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Angular momentum and action in surface gravity waves: Application to wave-current interaction

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Abstract. When small-amplitude surface gravity waves progress in deep water, all the fluid particles are observed to orbit in circles within the depth of wave influence. Each fluid particle therefore has angular momentum with respect to the center of its orbit, and the angular momentum vectors are directed parallel to the crests and troughs or perpendicular to the wave number. The angular momentum per unit volume is calculated for each fluid particle and then, by vertical integration, the angular momentum per unit horizontal area is computed. The total energy and the magnitude of the angular momentum, both per unit area or both per unit volume, are found to be proportional, the factor of proportionality being the wave frequency; specifically, angular momentum magnitude equals energy divided by frequency. Wave action, which has become an increasingly popular quantity for interpreting surface gravity wave problems and is defined as wave energy divided by frequency, is the same thing as the magnitude of the angular momentum. A general equation for the conservation of angular momentum along wave rays is given, and it contains on the right-hand side two torque terms; one changes the magnitude and the other changes the direction of the angular momentum. In applying this equation to the wave-current refraction problem there is only one torque, which changes the direction of the angular momentum, and it is explicitly determined as a function of the horizontal shear in the current. The magnitude of the angular momentum, and therefore also the wave action, will be conserved along the rays if there are no torques that could alter it, as in pure wave-current refraction. Our conservation equation for angular momentum is easily adapted to describing wave generation and dissipation by including the appropriate torques on the right side, and this may prove to be helpful for calculations of wave evolution.

Introduction

Among all the waves in physics the small-amplitude surface gravity wave progressing in deep water is the only kind of wave in which material particles perform a circular motion [*Feynman et al.*, 1963]. Consequently, surface gravity waves have angular momentum, which is not a characteristic of wave motion that is regularly discussed. Since each fluid particle in a progressive surface gravity wave moves in a circle, it is obvious that each particle has angular momentum with respect to the center of its orbit. Very few journal publications mention the orbital angular momentum property of surface gravity waves, and the interest in this topic is relatively recent [e.g., *Brinch-Nielsen and Jonsson*, 1985]. In fact, angular momentum is generally not even mentioned in fluid dynamics texts; an exception is the work by *Kundu* [1990].

Taking the particle orbits to be circles could be viewed as a theoretical assumption, but it is ultimately based on direct observations of small-amplitude surface waves in deep water; for example, see the streak photographs of small neutrally buoyant particles in the laboratory experiments discussed by *Sommerfeld* [1964] and *Wiegel* [1964].

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In shallow water it is observed that the particle orbits are more nearly elliptical, with the major axes of the ellipses being parallel to the bottom or to the mean surface. For simplicity we restrict the discussion below to waves in deep water, meaning that the average total depth should exceed about one wave length. Extensions of the present analysis to elliptical particle orbits are possible and will be presented elsewhere.

If the wave amplitude is finite, the particle orbits are circles that are not quite closed, and this leads to a net (Stokes) drift of the particles in the direction of wave propagation or in other words to a linear momentum of the waves. We will neglect this finite amplitude effect, because we are primarily interested in the angular and not the linear momentum. Also, the influence of the Coriolis force, acting upon the orbiting particles on the rotating Earth, will not be taken into account here. The neglect of the Coriolis force is an assumption, normally not stated, that occurs in most investigations of surface waves, and it is based probably on the small scales of the motion.

The angular momentum vectors associated with surface waves lie in horizontal planes, and they are parallel to the crests and troughs; they always remain perpendicular to the wave number vectors. As will be shown below, the magnitude of the angular momentum per unit horizontal area, that is, integrated over depth, is proportional to the total energy per unit horizontal area, the factor of proportionality being the wave frequency. Also, it will be seen that the angular momentum magnitude is the same thing as a quantity called wave action, which *Bretherton and Garrett* [1969] first introduced into the fluid dynamics literature by means of a Hamiltonian formulation.

When surface gravity waves refract in a variable current, the wave energy is not conserved, because energy can be exchanged between the wave and current motions. It turns out that the orbital angular momentum of the waves is conserved in magnitude but not in direction during wave-current refraction, because a torque must exist in order to change the direction of the angular momentum vectors as they propagate along the wave rays. The rays, and the wave number along the rays, are determined by the ray equations (see equation (6)).

A general (vector) equation for the conservation of angular momentum is presented in (7) and is discussed in relation to the existing (scalar) equation for conservation of wave action. Conservation of angular momentum is a concise method of deriving conservation of wave action, and it has the advantage of being easily extended to include wave generation and dissipation processes.

The torque term in the angular momentum equation is made explicit for the wave-current refraction problem by relating it to the horizontal current shear. Wave-current interaction is an important area of research in which interest has been growing over the past 20 years or so [*Jonsson*, 1990; *Holthuijsen and Tolman*, 1991; *Sheres and Kenyon*, 1990; *Sheres et al.*, 1985; *Kenyon*, 1971]. Torque and angular momentum are well-established physical concepts that are employed here in a new context, resulting in an increased understanding of the refraction process when surface gravity waves encounter variable currents.

Because of the orbital nature of the wave angular momentum, it is convenient in what follows to base the calculations of angular momentum and energy on the motion of individual fluid particles. As documented in the appendices, the computation of the wave potential energy (Appendix B) from the point of view of the fluid particle appears to be new, and the calculation of the kinetic energy (Appendix A) is not as lengthy as it is in the standard (Eulerian) perturbation method.

What follows next is not a new theory of surface gravity waves per se. We believe that our main contribution is a novel way to interpret wave action and wave-current refraction in terms of angular momentum and torque, and we anticipate that these old tools in a new setting will prove helpful for understanding other wave propagation problems, particularly wave generation and dissipation mechanisms. Our approach supports and reinforces the concept of wave action, which has turned out to be very beneficial for studying processes affecting surface gravity waves [*Holthuijsen and Tolman*, 1991].

Angular Momentum

The magnitude A of the angular momentum vector, per unit volume, of a surface particle in circular motion with constant radius a is

$$A = \rho|\mathbf{r} \times \mathbf{v}| = \rho\omega a^2 = \text{const} \qquad (1)$$

where $|\mathbf{v}| = \omega a$, $|\mathbf{r}| = a$, and the angular momentum is taken with respect to the center of the orbit. There is only circular motion, so the observer is stationary with respect to the rotating particle or moving with the current if there is one. Thus ω is called the intrinsic frequency. According to (1), $\mathbf{A}$ is independent of time.

Comparing (1) with the total energy per unit volume from (B4) in the second appendix, we have

$$A = E/\omega \qquad (2)$$

which shows that for the surface particle the magnitude of the angular momentum and the total energy, both per unit volume, are proportional, the factor of proportionality being the wave frequency.

Now for plane waves the directions of the angular momentum vectors of all the fluid particles are parallel, so it is easy to sum them up in the vertical direction, for example, in order to compute the angular momentum per unit horizontal area, with magnitude $\bar{A}$. Replacing a by $r(z)$ in (1) and integrating over depth gives

$$\bar{A} = \rho g a^2/2\omega \qquad (3)$$

where (A4) has been used for $r(z)$.

Relating (B5) and (3) produces the analogue to (2)

$$\bar{A} = \bar{E}/\omega \qquad (4)$$

that the angular momentum and energy, both per unit area, are proportional, with the frequency again being the proportionality factor. It follows also that

$$A(z) = E(z)/\omega \qquad (5)$$

at any particular mean depth of a fluid particle z. Following accepted practice, from now on ω will be written as ω'.

Wave Action

Bretherton and Garrett [1969] used the Hamiltonian formulation to develop expressions for the conservation of wave action to be applied along the wave rays when slowly changing wave trains of small amplitude propagate in various inhomogeneous moving media. They defined wave action as the total wave energy per unit area divided by the intrinsic frequency, the frequency measured by an observer moving with the local mean velocity of the medium. Since 1969 the concept of wave action conservation has become an increasingly popular method for describing theoretically the complete evolution of surface gravity waves from generation and propagation to dissipation. ("Action" has an extensive history in physics and astronomy to which the reader will be led by citations either in the paper by Bretherton and Garrett or in a standard mechanics text such as those by *Joos* [1986] or *Born* [1960].) An interesting alternative derivation of the conservation of wave action not based on the Hamiltonian approach is given by *Christoffersen* [1982].

For comparison purposes we begin by considering surface gravity waves traveling in a homogeneous medium that is not moving. Then the intrinsic frequency equals the wave frequency, and wave action is identical to the magnitude of the angular momentum per unit area, because they are both equal to the same quantity: the total energy per unit area divided by the frequency (and the angular momentum does not change direction). Actually, *Naeser* [1979] was the first to point out a connection between wave action and angular momentum, but his approach is different and not based on orbiting fluid particles, as we have done.

In a moving medium which is homogeneous the equality between wave action and the magnitude of the angular momentum still holds, because both quantities are again equal to the same thing, which in this case is the total energy per unit area divided by the intrinsic frequency. This can be most easily understood from the point of view of an observer traveling with the moving medium. Conservation of wave action would equal conservation of angular momentum in this situation also.

The analogy between angular momentum and wave action does not hold in general, however, when the medium is inhomogeneous, whether it is moving or not. If the medium is inhomogeneous, the analogy breaks down as a consequence of the fact that angular momentum is a vector with changing direction and wave action is a scalar. When waves refract in an inhomogeneous medium, the wave number and angular momentum vectors will usually change direction in the horizontal plane along the wave rays. Consequently, a scalar equation for the conservation of wave action along the rays could not also be a vector equation for the conservation of angular momentum during the refraction process. In fact, angular momentum cannot be conserved along the rays in an arbitrary refraction case, because a source term (or torque) must exist in order to change the direction of the angular momentum.

Angular Momentum Equation

The ray equations for a steady medium are [*Kenyon*, 1971; *Landau and Lifshitz*, 1959]

$$d\mathbf{x}/dt = \partial\omega/\partial\mathbf{k} \tag{6a}$$

$$d\mathbf{k}/dt = -\partial\omega/\partial\mathbf{x} \tag{6b}$$

where $\omega = \omega(\mathbf{k}, \mathbf{x})$ is a known function of wave number $\mathbf{k}$ and position $\mathbf{x}$, and it is conserved. The rays and the wave number along the rays are determined by the simultaneous integration of (6) with respect to time. The ray is the path of the group velocity of the waves described by $\mathbf{x}(t)$; ω is the wave frequency measured by a stationary observer that is not moving with the current.

The wave amplitude along the rays is determined by the equation for orbital angular momentum $\mathbf{A}$

$$d\mathbf{A}/dt = \boldsymbol{\tau} \tag{7}$$

where $\boldsymbol{\tau}$ is the torque. If we know the torques along the rays and the initial value of the angular momentum, then the angular momentum can be found all along the rays from a time integration of (7). Next by obtaining the intrinsic frequency from the ray equations (via k as $\omega'(k)$) we can calculate the wave energy and therefore the wave amplitude along the rays, because angular momentum equals energy divided by intrinsic frequency, and energy is proportional to the square of the wave amplitude.

It is convenient for what follows to split the torque on the right side of (7) into two perpendicular components: $\boldsymbol{\tau} = \boldsymbol{\tau}_m + \boldsymbol{\tau}_d$, where $\boldsymbol{\tau}_m$ changes only the magnitude and $\boldsymbol{\tau}_d$ changes only the direction of the angular momentum vector. When there is no torque at all ($\boldsymbol{\tau} = 0$), (7) shows that total angular momentum is conserved along the rays. If there is no torque that will alter the magnitude of the angular momentum ($\boldsymbol{\tau}_m = 0$), then the magnitude of the angular momentum is conserved along the rays even though the direction of the angular momentum might change.

The orbital angular momentum for surface gravity waves can be represented by

$$\mathbf{A} = A(\hat{\mathbf{z}} \times \hat{\mathbf{k}}) \tag{8}$$

where $\hat{\mathbf{z}}$ is a unit vector always pointing up, perpendicular to the horizontal (equilibrium) surface, and $\hat{\mathbf{k}}$ is a unit vector in the direction of the wave number. Then the time rate of change of the angular momentum is (since $\hat{\mathbf{z}}$ is constant in magnitude and direction)

$$\frac{d\mathbf{A}}{dt} = \frac{dA}{dt}(\hat{\mathbf{z}} \times \hat{\mathbf{k}}) + A\hat{\mathbf{z}} \times \frac{d\hat{\mathbf{k}}}{dt} \tag{9}$$

The second term in (9) can be evaluated further by noting that the unit vector $\hat{\mathbf{k}}$ has constant length and can therefore only change direction

$$\frac{d\hat{\mathbf{k}}}{dt} = \frac{d\theta}{dt}(\hat{\mathbf{z}} \times \hat{\mathbf{k}}) \tag{10}$$

where θ is the polar angle of the wave number vector. Thus the rate of change of the angular momentum in (9) is divided into two perpendicular parts; the first, in the direction of the angular momentum, is the rate of change of the magnitude of the angular momentum, and the second, in the direction of the wave number, describes the rate of change of the direction of the angular momentum.

Now by taking the time derivative of $\tan\theta = k_y/k_x$ (i.e.,

$$\frac{d}{dt}\tan\theta = \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{k^2}{k_x^2}\frac{d\theta}{dt}$$

for the left side and

$$\frac{d}{dt}\frac{k_y}{k_x} = \frac{1}{k_x^2}\left(k_x\frac{dk_y}{dt} - k_y\frac{dk_x}{dt}\right)$$

for the right side) we get

$$\frac{d\theta}{dt} = \frac{1}{k^2}\left(\mathbf{k} \times \frac{d\mathbf{k}}{dt}\right) \cdot \hat{\mathbf{z}} \tag{11}$$

Finally, combining (10) and (11) with the last term in (9), which we identify with $\boldsymbol{\tau}_d$, we get

$$\boldsymbol{\tau}_d = A\left\{\frac{1}{k}[\hat{\mathbf{k}} \times (\partial\omega/\partial\mathbf{x})] \cdot \hat{\mathbf{z}}\right\}\hat{\mathbf{k}} \tag{12}$$

where the second ray equation in (6) has also been used. Another way to write (12) is

$$\boldsymbol{\tau}_d = \mathbf{A} \times \boldsymbol{\Omega} \tag{13}$$

where

$$\boldsymbol{\Omega} = \left\{\frac{1}{k}[\hat{\mathbf{k}} \times (\partial\omega/\partial\mathbf{x})] \cdot \hat{\mathbf{z}}\right\}\hat{\mathbf{z}} \tag{14}$$

Equation (12) or (13) gives the functional form for the torque that is necessary to change the direction of the angular momentum in terms of spatial derivatives of the given function ω. It is evident that the torque in (12) can be computed along the rays by means of the information obtained from the ray equations (6). In (9) the two terms are perpendicular to each other; therefore, to change the magnitude of the angular momentum (or the wave action) along the rays, we need to have a torque in the direction of the angular momentum, or along the wave crests, and to change the direction of the angular momentum we need a torque perpendicular to the angular momentum, or in the direction of the wave number.

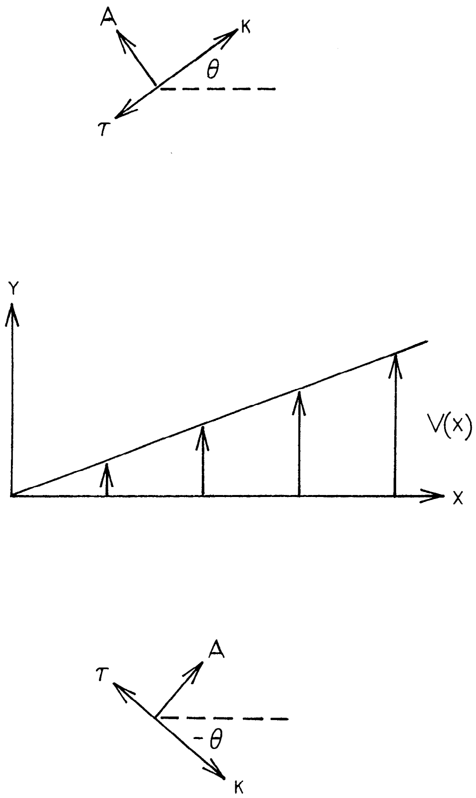

Figure 1. Illustration of two configurations for the torque, angular momentum, and wave number vectors during wave-current refraction in a shear current when the waves travel with the current (top) and when the waves travel against the current (bottom). The shear $V(x)$ is shown in the middle.

Wave-Current Refraction

In applying the above analysis to the problem of wave-current refraction it is only necessary to specify the function $\omega(\mathbf{k}, \mathbf{x})$

$$\omega = \omega' + \mathbf{k} \cdot \mathbf{U} \tag{15}$$

where ω is the frequency relative to a fixed coordinate system and ω' is the frequency relative to a coordinate system moving with the current velocity $\mathbf{U} = \mathbf{U}(\mathbf{x})$, which is a given function of position $\mathbf{x}$. Usually, ω' is called the intrinsic frequency, and for surface gravity waves in deep water it is specified by $\omega' = (gk)^{1/2}$, where g is the acceleration of gravity. If the current velocity is independent of time and the vertical coordinate, then it is straightforward to integrate (6) to find the rays and the wave number along the rays [*Kenyon*, 1971]; the integration can easily be done numerically for an arbitrary current distribution, and it can be carried out analytically for certain simple current shears. Now with (7) the wave amplitude can be calculated along the rays as well. The current shear produces a torque that changes the direction of the angular momentum along the rays, and this effect can be evaluated by inserting (15) into (12). Assuming that there are no other torques acting, then the magnitude of the angular momentum, or the wave action, will be conserved along the rays.

As a particular example, we compute the torque for a simple current shear. Let the current velocity be $\mathbf{U} = 0,\ V(x)\hat{\mathbf{y}}$ and $V(x) = sx$, where $s = $ const. Then (15) becomes

$$\omega = \omega' + k_y s x$$

and we compute

$$\partial \omega / \partial \mathbf{x} = k_y s \hat{\mathbf{x}}$$

so that the torque in (12) becomes

$$\boldsymbol{\tau}_d = -As \sin^2 \theta \hat{\mathbf{k}}$$

where $\sin \theta = k_y/k$. The torque is larger for larger given values of the angular momentum A, current shear s, and angle θ, all of which are physically reasonable. In addition, it can be seen (Figure 1) that the torque has the right sign to try to bend the angular momentum vector, and therefore also the wave number vector, away from regions of increasing current speed when the waves travel with the current ($\theta \geq 0$, top of Figure 1) and toward regions of increasing current speed when the waves travel against the current ($\theta \leq 0$, bottom of Figure 1), as we already know should happen from the ray equations [*Kenyon*, 1971]. Therefore the angular momentum and ray equations are consistent. There is no way the torque could bend the wave number vector any slower or faster than dictated by the ray equations. Equations (11) or (12) show that the time rate of change of the angular momentum is directly proportional to the time rate of change of the wave number vector when there are no additional torques that can alter the magnitude of the angular momentum.

Discussion

The wave orbital angular momentum is obtained above by starting with the fluid particle, and this is different from the existing procedures, although angular momentum is a property of surface gravity waves that has been discussed very little in the past. The angular momentum magnitude equals the energy divided by the frequency, and therefore it also equals the wave action. A similar equation to ours relating the magnitude of the angular momentum to the total energy of surface gravity waves holds also for the angular momentum and energy of a planet that has a circular orbit around the Sun [*Kenyon*, 1993], and the frequency is again a factor of proportionality (generalization to an elliptical orbit is straightforward for both the gravity wave and solar system problems).

When surface gravity waves refract in a spatially variable current, angular momentum is not conserved, because a torque must be present in order to change the direction of the angular momentum vectors as they propagate along the rays. The torque is directly related to the horizontal shear in the current, as just demonstrated. If no other torques are present, which could change the magnitude of the angular momentum, such as those that might be related to the forces of the wind or friction, then the magnitude of the angular momentum and the wave action will remain constant along the rays. This is a simple way to explain how it is that wave action can be conserved along the rays.

The usefulness of the angular momentum formulation extends beyond the particular application of wave-current interaction that we exhibited here. For example, by constructing the appropriate torque terms on the right-hand side our angular momentum equation can also describe wave generation and dissipation processes. Consider first wave generation, and let a uniform external wind blow parallel to the direction of travel of a plane wave. This situation provides the possibility of creating a torque which can increase the magnitude of the orbital angular momentum of the surface fluid particles through a frictional interaction between air and water without changing the direction of their angular momentum vectors. The torque is the local product of the wind force times the orbit radius of the surface particles, and in general the torque is positive, meaning that the waves will grow.

Similarly the process of wave dissipation can be handled by the angular momentum equation. The force of internal friction within the medium transmitting the waves, whatever its true formulation is, has the right sense always to decrease the magnitude of the angular momentum of the surface gravity waves, and it too leaves the direction of the angular momentum unaltered. The torque in this case, which is always negative, is the cross product of the friction force and the radius vectors of the fluid particles.

Appendix A: Kinetic Energy

Consider a surface fluid particle moving clockwise around its circular orbit, which is observationally consistent with a small-amplitude surface gravity wave progressing from left to right. The mean position of the particle is $z = 0$, $x = x_0$, and the radius of the orbit is a (which is also the wave amplitude). Then the horizontal (u) and vertical (w) components of the particle velocity are

$$u = \omega a \cos \omega t \qquad w = -\omega a \sin \omega t \tag{A1}$$

where ω is the angular frequency. In (A1) it is assumed that there is no net horizontal drift of the particle, that is, that the orbits are closed circles.

The kinetic energy per unit volume $K.E.$ of the surface particle in (A1) is

$$K.E. = \tfrac{1}{2}\rho(u^2 + w^2) = \tfrac{1}{2}\rho\omega^2 a^2 = \text{const} \tag{A2}$$

where ρ is the constant fluid density. Equation (A2) shows that the particle kinetic energy is constant, independent of time. Also, the kinetic energy in (A2) is the same for all surface particles, assuming the surface wave has constant amplitude (likewise the kinetic energy at any particular mean depth z is the same for all particles, see (A3)).

At any particular mean depth z below the surface the kinetic energy per unit volume is

$$K.E.(z) = \tfrac{1}{2}\rho\omega^2 r^2(z) \tag{A3}$$

where $r(z)$ is the radius of the fluid particle's circular orbit at mean depth z given by

$$r(z) = ae^{kz} \tag{A4}$$

where k is the wave number and $z < 0$.

Using (A3) and (A4), the kinetic energy per unit horizontal area $\overline{K.E.}$ can be computed by a vertical integration of (A3)

$$\overline{K.E.} = \int_{-\infty}^{0} \tfrac{1}{2}\rho\omega^2 r^2(z)\, dz = \tfrac{1}{4}\rho(\omega^2/k)a^2 = \tfrac{1}{4}\rho g a^2 \tag{A5}$$

where the dispersion relation $\omega^2 = gk$ for linear deep water surface waves has been used in (A5). Equation (A5) agrees with the classical result for kinetic energy per unit area of a small-amplitude surface gravity wave progressing in deep water.

The kinetic energy per unit horizontal area has been computed for a small-amplitude surface gravity wave progressing in deep water by starting with the circularly orbiting fluid particles. The result is the same as that obtained classically, but the effort involved is much less. Usually a perturbation analysis must be carried out first, which is the lengthy part.

Appendix B: Potential Energy

The potential energy of an orbiting fluid particle is the work done against the restoring force encountered in bringing the particle from its resting position, at the center of the orbit, out to the circumference of the orbit. The restoring force is a pressure gradient, which can be related to the variable height of the wave surface and to the vertical acceleration of the fluid. The inward pressure gradient balances the outward centrifugal force on the particle [*Kenyon*, 1991].

A surface gravity wave progresses from left to right. Consider a surface particle moving vertically upward where the surface has maximum downward slope from left to right. The horizontal pressure force per unit volume, positive to the right, is

$$-\delta p/\delta x = -\rho g \delta\zeta/\delta x = \rho g k a \tag{B1}$$

where the surface elevation ζ has been taken to be a sine wave, $\zeta = a \sin(kx - \omega t)$, and x is positive to the right.

When the surface particle is at this position of its orbit, the potential energy per unit volume $P.E.$ is the work done against the pressure force (B1) to bring the particle out from the orbit's center ($x = 0$) to the orbit's circumference ($x = a$), that is,

$$P.E. = \int_0^a -(\delta p/\delta x)\, dx = \rho g k \int_0^a x\, dx = \tfrac{1}{2}\rho g k a^2$$
$$= \text{const.} \tag{B2}$$

Now it can be shown that (B2) is general, that is, that it holds for all positions of the particle around its orbit and not just for the particular position used for illustration in deriving it. First, the outward centrifugal force for any position of the particle on the orbit's circumference is $\rho\omega^2 a$. Second, there is a balance of two forces in effect at all times on each fluid particle between the outward centrifugal force and an equal but opposite (inward) pressure force [*Kenyon*, 1991], which is called the cyclostrophic balance. Therefore the amount of work done to bring the particle out from the orbit's center to the circumference will not depend on the position along the circumference, because the opposing pressure force is the same along all radii.

Comparing (B2) with (A2) and using the dispersion relation $\omega^2 = gk$, it can be seen that the kinetic and potential energies per unit volume of the surface particle are equal. Like the kinetic energy, the potential energy of a fluid particle in (B2) is a constant independent of time, and all particles at the same mean depth have the same potential energy, which can be seen by substituting $r(z)$ for a in (B2).

By replacing a with $r(z)$ and then integrating (B2) over depth the potential energy per unit horizontal area $\overline{P.E.}$ is

$$\overline{P.E.} = \tfrac{1}{4}\rho g a^2 \tag{B3}$$

where (A4) is also used. Equation (B3) agrees with the classical result for the potential energy per unit horizontal area, which is obtained by an entirely different method.

The total energy per unit volume E of a surface particle is the sum of the kinetic and potential energies per unit volume in (A2) and (B2), which are identical

$$E = K.E. + P.E. = 2K.E. = \rho\omega^2 a^2 = \text{const} \quad \text{(B4)}$$

Likewise the total energy per unit horizontal area $\bar{E}$ is obtained by adding (A5) and (B3), which are identical

$$\bar{E} = \frac{1}{2}\rho g a^2 \quad \text{(B5)}$$

Equation (B5) agrees with the classical result for the total wave energy per unit area of surface gravity waves in deep water.

Our method of computing the potential and kinetic energies based on the particle orbits appears to be new, and it puts the potential and kinetic energies on an equal footing, so to speak, which differs from the usual way of doing things. For example, a common way of computing the potential energy of a surface gravity wave is to calculate the work done against gravity to change a flat surface into a sinusoidal one and then to take a horizontal average over a wave length. This method is independent of the motion beneath the surface; in fact, it can apply to finite amplitude waves of arbitrary shape in water of arbitrary depth. In contrast to this, the kinetic energy is normally computed by applying a perturbation analysis, thus limiting the calculation to infinitesimal waves, to the motion beneath the surface and then taking a vertical integration over the entire water column. Therefore the generally accepted procedures for calculating the kinetic and potential energies are quite different, involving different assumptions and different amounts of analytic labor.

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