Wave Force on an Ocean Current

KERN E. KENYON

Del Mar, California

DAVID SHERES

San Diego, California

(Manuscript received 16 June 2004, in final form 14 June 2005)

ABSTRACT

Linear momentum of surface gravity waves changes with time during refraction by a horizontally variable current, as is predicted by ray theory; the momentum change per unit time requires a force by the current on the waves. According to Newton's third law, the waves apply an equal but opposite force back on the current. The wave force of linear waves on the current is calculated for a steady horizontal shear current and it is found to be directly proportional to the wave momentum times the shear in the current. For a current like the Gulf Stream it is theoretically possible for the wave force on the current to be as large as the Coriolis force on the current to the depth of wave influence; the effect on equatorial surface currents is likely to be even more significant. Considering the reasonable conjecture that the orbital angular momentum of the waves cannot be exchanged with the current, the growth or decay of the wave amplitude in the shear current is computed as well. An exponential growth or decay of the amplitude is obtained with the *e*-folding scale being proportional to the current shear. A comparison between the calculated wave force and the Coriolis force for reported data describing the reflection of waves by the Gulf Stream is presented. The potential effects of the wave force on the surface extent of such currents and their observations by remote sensing, including possible bias in estimation of their transport capacity, are discussed. Instances of potential positive and negative feedback acting during the interaction between the waves and the current are outlined.

1. Introduction

When a plane surface gravity wave refracts in a steady but horizontally variable ocean current, the application of geometrical optics theory has predicted that the wave rays are curved and the direction and magnitude of the wavenumbers continuously change along the rays, in general, if there are no abrupt discontinuities in current speed (Kenyon 1971). For example, this happens conceptually when a sinusoidal wave enters a following current and ends up being totally reflected by the current (Fig. 1). An analogous situation (total internal reflection) can take place when a wave travels against a horizontal shear current with a single maximum speed (Fig. 2). What is the rate of exchange of energy and linear momentum between the waves and the current during refraction? Does the wave angular momentum play any role in the wave–current interaction? What rate of growth or decay is experienced by the wave amplitude? How large is the wave force on the current, compared to the Coriolis force on the same current, for example? Some of these questions have not been asked before, but all of them will be answered below in a straightforward manner with the help of an elementary model involving both the ray equations and the linear and angular momentum balances for the waves.

Surface gravity waves possess linear momentum, like light and sound waves do. For surface waves the momentum is proportional, through the particle (fluid) density, to the Stokes drift velocity, and it is a vector quantity that points horizontally in the direction of wave propagation. Also, exactly like light waves and qualitatively like sound waves, the magnitude of the linear momentum of surface waves is equal to the total energy divided by the phase speed (Kenyon 1969).

The exchange of surface gravity wave momentum with the environment, with an emphasis on coastal

Corresponding author address: David Sheres, 4604 East Talmadge Dr., San Diego, CA 92116.
E-mail: dsheres@cox.net

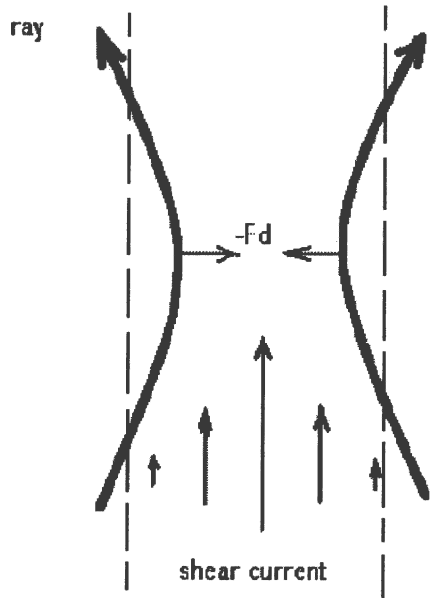

FIG. 1. A horizontal shear current flows between the dashed lines from the bottom to the top of the diagram. Two selected wave rays enter the current on either side near the bottom of the figure and exit near the top, showing that these particular surface gravity waves are totally reflected by refraction in the current shear. Where there is no current the rays are straight lines; inside the current the rays are curved lines. The force that the waves exert on the current is indicated by the two short arrows, labeled $-\mathbf{F}_d$, that point toward each other and toward the center of the current.

work, has been discussed by a relatively small number of investigators using different methods. Putnam et al. (1949) employed the concept of momentum flux due to breaking waves to estimate the velocity of alongshore flow; the water velocity of both the momentum and its flux were taken to be the shallow water wave speed since the group and phase speeds are the same. Longuet-Higgins and Stewart (1964) obtained a wave momentum flux into the wave-facing vertical side of a water parcel by averaging the square of the wave particle velocity multiplied by the density. They added to that the pressure difference on the same side of the water parcel, with and without waves, and called the combination "radiation stress." They then postulated the exchanges between the radiation stress and the current shear and calculated these exchanges using a modified conservation equation for waves. Garrett (1976) calculated the force applied by a wave train on a shear flow, in the context of explaining the relatively small scale Langmuir circulation; he used the momentum conservation equation for the combined wave and current flow, as well as the action conservation equation and the ray refraction equation for the waves. He also used wave dissipation for momentum transfer to the circulation to increase the shear for feedback. Sheres and Kenyon (1990) discussed momentum exchange between the Southern California coastal circulation and incoming swell refracted by it, using radiation stress; they compared the potential effects of the waves to that of the wind on the circulation in the area. A recent paper by Buhler and McIntyre (2003) found that waves refracted by an inhomogeneous flow apply a force to the flow within a context aimed at understanding internal gravity waves propagating in spatially variable winds in the atmosphere. Their model was based on pseudomomentum, and the model equations adopted by them can be interpreted in either of two ways, compressible flow or shallow water wave motion, by a well-known analogy based on the mathematical similarity in the respective equations of motion and continuity.

Here we compute the effects of surface gravity wave momentum changes due to refraction by inhomogeneous ocean currents in deep water with the wave momentum being based on the well-established Stokes drift. We show that changes in direction of the waves due to refraction in the shear current imply a force on the current. A major tool used is the well-known ray equations describing the wave refraction due to the horizontal shear in the steady current. We assume monochromatic low-slope deep water waves that do not exhibit nonlinear effects.

During wave–current refraction, since the wavenumber changes direction along a ray, then the linear momentum of the waves must change direction along the rays also, because it remains parallel to the wavenumber; in general, the magnitude of the linear momentum continuously alters along a ray too. Thus, there is a time rate of change of the vector linear momentum during wave–current interaction. It takes a force to cause a change with time in the wave linear momentum, whether that change occurs to the magnitude or to the direction of the momentum (or to both). Consequently there is a force on the waves by the current, and, according to Newton's third law, there must be an equal but opposite force by the waves back on the current.

We will set up the balance equation for the linear momentum of the surface waves and use it, along with the ray equations, to compute the total wave force on a current, one component being normal to and the other parallel to the wavenumber. To the best of our knowledge the computation of the wave force on an ocean current has never been attempted before for refracting surface gravity waves.

Only very recently have ocean waves and currents been put together into the same numerical model (e.g., Mellor 2003; Perrie et al. 2003). As normally occurs in the analytical approach to refraction problems the en-

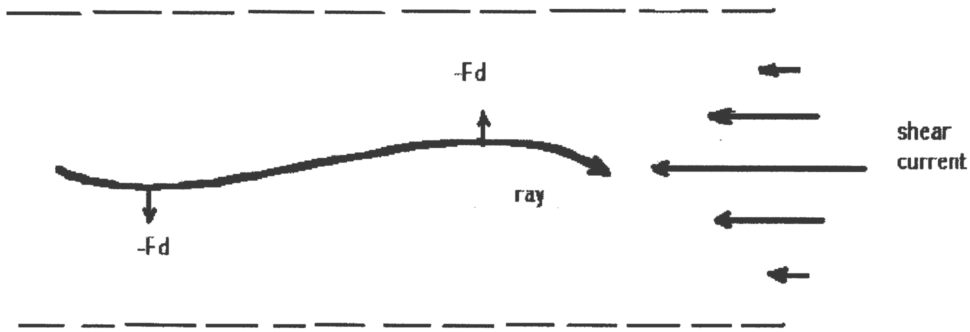

FIG. 2. The same horizontal shear current as in Fig. 1 but the waves now travel against the flow. One ray is shown illustrating that these particular waves are totally internally reflected by refraction in the shear current. The wave force on the current is indicated by two short arrows, labeled $-\mathbf{F}_d$, that point away from each other and away from the center of the current.

vironment is given and fixed and then the changes in the properties of an initial set of waves that move through the environment are calculated. Therefore, it is the horizontally variable current that is the given quantity in the present problem, and the current cannot move in response to the force of the waves. In other words, we do not have a truly interactive system involving waves and currents, but that is what the numerical models will be capable of achieving in the future.

In contrast to most wave types in physics (the Rayleigh wave in the solid earth being an exception), surface and possibly internal gravity waves have orbital angular momentum, which is related to the circular or elliptical motion of material (fluid) particles. [A recent interest in light's orbital angular momentum has arisen (Padgett et al. 2004).] The angular momentum vectors of surface gravity waves always point along the crests and troughs. To change the magnitude of the wave angular momentum (or wave action) we need a torque in the direction of the angular momentum, which is parallel to the crests and troughs along the water surface (Kenyon and Sheres 1996). However, a horizontal shear current that is uniform vertically can potentially supply a torque only in the direction perpendicular to the mean free surface. Therefore, the horizontal shear in the current cannot alter the magnitude of the wave angular momentum. Thus the magnitude of the angular momentum must be conserved during this type of wave–current refraction.

Conserving the magnitude of the orbital angular momentum, which is equivalent to conserving wave action (Kenyon and Sheres 1996), is a key element of the present analysis that leads us to predict the time rate of change of the wave amplitude as well as the exchange rates of energy and linear momentum between the waves and the current. To change the direction of the wave angular momentum in the horizontal plane a torque is required, and this torque has been calculated before for a linear horizontal current shear (Kenyon and Sheres 1996). The shear current causes a torque on the waves about vertical axes, and there must be an equal but opposite torque of the waves back on the current about the same axes (e.g., Ference et al. 1956, p. 98).

Throughout its moderately long history the theoretical description of the evolution of wind waves has been heavily biased toward the energy balance of these waves, and more recently the action balance. All conceivable physical processes involved in the generation, propagation, and decay of surface gravity waves have been collected together as separate terms in one equation, the energy equation (or radiative transfer equation), as a starting point for understanding the waves. However, as in all physics problems, energy levels can only be changed by work. But surface gravity waves also carry both linear and angular momentum. What can the energy equation say about them? It says very little. Linear momentum is altered per unit time by a force and angular momentum by a torque. Linear momentum and angular momentum balance equations for the waves are therefore needed in addition to the existing energy balance equation. The angular momentum balance for surface waves has been written down and applied to wave–current interaction by Kenyon and Sheres (1996). Now, we will do the same thing for the linear momentum balance and apply it also to the wave–current refraction problem.

When surface gravity waves reflect off a vertical wall, they exert a force on that wall, just like sound and light waves do when they reflect off solid surfaces. The steady force on the wall is related to the time rate of change of the wave momentum upon reflection from the wall. There is still some controversy surrounding the reflection of sound waves from a rigid surface

(Faber 1995), but this controversy pales in comparison to the rather extreme story behind the calculation of the force during the reflection of surface gravity waves from a vertical wall. For example, as of 1951 there were published more than 25 separate theoretical formulas for the wave force on a vertical wall that neither agreed among themselves nor with laboratory data published a bit later (Rundgren 1958). At present a few more formulas for the wave force on a wall have been put into the public domain that are still different from the earlier ones (e.g., Kenyon 2004a), but at least the laboratory measurements established that reflecting surface waves do cause a force on a vertical wall. This gives us some confidence to predict that, if surface waves are truly reflected from a current by refraction, then they will definitely exert a force on that current.

2. Linear wave momentum

Stokes discovered in 1847, as a by-product of the complete solution of the surface gravity wave problem to second order in a perturbation expansion of the fluid dynamics equations and boundary conditions, that the orbiting fluid particles do not return exactly to their starting positions at the end of a wave period but are displaced a tiny bit in the direction of propagation of the wave. The net forward displacement of the particles in a wave period is equivalent to a steady streaming velocity that has come to be known as the Stokes drift velocity, or simply Stokes drift.

Therefore, for more than 150 years it could have been said that surface gravity waves have linear momentum, by identifying the momentum with the Stokes drift multiplied by the particle density, but historically it has not been talked about in quite this way for the most part. It is just as conceivable that Stokes himself could have made such a statement, based on the analogy with light waves, because Maxwell and he were good friends (Mahon 2003, p. 63), and it was Maxwell who predicted the force of light waves on objects that intercept the light from his famous electromagnetic equations published in 1873. It is true that the experimental verification of the light pressure came considerably later in 1900, but then Stokes did not die until 1903. Apparently Stokes did not revisit the subject, and no connection was made between Stokes drift and the linear momentum of surface gravity waves for well over 100 years (e.g., see the references in Barnett and Kenyon 1975).

The linear momentum of surface gravity waves is directly proportional to Stokes drift through the particle (or fluid) density. What Stokes derived is the following:

$$U = c(ak)^2 e^{2kz}, \tag{1}$$

where U is the Stokes drift, c is the phase speed of a monochromatic progressive surface wave in still water, a is the amplitude, k is the wavenumber, and z is the mean depth measured positively upward from the still water level ($z = 0$). Notice that (1) does not depend on either time or the horizontal coordinates. Under normal usage, where no currents are involved, the reference frame selected is fixed to the coast of the ocean or shore of a lake, and then the speed of the crests and troughs, the wave phase speed, is measured relative to that frame. Equation (1) is valid for waves of moderately small amplitude (compared to the wavelength) in deep water, meaning theoretically that the total depth of water is infinite. [Stokes (1847) also gave the generalization of (1) for finite constant mean depth.] Recent derivations of (1) may be easier to find and to follow than the original (e.g., Kundu 1990).

One characteristic of the Stokes drift (1) has been appreciated for a long time. Since (ak) is proportional to the average slope of the wave surface, which is normally quite small compared to unity, the Stokes drift is much smaller than the phase speed because it depends on the square of a small quantity. But what has probably never been said before is that the Stokes drift propagates with the speed of the wave. This feature gives the tiny Stokes drift a greater significance, especially with its relation to the linear momentum of the waves and to the flux of momentum in particular. Thus the linear and angular momenta and the energy all travel at the wave speed, and for a wave group it is obviously the group speed.

From (1) the magnitude of the linear momentum per unit volume m is defined as

$$m = \rho U = \rho c(ak)^2 e^{2kz}, \tag{2}$$

where ρ is the density of a fluid particle, which is also the density of the entire fluid for a fluid of constant density as assumed here. The direction of the linear momentum is the same as the direction of wave propagation. Moving from (1) to (2) appears, on the face of it, to be a simple physical step specifically linking the linear wave momentum with the Stokes drift.

By a vertical integration of (2) over the depth of wave influence we obtain the linear momentum per unit horizontal area

$$M = \int_{-\infty}^{0} m\, dz = \frac{1}{2}\rho\omega a^2, \tag{3}$$

a quantity that characterizes the wave as a whole, where $c = \omega/k$. Also the well-known dispersion relation for deep water waves is

$$\omega^2 = gk, \tag{4}$$

where ω is the frequency of the waves, g is the acceleration of gravity, and with (4) the horizontal momentum per unit horizontal area in (3) can be written

$$M = \frac{1}{2}\rho\sqrt{gk}a^2,$$

which is determined by measuring wavelength and amplitude that are both independent of the reference frame used for the measurements (i.e., fixed frame or frame moving with a current).

When we come to calculate the wave force on an ocean shear current in section 4, the waves have propagated from still water into a spatially variable current where they are then refracted. The momentum M is an intrinsic property that the waves carry with them. The frequency and velocity of the waves, c and ω, will now be with respect to the current; to note that, they will be from now on written as c' and ω' with

$$\omega'^2 = gk. \tag{4a}$$

Here ω' is called the intrinsic frequency.

3. Wave momentum balance

First, we put down the linear momentum balance equation for the wave motion

$$\frac{d\mathbf{M}}{dt} = \mathbf{F}, \tag{5}$$

which states that the time rate of change of the wave momentum per unit horizontal area equals the applied force per same unit area. (For example, the applied force can come from a shear current while the waves are being refracted by the current. Since any current that we will consider is a steady one, the time rate of change of the current momentum is zero by definition.) When the applied force vanishes, the wave momentum is and remains a constant. In principle, if we know the applied force along the rays as well as the initial value of the linear momentum, then the linear momentum can be found at all points of the rays by a time integration of (5)–(7). From the initial value of the momentum and wavenumber, as well as the given frequency function (4) [dispersion relation], the wave amplitude can be obtained, in principle, along the rays from (3)–(7), assuming that we know the force $\mathbf{F}$ along the ray. If we do not know $\mathbf{F}$, we can use the conservation of angular momentum magnitude (action) along the ray to calculate the wave amplitude, as shown in section 5.

To the momentum equation we add the standard ray equations for wave propagation in a steady spatially variable medium (e.g., Kenyon 1971; Landau and Lifshitz 1959)

$$\frac{d\mathbf{x}}{dt} = \frac{\partial\omega}{\partial\mathbf{k}}, \tag{6}$$

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial\omega}{\partial\mathbf{x}}, \quad \text{and} \tag{7}$$

$$\omega = \omega' + \mathbf{k}\cdot\mathbf{U}, \tag{8}$$

where $\omega(\mathbf{k}, \mathbf{x})$ is a known function of the wavenumber vector $\mathbf{k}$ and the horizontal position vector $\mathbf{x}$; it is the wave frequency measured by a stationary observer not moving with the current and is conserved for a steady current. We assume for simplicity that the transmitting medium is independent of the vertical coordinate z. The rays and the wavenumber along the rays are determined by the simultaneous integration with respect to time of (6) and (7). Equation (6) gives the rays, which are paths traced out by points that move with the group velocity $\mathbf{c}_g = \partial\omega/\partial\mathbf{k}$. Behind the ray equations lies the central assumption that the inhomogemeous medium varies slowly within the distance of a wavelength.

It is convenient to split the total applied force on the right-hand side of (5) into two mutually perpendicular components:

$$\mathbf{F} = \mathbf{F}_m + \mathbf{F}_d, \tag{9}$$

where $\mathbf{F}_m$ changes only the magnitude of the momentum and $\mathbf{F}_d$ changes only the direction of the momentum. The linear momentum per unit area of the surface waves can be represented by

$$\mathbf{M} = M\hat{\mathbf{k}}, \tag{10}$$

where M is the magnitude of the momentum per unit area and $\hat{\mathbf{k}}$ is a unit vector pointing in the direction of the wavenumber, that is, in the direction of wave propagation.

Taking the time rate of change of the momentum per unit area (10) gives

$$\frac{d\mathbf{M}}{dt} = \frac{dM}{dt}\hat{\mathbf{k}} + M\frac{d\hat{\mathbf{k}}}{dt} = \mathbf{F}_m + \mathbf{F}_d \tag{11}$$

using (9); therefore from the definitions of the components of the force per unit area we have

$$\frac{dM}{dt}\hat{\mathbf{k}} = \mathbf{F}_m \tag{12}$$

and

$$M\frac{d\hat{\mathbf{k}}}{dt} = \mathbf{F}_d. \tag{13}$$

Equation (12) will be dealt with below. Equation (13) can be further developed now by noting that since $\hat{\mathbf{k}}$ is a unit vector that cannot change its magnitude but only its direction

$$\frac{d\hat{\mathbf{k}}}{dt} = \frac{d\theta}{dt}(\hat{\mathbf{z}} \times \hat{\mathbf{k}}), \quad (14)$$

where θ is the polar angle of the wavenumber vector, measured counterclockwise from the $\mathbf{x}$ axis, and $\hat{\mathbf{z}}$ is a unit vector that points vertically up normal to the mean free surface.

To relate the time rate of change of the angle θ on the right-hand side of (14) to quantities we know, we first calculate

$$\frac{d}{dt}\tan\theta = \sec^2\theta \frac{d\theta}{dt} = \frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{k^2}{k_2^x}\frac{d\theta}{dt}. \quad (15)$$

Also,

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{k_y}{k_x} = \frac{1}{k_x^2}\left(k_x\frac{dk_y}{dt} - k_y\frac{dk_x}{dt}\right). \quad (16)$$

Equating the right-hand sides of (15) and (16) produces the desired result

$$\frac{d\theta}{dt} = \frac{1}{k^2}\left(\mathbf{k} \times \frac{d\mathbf{k}}{dt}\right) \cdot \hat{\mathbf{z}} \quad (17)$$

because now the second ray equation, (7), can be substituted into the right-hand side of (17) for the time rate of change of the wavenumber vector.

Combining (7), (14), and (17) we get for the applied force component that changes the direction of the wave momentum

$$\mathbf{F}_d = -\frac{M}{k}\left[\left(\hat{\mathbf{k}} \times \frac{\partial\omega}{\partial\mathbf{x}}\right) \cdot \hat{\mathbf{z}}\right](\hat{\mathbf{z}} \times \hat{\mathbf{k}}), \quad (18)$$

and then the right-hand side of (17) can be evaluated completely once we specify the frequency function $\omega(\mathbf{k}, \mathbf{x})$. For example, we can put (8) into (18).

4. Wave force on a shear current

For the most general wave–current interaction involving a steady spatially variable current the frequency function is specified by (8), where on the left-hand side is the frequency observed in the steady frame, and on the right-hand side is the frequency observed relative to the current $\mathbf{U}(\mathbf{x})$. On the far right of (8) is the so-called Doppler frequency shifting term. Usually ω' is called the intrinsic frequency, and for deep-water waves its functional dependence on the wavenumber is given by the dispersion relation (4a).

Besides going from the most general situation to the special case of deep water waves, we now take the special case of a horizontal shear current for ease of illustration,

$$\mathbf{U}(\mathbf{x}) = V(x)\hat{\mathbf{y}}, \quad (19)$$

which is a steady current that flows only in the y direction and varies only in the x direction; $\hat{\mathbf{y}}$ is a unit vector in the y direction. A further specialization of (19) is to a linear shear current

$$V(x) = sx, \quad (20)$$

where s is the constant shear in units of inverse time. With (20) and (19), (8) becomes

$$\omega(\mathbf{k}, \mathbf{x}) = \sqrt{gk} + k_y sx. \quad (21)$$

From (21) we compute

$$\frac{\partial\omega(\mathbf{k}, \mathbf{x})}{\partial\mathbf{x}} = k_y s\hat{\mathbf{x}}, \quad (22)$$

where $\mathbf{x}$ is a unit vector in the x direction. If we put (22) into (18), we get

$$\mathbf{F}_d = Ms\sin^2\theta(\hat{\mathbf{z}} \times \hat{\mathbf{k}}), \quad (23)$$

where θ is the angle of the wavenumber measured counterclockwise from the x axis. In (23) the momentum is with respect to the current; that is, M is given by (3) and the frequency is interpreted to be the intrinsic frequency. Equation (23) gives the force of the current on the wave, and by Newton's third law the force of the wave on the current is equal but opposite, that is, $-\mathbf{F}_d$.

The properties of the direction changing wave force on the shear current can be seen quickly from (minus) (23). Maximum wave force occurs when the wave travels parallel to the current ($\theta = \pi/2$), and at that position the wave tries to push the current in the direction normal to the flow. When the wave travels perpendicular to the current ($\theta = 0$), the direction changing force vanishes. For any direction of the wavenumber the force is always perpendicular to it (and therefore not perpendicular to the flow in general). Aside from the dependence on the polar angle θ the magnitude of the force is directly proportional to the linear momentum of the wave times the shear in the current. Note that the direction-changing force is independent of the group velocity.

Since M depends of the frequency according to (3), then the wave force on the current depends directly on the frequency for constant amplitude waves. This is in contrast to the total wave force on a vertical wall during perfect reflection, which does not depend on the wave frequency for constant amplitude waves (Kenyon 2004a). It is known now that shoaling waves progressing shoreward and, ultimately being absorbed (i.e., not reflected), exert a force on the bottom, and that force is independent of the wave frequency as well (Kenyon 2004b). However, the frequency dependence of the wave force on a current is consistent with ray theory for wave–current refraction by which the high frequencies are predicted to be more affected than the low frequencies. For a given initial amplitude and angle of incidence there is more bending or curvature to the rays for the higher frequencies. [In optics it is well known that reflection of light waves is independent of the frequency, whereas refraction depends on frequency.]

Characteristics of the magnitude changing wave force on a shear current will be discussed in the following section and will be compared with those of the direction-changing wave force in (23).

5. Wave growth or decay in a shear current

As discussed in the introduction, during pure wave–current interaction the magnitude of the angular momentum of the waves (i.e., the wave action) must remain constant provided the horizontal shear current is uniform vertically. So we begin with the time rate of change of the angular momentum magnitude being set equal to zero:

$$\frac{dA}{dt} = 0. \tag{24}$$

Since the magnitude of the angular momentum A and the total energy E of the waves are related through the intrinsic frequency (Kenyon and Sheres 1996), $A = E/\omega'$, from (24) comes

$$\frac{dA}{dt} = \frac{d}{dt}\left(\frac{E}{\omega'}\right) = 0, \tag{25}$$

which can be written as

$$\frac{1}{E}\frac{dE}{dt} = \frac{1}{\omega'}\frac{d\omega'}{dt}. \tag{26}$$

The right-hand side of (26) is worked out shortly in terms of quantities we know through the ray equations. Then (26) gives the wave energy loss or gain due to interchanges with the current.

Wave linear momentum magnitude and total energy, both per unit area, are related through the wave phase speed (Barnett and Kenyon 1975)

$$M = \frac{E}{c'}, \tag{27}$$

where c' is the phase speed relative to the current. From (27) the time rate of change of the momentum magnitude per unit area can be calculated for deep water waves with dispersion relation (4a) and the angular momentum magnitude (action) conservation equation in (26):

$$\frac{dM}{dt} = \frac{d}{dt}\left(\frac{E}{c'}\right) = \frac{d}{dt}\left(\frac{E}{g/\omega'}\right) = \frac{2E}{g}\frac{d\omega'}{dt}. \tag{28}$$

Developing the right-hand side of (28) through the ray equations then gives the time rate of change of the linear momentum per unit area of the waves or the wave force on the current in the direction parallel to the wavenumber (the magnitude-changing force).

We get from (4a)

$$\frac{1}{\omega'}\frac{d\omega'}{dt} = \frac{1}{2k}\frac{dk}{dt} \tag{29}$$

and, since $k^2 = k_x^2 + k_y^2$,

$$\frac{dk}{dt} = \frac{1}{k}\left(k_x\frac{dk_x}{dt} + k_y\frac{dk_y}{dt}\right), \tag{30}$$

where k_x and k_y are the wavenumber components along the x and y axes, respectively. The second ray equation, (7), in component form is

$$\frac{dk_x}{dt} = -\frac{\partial\omega}{\partial x} \quad \text{and} \tag{31}$$

$$\frac{dk_y}{dt} = -\frac{\partial\omega}{\partial y}, \tag{32}$$

which when inserted into (29) produces

$$\frac{dk}{dt} = -\frac{1}{k}\left(k_x\frac{\partial\omega}{\partial x} + k_y\frac{\partial\omega}{\partial y}\right). \tag{33}$$

When we combine (33) and (29) with (26) we get

$$\frac{1}{E}\frac{dE}{dt} = -\frac{1}{2k}\left(\cos\theta\frac{\partial\omega}{\partial x} + \sin\theta\frac{\partial\omega}{\partial y}\right), \tag{34}$$

where $k_x/k = \cos\theta$, $k_y/k = \sin\theta$. Given that the total energy per unit horizontal area of the waves is

$$E = \frac{1}{2}\rho g a^2, \tag{35}$$

the amplitude growth rate from (34) is

$$\frac{1}{a}\frac{da}{dt} = -\frac{1}{4k}\left(\cos\theta\frac{\partial\omega}{\partial x} + \sin\theta\frac{\partial\omega}{\partial y}\right). \tag{36}$$

For the linear shear current (20), (36) reduces through (22) to

$$\frac{1}{a}\frac{da}{dt} = -\frac{s}{8}\sin 2\theta, \tag{37}$$

which predicts an exponential time rate of growth or decay for the wave amplitude, depending on the sign of the shear in the current and the direction of the wavenumber relative to that of the current. A numerical example may be helpful. Take $s = 8 \times 10^{-4}$ s^{-1}, a rather large value for the current shear, and $\theta = \pi/4$, then $|da| = 0.1a$ when $dt \approx 16$ min, assuming s and θ remain constant over the 16-min interval. To obtain the amplitude over the whole length of a ray inside a shear current a numerical integration would need to be carried out, in general, where θ is supplied for all positions along the ray.

Similarly (28) can be evaluated using (29) and (33) to yield

$$\frac{dM}{dt} = -\frac{E\omega'}{gk}\left(\cos\theta\frac{\partial\omega}{\partial x} + \sin\theta\frac{\partial\omega}{\partial y}\right)$$
$$= -\frac{M}{ck}\left(\cos\theta\frac{\partial\omega}{\partial x} + \sin\theta\frac{\partial\omega}{\partial y}\right), \qquad (38)$$

which reduces to

$$\frac{dM}{dt} = -\frac{Ms}{2}\sin 2\theta \qquad (39)$$

for the linear shear current (20). Therefore, the magnitude changing force of the current on the wave (the force component in the direction of the wavenumber) is

$$\mathbf{F}_m = -\frac{Ms}{2}\sin 2\theta\hat{\mathbf{k}}. \qquad (40)$$

For the force of the wave on the current we apply a minus sign to (40) according to Newton's third law.

Comparing (40) with (23) we see first of all that both components of the wave force on a shear current have the same order of magnitude in general; that is, each component is proportional to the momentum times the current shear. Also, both components are zero when $\theta = 0$. However, the direction changing force has its maximum value at $\theta = \pi/2$, whereas the magnitude changing force is maximum value at $\theta = \pi/4$.

Both $\mathbf{F}_d$ and $\mathbf{F}_m$ are proportional to M, which is proportional to the frequency multiplied by the amplitude squared [(3)], $M = 0.5\rho\omega' a^2$. Thus for constant amplitude waves the force applied by the waves on the current is proportional to the frequency. As the frequency increases, the wavelength decreases [(4)], and the wave slope, ak, increases. Beyond a limiting high value of the slope the waves will break. When we consider waves with constant slope, $\mathbf{F}_d$ and $\mathbf{F}_m$ are proportional to the inverse cube of the frequency. This is because for $ak =$ const and, using (4), $M \propto \omega' a^2 \propto (\omega')^{-3}$. Thus, an increase in frequency will reduce the force of the waves on the current significantly provided the slope remains constant. The wave force on the current is concentrated from the surface down to the depth of wave influence, which decreases with increasing frequency.

6. Application

We would like now to attempt an application of the wave force formula to a real ocean current, such as the Gulf Stream. However, such an effort cannot be completed, mainly because all the necessary observations are not available at this time. So we begin by making qualitative estimates and comparisons.

First, contrast the maximum value of the wave force per unit volume, when the wave direction is parallel to the shear flow, by adapting (23)

$$f_d = ms = \rho U_s, \qquad (41)$$

where U is given by (2), to the magnitude of the Coriolis force, f_c, on the same current

$$f_c = \rho f u, \qquad (42)$$

where f is the Coriolis parameter and u is the speed of the current. Both force components in (41) and (42) act normal to the flow and are either parallel or antiparallel depending on the hemisphere and the wave direction relative to the current. Although for major ocean currents u is considerably larger than U, there are environmental situations for which s is an order of magnitude or more larger than f, even at midlatitudes (Sheres et al. 1985). Therefore, it is within the realm of possibility that the sideways wave force on the upper part of a current could be as large, and in some cases larger, as the Coriolis force on that current. This can be significant since it is generally believed that ocean currents are in geostrophic balance on the whole, meaning that the Coriolis force is balanced by an equal but opposite horizontal pressure gradient related to the density field. It would be very difficult to confirm the action of the wave force; however, in situ measurements and satellite imagery by Grodsky et al. (2000) support such a possibility, and their measurements on 28 August 1991 show 150-m waves reflected by the Gulf Stream that had a maximum speed of 2 m s^{-1}. Amplitude data for the 150-m-long waves was not available; however, the synthetic aperture radar (SAR) images for that date in Grodsky et al. (2000) clearly showed a narrow band (in frequency), for these waves.

The Coriolis force at 38°N on the Gulf Stream flowing with an average speed of 1 m s^{-1} is from (42):

$$f_C = 0.89 \times 10^{-4} \rho \text{ N m}^{-3}.$$

The maximum wave force would equal the Coriolis force for this case of wave reflection on 28 August, with 150-m wavelength and shear of 1.5×10^{-4} s $^{-1}$ [measured on 29 August; adapted from Fig. 6d of Grodsky et al. (2000)], if the wave amplitude would be 4.7 m, calculated using (41). In the equatorial regions where the Coriolis force is much smaller, 150-m-long waves with amplitude of 1.5 m will exert a force equal to the Coriolis force at 3.56° latitude, as they are reflected from the same shear as above. The maximum wave force occurs at the surface and diminishes exponentially with depth; clearly the longer the waves the deeper the wave force will affect the current.

Could the wave force play a role in the initiation of Gulf Stream meanders? This is an intriguing, seemingly unlikely, possibility and confirming it would be very difficult; here is what we do know, however. The wave ray radius of curvature was estimated at $R = C_g \text{ s}^{-1}$ (C_g the wave group velocity magnitude) by Kenyon (1971); a Gulf Stream shear of 1.5×10^{-4} s^{-1} and a swell wavelength 150 m give a ray radius of curvature of 51 km, comparable to the radius of curvature of the Gulf Stream meanders. The maximum force applied by waves at and near the surface can be comparable to the Coriolis force, as shown above. Grodsky et al. (2000) showed a total reflection of swell by a Gulf Stream meander with the parameters similar to those above, using SAR and infrared imagery as well as a numerical ray model. A lot more information will be required in order to support or negate this application.

Another example can be imagined involving a broad source of waves. Let a wind blow opposite to the direction of the current generating waves all along the axis of the current that move counter to the stream. Then refraction can cause total internal reflection of most of the waves and, due to the momentum exchange between the waves and the current in which the waves push outward normal to the flow (Fig. 2, the wave and momentum direction vary along the ray and are not shown), the net tendency will be to broaden the current region normal to the axis. An event of wave trapping in the Gulf Stream was described by Kudryavtsev et al. (1995). Changes of current width (such as that of the Gulf Stream) during such an event could conceivably be observed by remote sensing.

More hypothetical (for the Gulf Stream perhaps) would be the case of two broad sources of following waves that approach the current from both sides simultaneously and are then reflected due to refraction in the current (Fig. 1, the wave and momentum direction vary along the ray and are not shown). Alternatively one could think of a single band of wind that blows in the direction of the stream and generates waves inside the current that refract out both sides of the current. From the wave–current momentum exchange we anticipate a thinning of the current under these conditions.

A thinning of a current by two broad sources of following waves might have a more likely application in the equatorial ocean areas, where the Coriolis force is small because of the low latitude. For example, in the North Pacific, and in the yearly mean, there is a westward equatorial surface current with northeast trade winds on its north side and southeast trades on its south side. Both winds and currents exist over most of the length of the ocean. Therefore, the winds create two broad sources of following waves that propagate into the current from both sides simultaneously. The momentum exchange between waves and currents may help hold this current together, confined to a relatively narrow latitude band, to a depth of wave influence, in view of the fact that the Coriolis force is weak.

One further conceptual step leads to a feedback mechanism between the waves and the current. A thinning current produces increasing horizontal shear, which in turn increases the force of the waves on the current by (19). Further thinning follows, and the feedback is a positive one. Of course, a negative feedback would take place for waves propagating against the Gulf Stream because the wave force tends to broaden the current, reducing the current shear and weakening the wave force on the current. This feedback mechanism (positive or negative), with the wave force changing the shear that determines its magnitude [see (41)], is likely to operate in all the interactions described above.

Since the wave force on the current only exists over the depth of wave influence, which is comparable to a wavelength, whereas the depth scale of the current may be greater than that, then the force might push the top of the current sideways and leave the bottom part where it was. One consequence would be to change the slope of the isotherms in a vertical section when the water density is controlled mainly by temperature. For example, for waves propagating into the Gulf Stream from the southeast (e.g., from a hurricane) and then reflecting away, the warm surface water would be pushed to the west, resulting in isotherms that have a different slope in vertical sections in which the flow is directed into the paper; this will affect Gulf Stream velocity estimates based on density measurements.

The potential changes in the Gulf Stream, or other current, surface extent described above would have an effect on remote sensing imagery of these currents; it will bias the estimates of their width, and therefore the amounts of water, heat, and so on transported by them. Estimates of such bias can be important for ocean dynamics and climate models.

Acknowledgments. We thank Bert Green for his review and helpful comments.

REFERENCES

Barnett, T. P., and K. E. Kenyon, 1975: Recent advances in the study of wind waves. *Rep. Prog. Phys.,* **38,** 667–729.

Buhler, O., and M. E. McIntyre, 2003: Remote recoil: A new wave–mean interaction effect. *J. Fluid Mech.,* **492,** 207–230.

Faber, T. E., 1995: *Fluid Dynamics for Physicists*. Cambridge University Press, 440 pp.

Ference, M., Jr., H. B. Lemon, and R. J. Stephenson, 1956: *Analytical Experimental Physics.* 2d ed. The University of Chicago Press, 623 pp.

Garrett, C., 1976: Generation of Langmuir circulations by surface waves—A feedback mechanism. *J. Mar. Res.,* **34,** 117–130.

Grodsky, S., V. Kudryavtsev, and A. Ivanov, 2000: Quasisynchronous observations of the Gulf Stream frontal zone with Almaz-1 SAR and measurements taken on board the R/V Akademik Vernadsky. *Global Atmos. Ocean Syst.,* **7,** 249–272.

Kenyon, K. E., 1969: Stokes drift for random gravity waves. *J. Geophys. Res.,* **74,** 6991–6994.

——, 1971: Wave refraction in ocean currents. *Deep-Sea Res.,* **18,** 1023–1034.

——, 2004a: Force and torque on a wall from reflected surface gravity waves. *Phys. Essays,* **17,** 95–102.

——, 2004b: Shoaling surface gravity waves cause a force and a torque on the bottom. *J. Oceanogr.,* **60,** 1045–1052.

——, and D. Sheres, 1996: Angular momentum and action in surface gravity waves: Application to wave–current interaction. *J. Geophys. Res.,* **101,** 1247–1252.

Kudryavtsev, V. N., S. A. Grodsky, V. A. Dulov, and A. N. Bol'shakov, 1995: Observations of wind wave field in the Gulf Stream frontal zone. *J. Geophys. Res.,* **100** (C10), 20 715–20 727.

Kundu, P. K., 1990: *Fluid Mechanics*. Academic Press, 638 pp.

Landau, L. D., and E. M. Lifshitz, 1959: *Fluid Mechanics*. Addison-Wesley, 536 pp.

Longuet-Higgins, M. S., and R. W. Stewart, 1964: Radiation stress and mass transport in water waves; a physical discussion with applications. *Deep-Sea Res.,* **11,** 529–562.

Mahon, B., 2003: *The Man Who Changed Everything*. John Wiley and Sons, 226 pp.

Mellor, G., 2003: The three-dimensional current and surface wave equations. *J. Phys. Oceanogr.,* **33,** 1978–1989.

Padgett, M., C. J. Courtial, and L. Allen, 2004: Light's orbital angular momentum. *Phys. Today,* **57,** 35–40.

Perrie, W., C. L. Tang, Y. Hu, and B. M. DeTracy, 2003: The impact of waves on currents. *J. Phys. Oceanogr.,* **33,** 2126–2140.

Putnam, J. A., W. H. Munk, and M. A. Traylor, 1949: The prediction of longshore currents. *Trans. Amer. Geophys. Union,* **30,** 337–345.

Rundgren, L., 1958: Water wave forces, a theoretical and laboratory study. *Trans. Roy. Inst. Technol.,* **122**.

Sheres, D., and K. E. Kenyon, 1990: Swell refraction by the Pt. Conception, California, eddy. *Int. J. Remote Sens.,* **11,** 27–40.

——, ——, R. L. Bernstein, and R. C. Beardsley, 1985: Large horizontal surface velocity shears in the ocean obtained from images of refracting swell and *in situ* moored current data. *J. Geophys. Res.,* **90,** 4943–4950.

Stokes, G. G., 1847: On the theory of oscillatory waves. *Trans. Camb. Philos. Soc.,* **8,** 441–455.