ; x : Stability (n = 10) ; ● : Action conservation ; (a) AK = 0.3 ; (b) AK = 0.4.

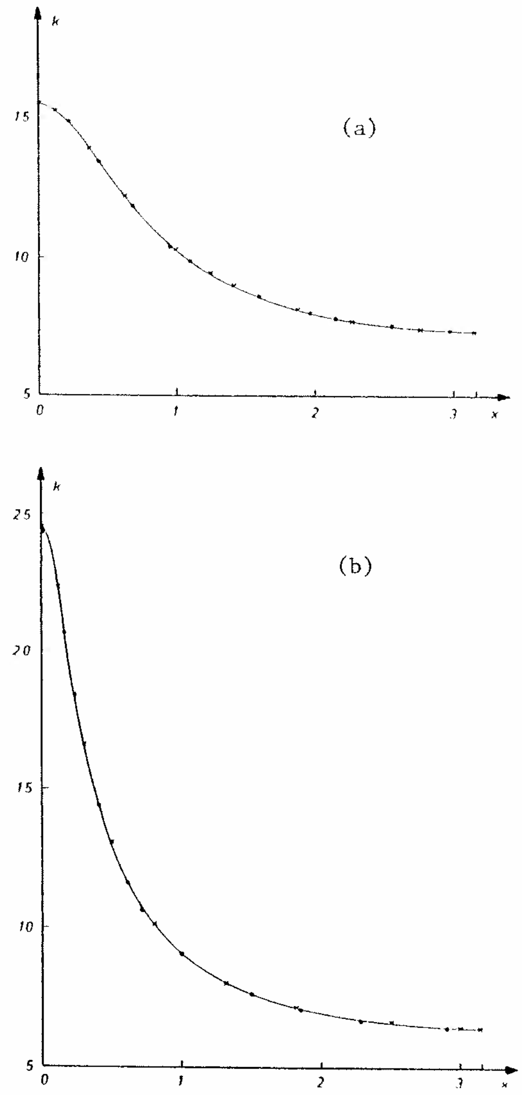

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Figure 3. Evolution of the wave number k of two dimensional short gravity waves propagating in the positive sense as a function of the horizontal coordinate x between the crest and the trough of the long wave. K = 1 ; x : Stability (n = 10) ; • : Ray equations ;
(a) AK = 0.3 ; (b) AK = 0.4.

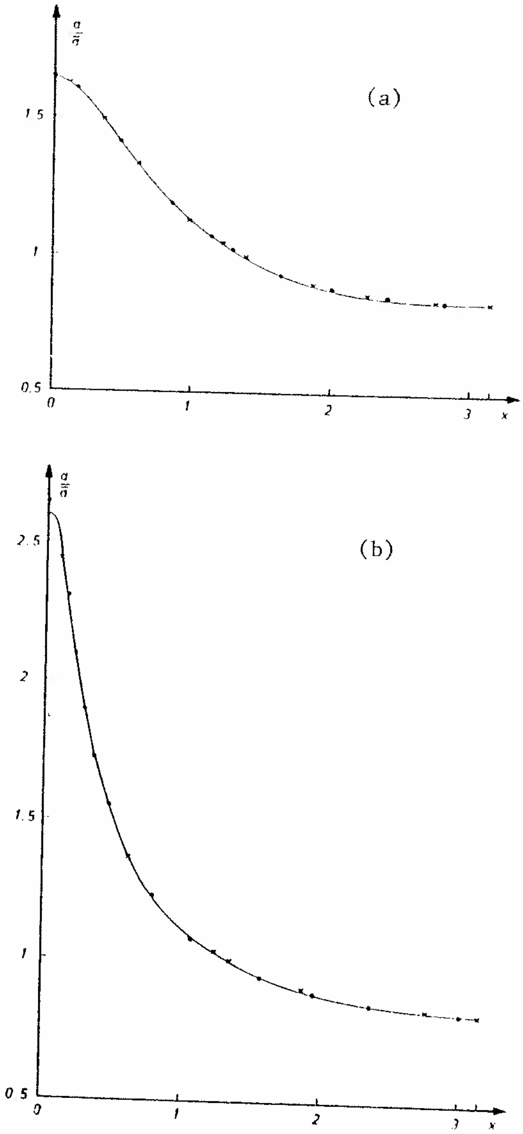

Figure 4. As Fig. 2 but for two dimensional short gravity capillary waves propagating in the positive sense.
K = 1 ; κ = 0.0003 ; x : Stability (n = 10) ; ● : Ray equations ;
(a) AK = 0.3 ; (b) AK = 0.4.

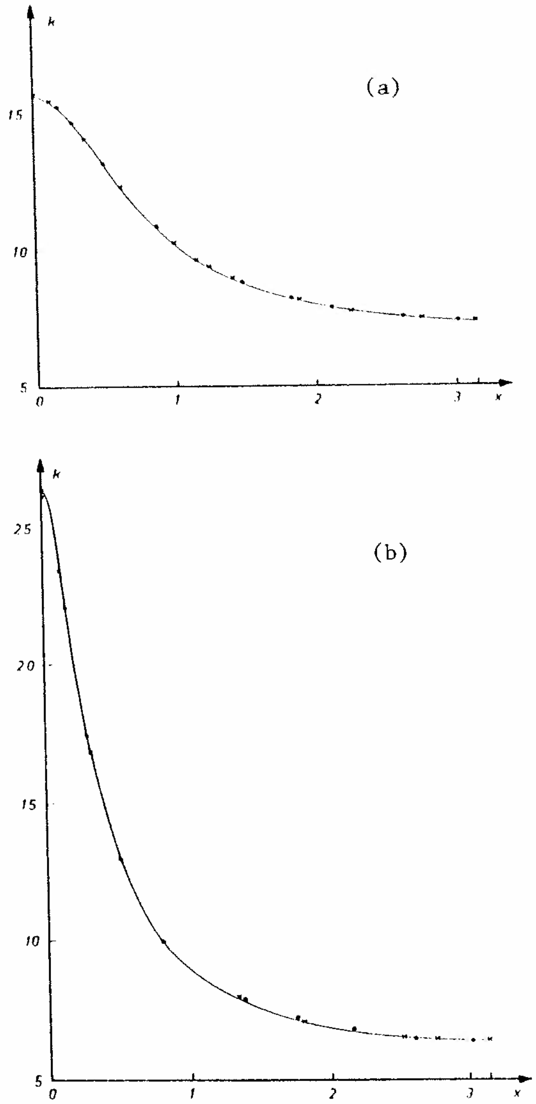

Figure 5. As Fig. 3 but for two dimensional short gravity capillary waves propagating in the <u>positive</u> sense.
K = 1 ; κ = 0.0003 ; x : Stability (n = 10) ; • : Action conservation ;
(a) AK = 0.3 ; (b) AK = 0.4.

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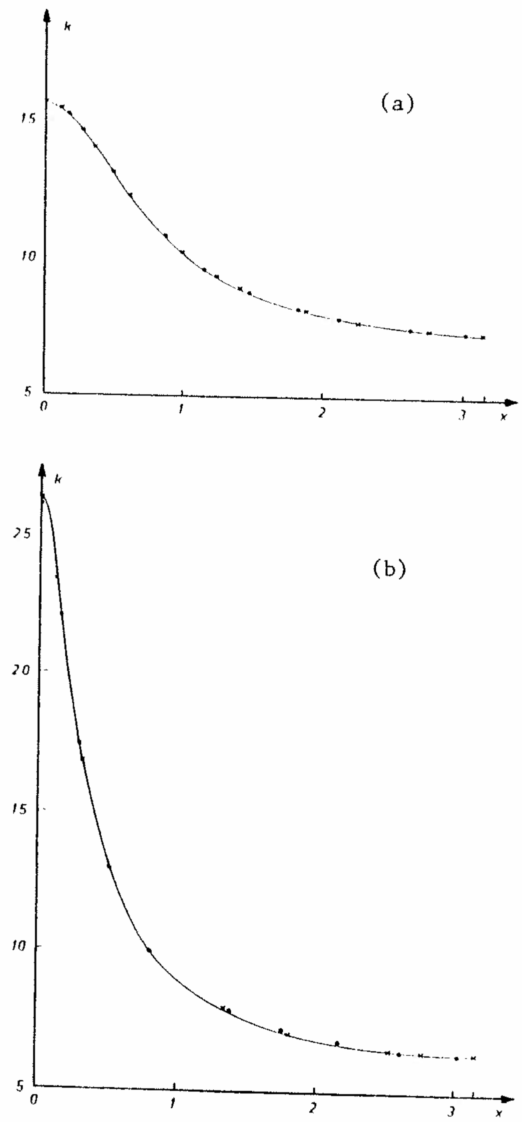


Figure 5. As Fig. 3 but for two dimensional short gravity capillary waves propagating in the <u>positive</u> sense.
K = 1 ; κ = 0.0003 ; x : Stability (n = 10) ; ● : Action conservation ;
(a) AK = 0.3 ; (b) AK = 0.4.

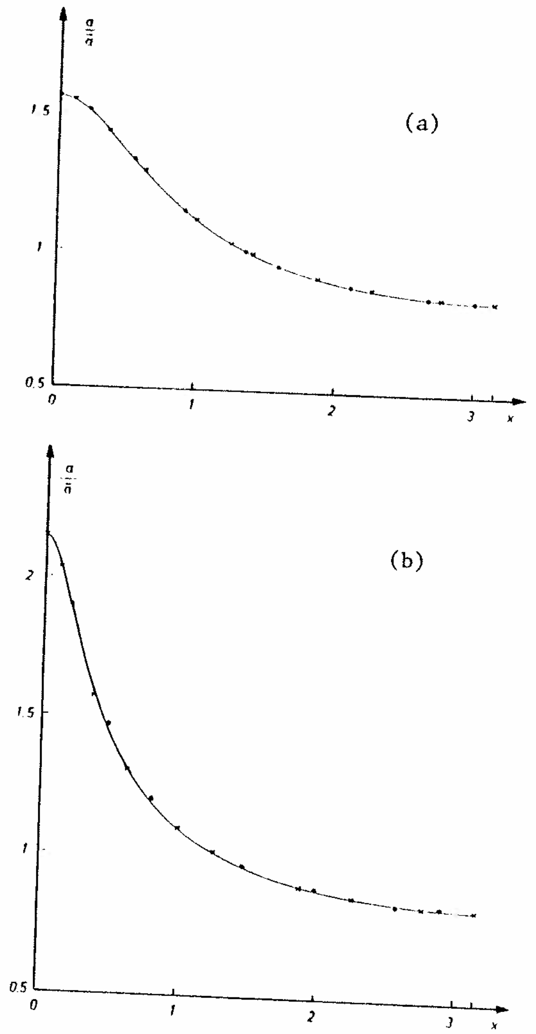
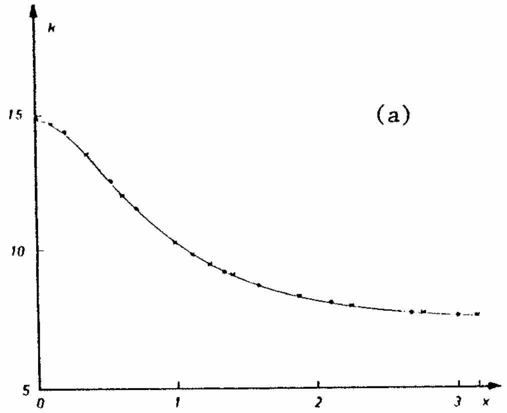


Figure 6. As Fig. 2 but for two dimensional short gravity capillary waves propagating in the negative sense.
K = 1 ; κ = 0.0003 ; x : Stability (n = 10) ; • : Action conservation ;
(a) AK = 0.3 ; (b) AK = 0.4.



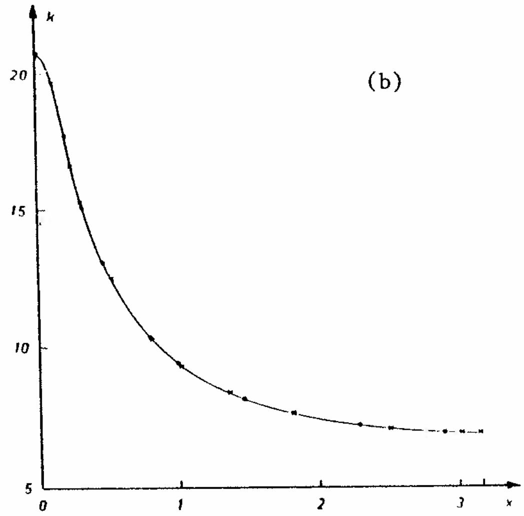


Figure 7. As Fig. 3 but for two dimensional short gravity capillary waves propagating in the negative sense.
K = 1 ; κ = 0.0003 ; x : Stability (n = 10) ; ● : Ray equations ;
(a) AK = 0.3 ; (b) AK = 0.4.

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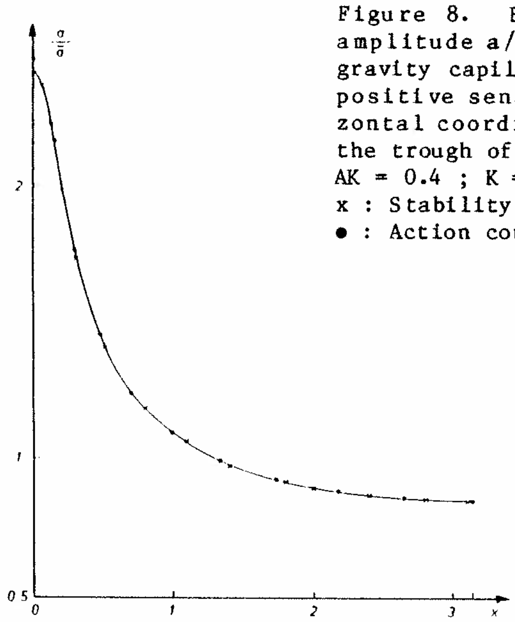

Figure 8. Evolution of the normalized amplitude $a/\bar{a}$ of three dimensional short gravity capillary waves propagating in the positive sense as a function of the horizontal coordinate x between the crest and the trough of the long wave.
AK = 0.4 ; K = 1 ; κ = 0.0003 ;
x : Stability (n = 10, q = 10) ;
• : Action conservation

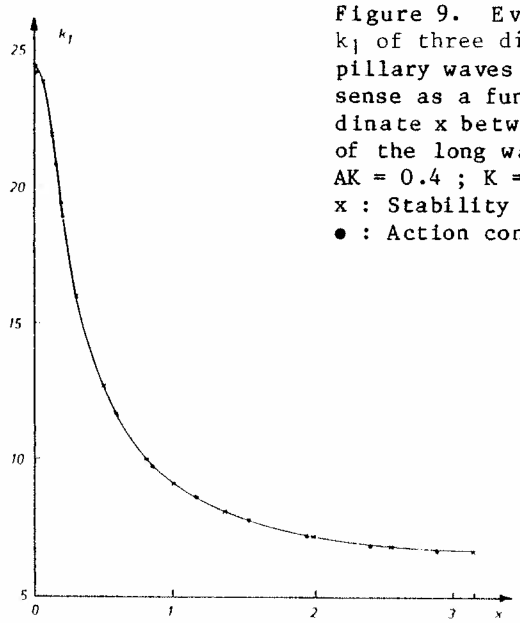

Figure 9. Evolution of the x-wave number k_1 of three dimensional short gravity capillary waves propagating in the positive sense as a function of the horizontal coordinate x between the crest and the trough of the long wave.
AK = 0.4 ; K = 1 ; κ = 0.0003 ;
x : Stability (n = 10, q = 10) ;
• : Action conservation

5. ACKNOWLEDGEMENT

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