On the Depth of Wave Influence

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ABSTRACT

Two physical ideas are combined to derive an expression for the depth of wave influence of a surface gravity wave without assuming the fluid motion is irrotational. One idea is that in the reference frame which makes the wave steady, the static pressure difference between crest and trough balances the dynamic pressure difference between crest and trough. The other idea is that at the depth of wave influence, the pressure under the crest equals the pressure under the trough due to the vertical acceleration of the fluid being downward at the crest and upward at the trough. The depth of wave influence is directly proportional to the wave length and independent of the wave height if the wave height is small compared to the wave length.

1. Introduction

The depth of wave influence of a surface gravity wave is the depth below which the pressure is constant with time and the velocity is always zero at a fixed position. The depth of wave influence is shown in Section 2 to be directly proportional to the wave length and independent of the wave height when wave height is small compared to wave length. The assumption of irrotational motion is not made, so the vorticity need not be zero.

The relationship involving the depth of wave influence is derived by combining two physical ideas. The first idea is that in the reference frame moving with the wave phase velocity, a steady state is possible because the static pressure difference between crest and trough balances the dynamic pressure difference between crest and trough (Einstein, 1916). This force balance depends on the depth of wave influence being a finite distance below the surface.

The second idea is that since the vertical acceleration of the fluid is downward at the crest and upward at the trough, pressure variations at a fixed position, due to the combined effects of the acceleration of gravity and of the vertical acceleration of the fluid, will vanish when the fixed position is sufficiently deep (Sverdrup, Johnson and Fleming, 1942, pp. 520–521). This idea provides the possibility that the depth of wave influence could be finite, but it gives no guidance about what the depth of wave influence is related to.

2. Depth of wave influence

Starting with the first idea, the static pressure difference between crest and trough, Δp_S, is

$$\Delta p_S = \rho g H, \tag{1}$$

where ρ is the fluid density, g the acceleration of gravity and H the wave height. The static pressure is greater at the trough than at the crest. It depends on the acceleration of gravity but it is independent of the fluid motion.

The dynamic pressure difference between crest and trough, Δp_D, depends on the fluid speed but not on the acceleration of gravity.

$$\Delta p_D = \rho u \Delta u. \tag{2}$$

The average horizontal speed of the fluid at the crest and trough is u, in the reference frame which makes the wave motion steady, and Δu is the difference in u between the crest and trough. Equation (2) comes from applying a simplified form of Bernoulli's relation, $p = \text{const} - \frac{1}{2}\rho u^2$, along the streamline at the wave surface.

Conservation of mass between crest and trough for an idealized model gives

$$\Delta u d = uH, \tag{3}$$

where d is the depth of wave influence measured with respect to the equilibrium free surface. The idealization made is that u and Δu are independent of depth down to the depth d; below d the fluid speed everywhere equals u, and $\Delta u = 0$.

Conservation of mass requires that the fluid speed under the trough is greater than that under the crest. Then the reduced form of Bernoulli's relation given above shows that the dynamic pressure at the trough is less than that at the crest. By balancing the static pressure difference (1) against the dynamic pressure difference (2) and using conservation of mass (3), the result is

$$u^2 = gd. \tag{4}$$

The square of the average speed u is directly proportional to the depth of wave influence d and is independent of the wave height H by (4).

Up to this point no restriction has been placed on the wave height. It can be seen from (3) and (2) that the dynamic pressure difference is directly proportional to the wave height, assuming the depth of wave influence is independent of wave height, and (1) shows that the static pressure difference is directly proportional to the wave height. Therefore, if the static and dynamic pressure differences between crest and trough are balanced for one wave height, they are balanced for all wave heights.

The second idea can be expressed

$$\rho(g + a_T)\left(d - \frac{H}{2}\right) = \rho(g - a_C)\left(d + \frac{H}{2}\right), \quad (5)$$

where a_T and a_C are the magnitudes, respectively, of the vertical fluid accelerations under the trough and crest. Equation (5) states that at the depth of wave influence the pressure under the trough equals that under the crest. The idealized model implies that a_T and a_C are independent of depth down to the depth d and zero below d.

The vertical accelerations a_T and a_C above d are

$$a_T = \frac{\left(u + \frac{\Delta u}{2}\right)^2}{R}$$

$$a_C = \frac{\left(u - \frac{\Delta u}{2}\right)^2}{R} \quad (6)$$

where R is the magnitude of the radius of curvature, which is taken to be the same at the crest and trough. Now, substitute (6) into (5), use (3), and assume that the wave height is small compared to the depth of wave influence. The result is

$$\frac{2u^2 d}{R} = gH, \quad \left(\frac{H}{d}\right)^2 \ll 1. \quad (7)$$

The magnitude of the radius of curvature at the crest and trough of a sine wave is

$$R = \frac{\lambda^2}{2\pi^2 H}, \quad \left(\frac{H}{\lambda}\right)^2 \ll 1 \quad (8)$$

where λ is the wave length and the wave height is assumed to be small compared to the wave length.

Finally, combining (7) and (8) with (4) and taking the square root gives the end result

$$d = \frac{\lambda}{2\pi} \quad (9)$$

that the depth of wave influence is directly proportional to the wave length and independent of the wave height.

Given the result (9), the two separate assumptions used in (7) and (8) can be reduced to the single assumption in (8), that the wave height is small compared to the wave length (i.e. the average slope is small compared to one).

3. Discussion

The depth of wave influence in Eq. (9) can be compared to the vertical e-folding scale derived from irrotational theory. Irrotational theory predicts the pressure and velocity variations of small slope surface gravity waves to decrease exponentially with increasing depth, and to be smaller than their surface values by the factor $1/e$ when the depth equals the wave length divided by 2π. Therefore, the depth of wave influence arrived at here is numerically equal to the e-folding depth scale obtained from irrotational theory.

Also Eq. (4) can be compared to the relation between phase speed and wave length for a deep-water surface gravity wave based on irrotational theory. First, the average speed u in (4), which is also the fluid speed below the depth of wave influence, is interpreted to be the phase speed c of the wave by moving in the reference frame with speed u. Second, use Eq. (9) to relate the depth of wave influence to the wave length. Then (4) becomes

$$c^2 = \frac{g\lambda}{2\pi} \quad (10)$$

which is equivalent to the dispersion relation obtained from irrotational theory when the height is small compared to the wave length and the wave length is small compared to the water depth.

It is clear that the depth of wave influence can be calculated for different idealizations than the one used here. For example, the fluid speed in the steady frame could have been chosen to vary linearly with depth below the surface instead of being taken independent of depth down to the depth of wave influence. However, if this were done, no qualitative difference in the end result (9) is expected. It is also clear that the explicit depth dependence of the fluid speed below the surface cannot be deduced on the basis of the two physical ideas used above.

The assumption that the wave height is small compared to the wave length is used here mostly for convenience. By not making this assumption in (8), and by not assuming the wave height is small compared to the depth of wave influence in (7), a more general (but more complex) relation for the depth of wave influence than (9) can be found.

The importance of deriving Eq. (9) in the way it was done here is that, by putting two separate ideas together, a relationship for the depth of wave influence can be obtained. No similar relation has come from either idea by itself (Einstein, 1916; Sverdrup *et al.*, 1942). Eq. (9) is therefore a demonstration that the existence of a surface gravity wave is due to the combination of the two physical ideas just referenced.

The result that the depth of wave influence of a progressive surface gravity wave is finite also carries over to a standing wave, because a standing wave can be considered to be composed of two progressive waves propagating in opposite directions. The idea of Sverdrup *et al.* (1942, pp. 520–521), which provides the possibility that the depth of wave influence could be finite, applies equally well to standing and progressive waves.

REFERENCES

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Sverdrup, H. U., M. W. Johnson and R. H. Fleming, 1942: *The Oceans: Their Physics, Chemistry, and General Biology*. Prentice-Hall, 1087 pp.